Lecture 2
Monoid duality

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Outline

- Forward and backward stochastic flows
- Pathwise duality
- Additive duality
- Cancellative duality
- Monoid duality

The split real line

The *split real line* is $\mathbb{R}_\mathfrak{s} := \big\{t\star : t \in \mathbb{R},\ \star \in \{-,+\}\big\}$.
We let $\underline{\tau} := t$ denote the *real part* and
$\mathfrak{s}(\tau) := *$ the *sign* of a *split real number* $\tau = t\star$.

We equip $\mathbb{R}_\mathfrak{s}$ with the *lexicographic order* and the associated *order topology*, which is generated by the open intervals

$$(\!(\sigma,\rho)\!) := \{\tau \in \mathbb{R}_\mathfrak{s} : \sigma < \tau < \rho\} \qquad (\sigma,\tau \in \mathbb{R}_\mathfrak{s}).$$

For $\tau_n, t+ \in \mathbb{R}_\mathfrak{s}$ one has $\tau_n \to t+$ iff
$\underline{\tau}_n \to t$ and $\tau_n \geq t+$ for all n large enough.

For $\tau_n, t- \in \mathbb{R}_\mathfrak{s}$ one has $\tau_n \to t-$ iff
$\underline{\tau}_n \to t$ and $\tau_n \leq t-$ for all n large enough.

The split real line

We also write

$$[\![\sigma, \rho)\!) := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau < \rho\} \qquad (\sigma, \tau \in \mathbb{R}_{\mathfrak{s}})$$

etc. The space $\mathbb{R}_{\mathfrak{s}}$ is first countable, Hausdorff, and separable, but not second countable and not metrisable. A subset $C \subset \mathbb{R}_{\mathfrak{s}}^d$ is compact iff it is closed and bounded.

Cadlag functions

Let T be a Hausdorff topological space and let $s, u \in \mathbb{R}$, $s < u$.

Lemma Let $f^+ : [s, u] \to T$ be a cadlag function and let $f^- : [s, u] \to T$ be its left-continuous modification, defined as

$$f^-(t) := \begin{cases} \lim_{r \uparrow t} f^+(r) & \text{if } t \in (s, u], \\ f^+(s) & \text{if } t = s. \end{cases}$$

Then setting

$$f(t\pm) := f^{\pm}(t) \qquad \big(t\pm \in [\![s+, t+]\!]\big)$$

defines a continuous function $f : [\![s+, t+]\!] \to T$, and each continuous function $f : [\![s+, t+]\!] \to T$ is of this form.

Remark A continuous function $f : [\![s-, t+]\!] \to T$ can jump at time s but this is not possible for a cadlag function.

Stochastic flows

Let $\mathbf{S}$ be a set and let $(\mathbb{X}_{s,u})_{s \leq u}$ be a collection of random maps $\mathbb{X}_{s,u} : \mathbf{S} \to \mathbf{S}$. We say that $(\mathbb{X}_{s,u})_{s \leq u}$ is a *stochastic flow* on $\mathbf{S}$ if

$$\mathbb{X}_{s,s} = 1 \quad \text{and} \quad \mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u} \quad \text{a.s.} \quad \forall s \leq t \leq u.$$

Often, one even has a.s. equality jointly for all $s \leq t \leq u$.
In particular, this holds for all stochastic flows we have seen so far.
A stochastic flow has *independent increments* if

$$\mathbb{X}_{t_0,t_1}, \ldots, \mathbb{X}_{t_{n-1},t_n} \quad \text{are independent} \quad \forall\ t_0 < \cdots < t_n.$$

If $(\mathbb{X}_{s,u})_{s \leq u}$ is a stochastic flow with independent increments, X_0 is an independent $\mathbf{S}$-valued random variable, and $s \in \mathbb{R}$, then

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t \geq 0}$.

A *backward stochastic flow* on a set **R** is a collection of random maps $(\mathbb{Y}_{u,s})_{u\geq s}$ such that

$$\mathbb{Y}_{s,s} = 1 \quad \text{and} \quad \mathbb{Y}_{t,s} \circ \mathbb{Y}_{u,t} = \mathbb{Y}_{u,s} \quad \text{a.s.} \quad \forall u \geq t \geq s.$$

If $(\mathbb{Y}_{u,s})_{u\geq s}$ is a backward stochastic flow on **R** with independent increments, Y_0 is an independent **R**-valued random variable, and $u \in \mathbb{R}$, then

$$Y_t := \mathbb{Y}_{u,u-t}(Y_0) \qquad (t \geq 0)$$

defines a Markov process $(Y_t)_{t\geq 0}$.

