Lecture 1
Interacting particle systems
and the backward process

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September 21-22, 2023

Outline

- Poisson construction of finite state space Markov processes
- Countable state spaces
- Interacting particle systems
- The backward process

Probability kernels

Let $\mathbf{S}$ be a finite set.
A *probability kernel* on $\mathbf{S}$ is a function $K : \mathbf{S} \times \mathbf{S} \to [0, 1]$ such that

$$\sum_{y \in \mathbf{S}} K(x, y) = 1 \qquad (x \in \mathbf{S}).$$

We can multiply probability kernels as matrices:

$$(KL)(x, z) := \sum_{y \in \mathbf{S}} K(x, y)L(y, z) \qquad (x, z \in \mathbf{S}).$$

We can also view kernels as linear operators that act on functions $f : \mathbf{S} \to \mathbb{R}$ as

$$Kf(x) := \sum_{y \in \mathbf{S}} K(x, y)f(y).$$

Continuous-time Markov chains

Let **S** be a finite set. A *Markov semigroup* is a collection of probability kernels $(P_t)_{t\geq 0}$ on **S** such that

$$\lim_{t\downarrow 0} P_t = P_0 = 1 \quad \text{and} \quad P_s P_t = P_{s+t} \qquad (s,t \geq 0).$$

Each such Markov semigroup is of the form

$$P_t = e^{tG} := \sum_{n=0}^{\infty} \frac{1}{n!}(tG)^n,$$

where the *generator* G is a matrix of the form

$$G(x,y) \geq 0 \quad (x \neq y) \quad \text{and} \quad \sum_{y \in \mathbf{S}} G(x,y) = 0.$$

One has $P_t(x,y) = 1_{\{x=y\}} + tG(x,y) + O(t^2)$ as $t \to 0$. We call $G(x,y)$ the *rate* of jumps from x to y $(x \neq y)$.

Random mapping representations

We view generators as linear operators that act on functions $f : \mathbf{S} \to \mathbb{R}$ as

$$Gf(x) := \sum_{y \in \mathbf{S}} G(x,y) f(y) \qquad (x \in \mathbf{S}).$$

Then $P_t f = f + tGf + O(t^2)$ as $t \to 0$.
Each generator G can be written in the form

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\},$$

where $\mathcal{G}$ is a finite set whose elements are functions $m : \mathbf{S} \to \mathbf{S}$ and $(r_m)_{m \in \mathcal{G}}$ are nonnegative rates.

We call this a *random mapping representation* of G.

Random mapping representation are usually far from unique.

Poisson construction of Markov processes

Each random mapping representation of G corresponds to a Poisson construction of the Markov process.

Let ρ be the measure on $\mathcal{G} \times \mathbb{R}$ defined by

$$\rho\big(\{m\} \times [s,t]\big) := r_m(t-s) \qquad (m \in \mathcal{G},\ s \le t).$$

Let ω be a Poisson point set on $\mathcal{G} \times \mathbb{R}$ with intensity ρ and let

$$\omega_{s,u} := \big\{(m,t) \in \omega : s < t \le u\big\} \qquad (s \le u).$$

Define random maps $(\mathbb{X}_{s,u})_{s \le u}$ by

$$\begin{aligned} \mathbb{X}_{s,u} &:= m_n \circ \cdots \circ m_1 \\ &\text{with} \quad \omega_{s,u} := \big\{(m_1,t_1),\ldots,(m_n,t_n)\big\} \\ &\text{and} \quad t_1 < \cdots < t_n. \end{aligned}$$

Poisson construction of Markov processes

The random maps $(\mathbb{X}_{s,u})_{s\leq u}$ form a *stochastic flow:*

$$\mathbb{X}_{s,s} = 1 \quad \text{and} \quad \mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u} \qquad (s \leq t \leq u).$$

This stochastic flow has *independent increments* in the sense that

$$\mathbb{X}_{t_0,t_1}, \ldots, \mathbb{X}_{t_{n-1},t_n} \quad \text{are independent} \quad \forall\ t_0 < \cdots < t_n.$$

Let X_0 be an $\mathbf{S}$-valued random variable, independent of ω, and let $s \in \mathbb{R}$. Then

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t\geq 0}$ with generator G.

The contact process

Let Λ be a finite set and let $\lambda(j,i) \geq 0$ $(i,j \in \Lambda)$.
Let $\mathbf{S} := \{0,1\}^\Lambda$ be the set of all functions $x : \Lambda \to \{0,1\}$.
For $x \in \mathbf{S}$ and $\Delta \subset \Lambda$, we let

$$x_\Delta = \big(x(i)\big)_{i \in \Delta}$$

denote the restriction of x to Δ.

For $x, y \in \mathbf{S}$ such that $x_{\Lambda \setminus \{i\}} = y_{\Lambda \setminus \{i\}}$ for some $i \in \Lambda$, let

$$G(x,y) := \begin{cases} \displaystyle\sum_{j \in \Lambda} x(j)\lambda(j,i) & \text{if } x(i) = 0,\ y(i) = 1, \\ \quad 1 & \text{if } x(i) = 1,\ y(i) = 0, \end{cases}$$

and let $G(x,y) := 0$ if x and y differ in more than one site.

The Markov process with generator G is called
the *contact process* with *infection rates* $\big(\lambda(i,j)\big)_{i \neq j}$
and *death rate* 1.

The nearest-neighbour contact process

Often, the lattice Λ has the structure of a graph.
Write $i \sim j$ if i and j are neighbours.

The nearest-neighbour contact process has infection rates

$$\lambda(j,i) = \lambda 1_{\{i \sim j\}} \qquad (i,j \in \Lambda),$$

where $\lambda \geq 0$ is the *infection rate*.

The random mapping representation

For each $i, j \in \Lambda$, we define a *branching map* $\mathtt{bra}_{ji} : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ as

$$\mathtt{bra}_{ji}(x)(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

and a *death map* $\mathtt{dth}_i : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ as

$$\mathtt{dth}_i(x)(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Then the generator of the contact process has the random mapping representation

$$\begin{aligned} Gf(x) = & \sum_{i,j \in \Lambda} \lambda(j,i) \big\{ f\big(\mathtt{bra}_{ji}(x)\big) - f\big(x\big) \big\} \\ & + \sum_{i \in \Lambda} \big\{ f\big(\mathtt{dth}_i(x)\big) - f\big(x\big) \big\} \qquad (x \in \{0,1\}^\Lambda). \end{aligned}$$

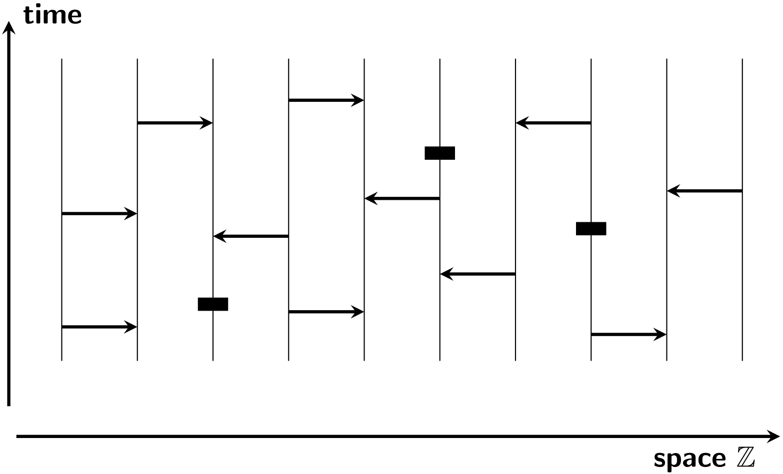
The graphical representation

We visualise the Poisson point set ω by drawing space Λ horizontally and time $\mathbb{R}$ vertically.

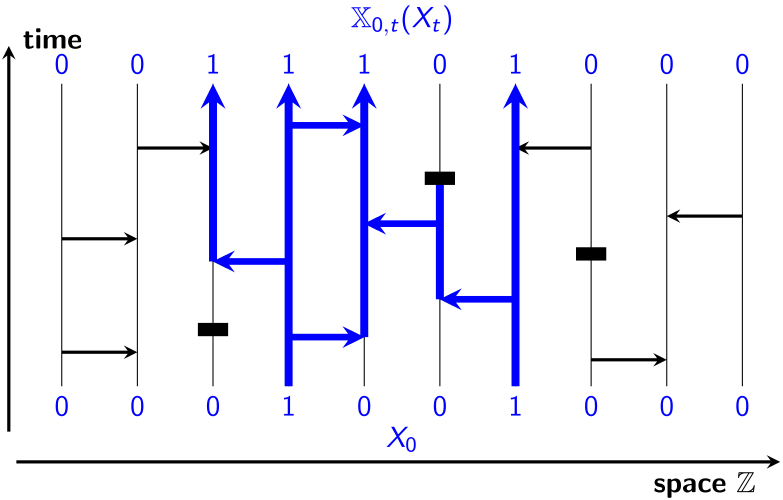
For each $(\mathtt{bra}_{ji}, t) \in \omega$ we draw an arrow from (j, t) to (i, t).
For each $(\mathtt{dth}_{ji}, t) \in \omega$ we draw a blocking symbol $\blacksquare$ at (i, t).

It is easy to check that if we evolve the process under the stochastic flow $(\mathbb{X}_{s,u})_{s \leq u}$, then in the state x, the site i flips from 0 to 1 at rate $\sum_{j \in \Lambda} x(j)\lambda(j, i)$, and from 1 to 0 at rate 1.

The graphical representation



The graphical representation



The cancellative contact process

Let $\oplus$ denote addition modulo 2.

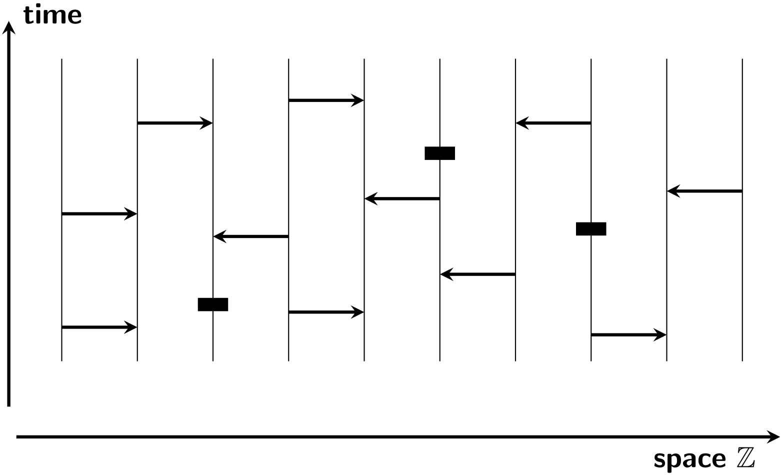
For each $i, j \in \Lambda$, we define an *annihilating branching map* $\mathtt{abr}_{ji} : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ as

$$\mathtt{abr}_{ji}(x)(k) := \begin{cases} x(i) \oplus x(j) & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

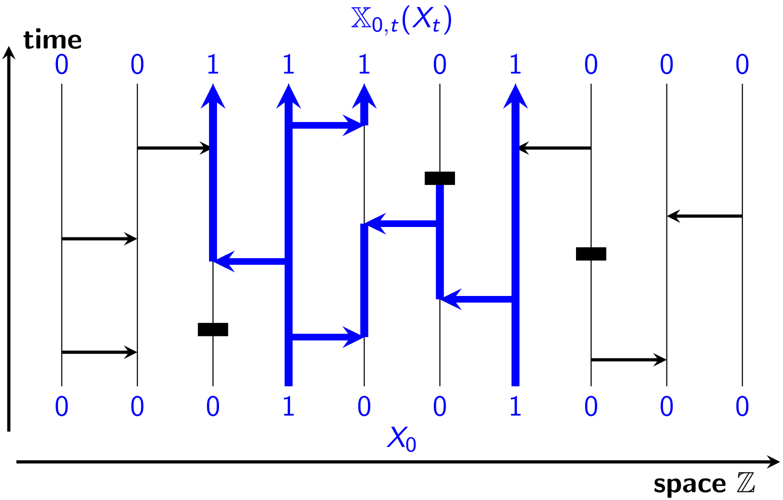
The *cancellative contact process* has the generator

$$\begin{aligned} Gf(x) = & \sum_{i,j \in \Lambda} \lambda(j,i) \big\{ f\big(\mathtt{abr}_{ji}(x)\big) - f\big(x\big) \big\} \\ & + \sum_{i \in \Lambda} \big\{ f\big(\mathtt{dth}_i(x)\big) - f\big(x\big) \big\} \qquad (x \in \{0,1\}^\Lambda). \end{aligned}$$

The graphical representation



The graphical representation



The threshold voter model

For $x, y \in \mathbf{S}$ that differ only in one site $i \in \Lambda$, let

$$G(x,y) := \begin{cases} 1 & \text{if } x(i) \neq x(j) \text{ for some } j \sim i, \\ 0 & \text{otherwise.} \end{cases}$$

and let $G(x,y) := 0$ if x and y differ in more than one site.

The Markov process with generator G is called
the *threshold voter model.*

For each $i \in \Lambda$, we define maps $\mathtt{min}_i$ and $\mathtt{max}_i$ by

$$\mathtt{min}_i(x)(k) := \begin{cases} \bigwedge_{j \sim i} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

and

$$\mathtt{max}_i(x)(k) := \begin{cases} \bigvee_{j \sim i} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Then the generator of the threshold voter model has the random mapping representation

$$\begin{aligned} Gf(x) = & \sum_{i \in \Lambda} \big\{ f\big(\mathtt{min}_i(x)\big) - f\big(x\big) \big\} \\ & + \sum_{i \in \Lambda} \big\{ f\big(\mathtt{max}_i(x)\big) - f\big(x\big) \big\} \qquad (x \in \{0,1\}^\Lambda). \end{aligned}$$

Second random mapping representation

Let $\mathcal{N}_i := \{i\} \cup \{j : j \sim i\}$.

For each $i \in \Lambda$ and $\Delta \subset \mathcal{N}_i$, we define a map $\mathtt{flip}_{i,\Delta}$ by

$$\mathtt{flip}_{i,\Delta}(x)(k) := \begin{cases} \bigoplus_{j\in\Delta} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Then the generator of the threshold voter model has the random mapping representation

$$Gf(x) = \sum_{i\in\Lambda} 2^{-|\mathcal{N}_i|} \sum_{\substack{\Delta\subset\mathcal{N}_i \\ |\Delta| \text{ is odd}}} \big\{ f\big(\mathtt{flip}_{i,\Delta}(x)\big) - f\big(x\big) \big\}.$$

Second random mapping representation

Note there are $2^{|\mathcal{N}_i|-1}$ odd subsets of $\mathcal{N}_i$.
We claim the threshold voter mode has the following description:

- Each site i is activated with Poisson rate 2.
- If i is activated, we uniformly chose an odd subset $\Delta \subset \mathcal{N}_i$ and apply $\mathtt{flip}_{i,\Delta}$.

If $x(j) = 0$ for all $j \in \mathcal{N}_i$ then $\bigoplus_{j\in\Delta} x(j) = 0$ for all odd $\Delta \subset \mathcal{N}_i$ so $\mathtt{flip}_{i,\Delta}$ does nothing.

If $x(j) = 1$ for all $j \in \mathcal{N}_i$ then $\bigoplus_{j\in\Delta} x(j) = 1$ for all odd $\Delta \subset \mathcal{N}_i$ so $\mathtt{flip}_{i,\Delta}$ does nothing.

In all other cases, $\Delta \cap \{j \in \mathcal{N}_j : x(j) = 1\}$ is uniformly chosen from all subsets of $\{j \in \mathcal{N}_j : x(j) = 1\}$ so $\bigoplus_{j\in\Delta} x(j)$ is uniformly distributed on $\{0, 1\}$ and there is a probability $1/2$ that $\mathtt{flip}_{i,\Delta}$ changes the local state at i.

We wish to define the (cancellative) contact process, the threshold voter model, and many more interacting particle systems also when Λ is countably infinite.

As before, we let $\mathbf{S} = \{0,1\}^\Lambda$, and we set

$$\mathbf{S}_{\rm fin} := \big\{x \in \mathbf{S} : |x| < \infty\big\} \quad \text{with} \quad |x| := \sum_{i \in \Lambda} x(i).$$

Note that $\mathbf{S}_{\rm fin}$ (contrary to $\mathbf{S}$) is countable.

We first extend the Poisson construction of Markov processes to countable state space.

Poisson construction on countable spaces

Let $\mathbf{S}$ be a countably infinite set and let $\overline{\mathbf{S}} := \mathbf{S} \cup \{\infty\}$ be its one-point compactification, i.e., $\mathbf{S} \ni x_n \to \infty$ iff for all finite $\mathbf{S}' \subset \mathbf{S}$ there is an $N < \infty$ such that $x_n \notin \mathbf{S}'$ for all $n > N$.

Let $\mathcal{G}$ be a countable set whose elements are functions $m : \mathbf{S} \to \mathbf{S}$. Let $(r_m)_{m \in \mathcal{G}}$ be nonnegative rates.

For $x \in \mathbf{S}$, we set

$$\mathcal{G}_x := \big\{ m \in \mathcal{G} : m(x) \neq x \big\}$$

and we assume that

$$\sum_{m \in \mathcal{G}_x} r_m < \infty \qquad (x \in \mathbf{S}).$$

As before let ρ be the measure on $\mathcal{G} \times \mathbb{R}$ defined by

$$\rho\big(\{m\} \times [s,t]\big) := r_m(t-s) \qquad (m \in \mathcal{G},\ s \leq t).$$

Let ω be a Poisson point set on $\mathcal{G} \times \mathbb{R}$ with intensity ρ.

Recall that a function f defined on a real interval is *cadlag* if it is right-continuous and the left-limit $f_{t-} := \lim_{s \uparrow t} f_s$ exists for all t.

Lemma For all $s \in \mathbb{R}$ and $x \in \mathbf{S}$, there exists a unique cadlag function $[s,\infty) \ni t \mapsto X_t \in \overline{\mathbf{S}}$ and time $0 < \tau \leq \infty$ such that

1. $X_s = x$,
2. $X_t = m(X_{t-})$ if $(m,t) \in \omega$ for some necessarily unique $m \in \mathcal{G}$ and $X_t = X_{t-}$ otherwise $\big(t \in [0,\tau)\big)$.
3. If $\tau < \infty$, then $\lim_{t \uparrow \tau} X_t = \infty$ and $X_t = \infty$ for all $t \geq \tau$.

Explosion

If $\tau < \infty$, then we say that the Markov process *explodes*.
To prove that the process is nonexplosive, it suffices to find a Lyapunov function.

Lemma Assume that there exists a function $L : \mathbf{S} \to [0, \infty)$ such that $L(x) \to \infty$ as $x \to \infty$, and a constant $K < \infty$ such that

$$GL(x) \leq KL(x) \qquad (x \in \mathbf{S}).$$

Then $\tau = \infty$ a.s. and the process started in $X_s = x$ satisfies

$$\mathbb{E}^x\big[L(X_t)\big] \leq e^{K(t-s)}L(x) \qquad (s \leq t,\ x \in \mathbf{S}).$$

We define random maps $(\mathbb{X}_{s,u})_{s \leq u}$ by

$$\mathbb{X}_{s,u}(x) := X_u \quad \text{where} \quad (X_t)_{t \geq s} \text{ satisfies 1–3.}$$

Finite contact processes

Recall the generator of the contact process

$$Gf(x) = \sum_{i,j\in\Lambda} \lambda(j,i)\big\{f\big(\mathtt{bra}_{ji}(x)\big) - f\big(x\big)\big\} \\ + \sum_{i\in\Lambda} \big\{f\big(\mathtt{dth}_i(x)\big) - f\big(x\big)\big\} \qquad (x \in \{0,1\}^\Lambda).$$

As before, let

$$\mathbf{S}_{\mathrm{fin}} := \big\{x \in \mathbf{S} : |x| < \infty\big\} \quad \text{with} \quad |x| := \sum_{i\in\Lambda} x(i).$$

We use the Lypapunov function $L(x) := |x|$. Provided that

$$K := \sup_{j\in\Lambda} \sum_{i\in\Lambda} \lambda(j,i) < \infty,$$

we have $GL(x) \leq KL(x)$ $(x \in \mathbf{S}_{\mathrm{fin}})$ and the contact process with state space $\mathbf{S}_{\mathrm{fin}}$ is well-defined.

Infinite initial states

The methods so far allow us to construct the (cancellative) contact process, the threshold voter model, and many more interacting particle systems for initial states $x \in \{0,1\}^\Lambda$ such that $|x| < \infty$.

We wish to allow initial states x with $|x| = \infty$.

We replace $\{0,1\}$ by a general finite set S, the *local state space*.
We let Λ be a countable set, called the *lattice*.
We let $\mathbf{S} := S^\Lambda$ denote the space of all functions $x : \Lambda \to S$.
We equip S^Λ with the product topology, under which it is compact.

Relevant sites

As before, for $x \in S^\Lambda$ and $\Delta \subset \Lambda$, we let

$$x_\Delta = \big(x(i)\big)_{i \in \Delta}$$

denote the restriction of x to Δ.

Lemma Let T be a finite set. Then a function $f : S^\Lambda \to T$ is continuous if and only if it depends on finitely many coordinates, i.e., there exists a finite set $\Delta \subset \Lambda$ and a function $f' : S^\Delta \to T$ such that $f(x) = f'\big(x_\Delta\big)$ $(x \in S^\Lambda)$.

Relevant sites

For any function $f : S^\Lambda \to T$, we call

$$\mathcal{R}(f) := \big\{ i \in \Lambda : \exists x, y \in S^\Lambda \text{ s.t. } f(x) \neq f(y) \text{ and } x_{\Lambda \setminus \{i\}} = y_{\Lambda \setminus \{i\}} \big\}.$$

the set of f-*relevant sites*.

If f is continuous, then $\mathcal{R}(f)$ is the smallest possible finite set $\Delta \subset \Lambda$ such that there exists a function $f' : S^\Delta \to T$ with $f(x) = f'(x_\Delta)$ $(x \in S^\Lambda)$.

If f is not continuous, then strange things can happen:

Example Set $S = T := \{0, 1\}$ and $f(x) := 1$ iff $\{i \in \Lambda : x(i) = 1\}$ is finite. Then $\mathcal{R}(f) = \emptyset$, but f is not constant.

Example Set $S = T := \{0, 1\}$ and $f(x) := 1$ iff $\{i \in \Lambda : x(i) = 1\}$ is finite and even. Then $\mathcal{R}(f) = \Lambda$.

Local maps

For any map $m : S^\Lambda \to S^\Lambda$ and $i \in \Lambda$, we define $m[i] : S^\Lambda \to S$ by

$$m[i](x) := m(x)(i) \qquad (x \in S^\Lambda).$$

Then m is continuous iff $m[i]$ is continuous for all $i \in \Lambda$.

By definition, a map $m : S^\Lambda \to S^\Lambda$ is *local* iff

1. m is continuous,
2. $\mathcal{D}(m) := \big\{ i \in \Lambda : \exists x \in S^\Lambda \text{ s.t. } m(x)(i) \neq x(i) \big\}$ is finite.

We will be interested in interacting particle systems with generator of the form

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\},$$

where $\mathcal{G}$ is a countable set whose elements are local functions $m : S^\Lambda \to S^\Lambda$ and $(r_m)_{m \in \mathcal{G}}$ are nonnegative rates.

Poisson construction of particle systems

Theorem Assume that

$$\sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(m)\ni i}} r_m\big(|\mathcal{R}(m([i]))|+1\big) < \infty.$$

Then almost surely, for each $s\in\mathbb{R}$ and $x\in S^\Lambda$, there exists a unique cadlag function $(X_t)_{t\geq s}$ such that

1. $X_s = x$,
2. $X_t = m(X_{t-})$ if $(m,t)\in\omega$ for some necessarily unique $m\in\mathcal{G}$ and $X_t = X_{t-}$ otherwise.

We define random maps $(\mathbb{X}_{s,u})_{s\leq u}$ by

$$\mathbb{X}_{s,u}(x) := X_u \quad \text{where} \quad (X_t)_{t\geq s} \text{ satisfies 1–2.}$$

If X_0 is independent of ω and $s\in\mathbb{R}$, then

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t\geq 0)$$

is a Markov process $(X_t)_{t\geq 0}$ with generator G.

The backtracking process

Idea of the proof Fix a finite "target" set T.

Then the set $\mathcal{C}(S^\Lambda, T)$ of continuous functions $f : S^\Lambda \to T$ is countable.

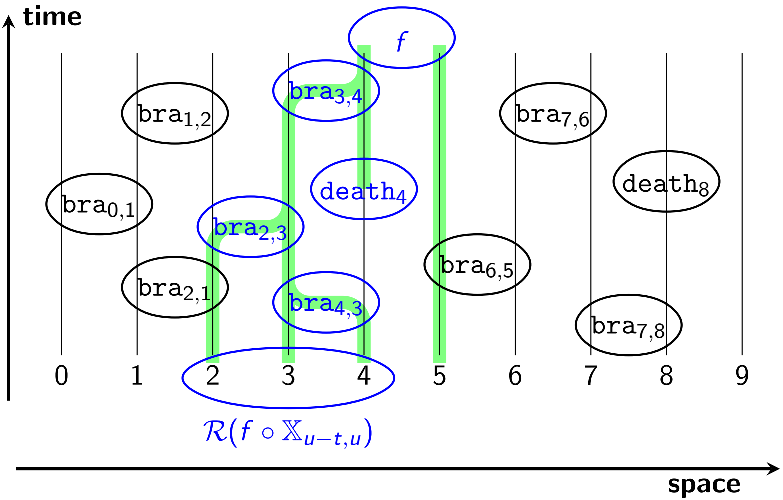
For fixed $f \in \mathcal{C}(S^\Lambda, T)$ and $u \in \mathbb{R}$, we want the *backtracking process*

$$(F_t)_{t\geq 0} := \big(f \circ \mathbb{X}_{u-t,t}\big)_{t\geq 0}$$

to be a well-defined Markov process with state space $\mathcal{C}(S^\Lambda, T)$ and generator

$$H\mathcal{F}(f) := \sum_{m\in\mathcal{G}} r_m \big\{\mathcal{F}\big(f \circ m\big) - \mathcal{F}\big(f\big)\big\}.$$

The backtracking process



The backtracking process

The condition

$$\sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(m)\ni i}} r_m < \infty$$

guarantees that

$$\sum_{\substack{m\in\mathcal{G}\\ f\circ m\neq f}} r_m < \infty \qquad \big(f\in\mathcal{C}(S^\Lambda,T)\big),$$

and the Lyapunov function

$$L(f) := |\mathcal{R}(f)| \qquad \big(f\in\mathcal{C}(S^\Lambda,T)\big),$$

satisfies $HL(f)\leq KL(f)$ with

$$K := \sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(m)\ni i}} r_m|\mathcal{R}(m([i])|.$$

Examples

The contact process is well-defined for arbitrary initial states $x \in \{0,1\}^\Lambda$ provided that

$$\sup_{i\in\Lambda} \sum_{j\in\Lambda} \lambda(j,i) < \infty.$$

Note that earlier, we proved that $|X_0| < \infty$ implies that $|X_t| < \infty$ for all $t \geq 0$ provided that

$$\sup_{j\in\Lambda} \sum_{i\in\Lambda} \lambda(j,i) < \infty.$$

Our theorem implies that the threshold voter model is well-defined provided the graph Λ is of uniformly bounded degree.
(This condition can be relaxed by a more clever choice of the Lyapunov function.)

The backward stochastic flow

We set

$$\mathbb{F}_{u,s}(f) := f \circ \mathbb{X}_{s,u} \qquad \big(u \geq s,\ f \in \mathcal{C}(S^\Lambda, T)\big).$$

Then $(\mathbb{F}_{u,s})_{u \geq s}$ is a *backward stochastic flow:*

$$\mathbb{F}_{s,s} = 1 \quad \text{and} \quad \mathbb{F}_{t,s} \circ \mathbb{F}_{u,t} = \mathbb{X}_{s,u} \qquad (u \geq t \geq s).$$

If F_0 is a random variable with values in $\mathcal{C}(S^\Lambda, T)$, independent of ω, and $u \in \mathbb{R}$, then

$$F_t := \mathbb{F}_{u,u-t}(F_0) \qquad (t \geq 0)$$

defines a Markov process $(F_t)_{t \geq 0}$ with generator H.
It is a consequence of our construction that this *backtracking process* has *caglad* sample paths, i.e., $t \mapsto F_t$ left-continuous and the right limit $F_{t+} := \lim_{s \downarrow t} F_s$ exists for all $t \geq 0$.

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