The cadlag property

Let $\Delta := \{(\sigma,\tau) \in \mathbb{R}^2_{\mathfrak{s}} : \sigma \leq \tau\}$. Let $(\mathbb{X}_{s,u})_{s\leq u}$ be a stochastic flow on a Hausdorff topological space $\mathbf{S}$.
We say that $(\mathbb{X}_{s,u})_{s\leq u}$ is *cadlag* if there exists a continuous function

$$\Delta \times \mathbf{S} \ni (\sigma,\tau,x) \mapsto \mathbb{X}^{\mathfrak{s}}_{\sigma,\tau}(x) \in \mathbf{S}$$

such that

$$\mathbb{X}_{s,t}(x) = \mathbb{X}^{\mathfrak{s}}_{s+,t+}(x) \qquad (s,t \in \mathbb{R},\ s \leq t,\ x \in \mathbf{S}).$$

Similarly, we call a backward stochastic flow $(\mathbb{Y}_{u,s})_{u\geq s}$ on $\mathbf{R}$ *cadlag* if there exists a continuous function

$$\Delta \times \mathbf{R} \ni (\sigma,\tau,y) \mapsto \mathbb{Y}^{\mathfrak{s}}_{\tau,\sigma}(y) \in \mathbf{R}$$

such that

$$\mathbb{Y}_{t,s}(y) = \mathbb{Y}^{\mathfrak{s}}_{t+,s+}(y) \qquad (t,s \in \mathbb{R},\ t \geq s,\ y \in \mathbf{R}).$$

The cadlag property

If a stochastic flow $(\mathbb{X}_{s,u})_{s\leq u}$ is cadlag, then a Markov process $(X_t)_{t\geq 0}$ defined as

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t \geq 0)$$

has cadlag sample paths.
However, if a backward stochastic flow $(\mathbb{Y}_{u,s})_{u\geq s}$ is cadlag, then a Markov process $(Y_t)_{t\geq 0}$ defined as

$$Y_t := \mathbb{Y}_{u,u-t}(Y_0) \qquad (t \geq 0)$$

has *caglad* sample paths.

Remark Without the split real line, it is quite tricky to define cadlag functions of several variables. Kolmogorov (1956) already pointed out that cadlag functions can be viewed as continuous functions on the split real line.

Pathwise duality

Let $\mathbf{S}, \mathbf{R}$, and T be sets, and let $\psi : \mathbf{S} \times \mathbf{R} \to T$ be a function. By definition, two maps $m : \mathbf{S} \to \mathbf{S}$ and $\hat{m} : \mathbf{R} \to \mathbf{R}$ are *dual* w.r.t. the *duality function* ψ if

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \qquad (x \in \mathbf{S},\ y \in \mathbf{R}).$$

A stochastic flow $(\mathbb{X}_{s,u})_{s \leq u}$ on $\mathbf{S}$ and a backward stochastic flow $(\mathbb{Y}_{u,s})_{u \geq s}$ on $\mathbf{R}$ are *dual* w.r.t. ψ if

$$\psi\big(\mathbb{X}_{s,u}(x), y\big) = \psi\big(x, \mathbb{Y}_{u,s}(y)\big) \qquad (s \leq u,\ x \in \mathbf{S},\ y \in \mathbf{R}).$$

Two Markov processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are *pathwise dual* if they can be constructed from a stochastic flow and a backward stochastic flow that are dual.

The backtracking process

Let $\mathbf{S}$ and T be compact metrisable spaces.
Equip $\mathcal{C}(\mathbf{S}, T)$ with the topology of uniform convergence.

Lemma If $(\mathbb{X}_{s,u})_{s \leq u}$ is a cadlag stochastic flow on $\mathbf{S}$, then

$$\mathbb{F}_{u,s}(f) := f \circ \mathbb{X}_{s,u} \qquad \big(u \geq s,\ \mathcal{C}(\mathbf{S}, T)\big)$$

defines a cadlag backward stochastic flow $(\mathbb{F}_{u,s})_{u \geq s}$ on $\mathcal{C}(\mathbf{S}, T)$.

The associated *backtracking process* $(F_t)_{t \geq 0}$ is pathwise dual to $(X_t)_{t \geq 0}$ with duality function

$$\psi_{\mathrm{back}}(x, f) := f(x) \qquad \big(x \in \mathbf{S},\ f \in \mathcal{C}(\mathbf{S}, T)\big).$$

Indeed:

$$\psi_{\mathrm{back}}\big(\mathbb{X}_{s,u}(x), f\big) = f \circ \mathbb{X}_{s,u}(x) = \psi_{\mathrm{back}}\big(x, \mathbb{F}_{u,s}(f)\big).$$

Invariant subspaces

Def a subspace $\mathcal{E} \subset \mathcal{C}(\mathbf{S}, T)$ is *invariant* under the backward stochastic flow $(\mathbb{F}_{u,s})_{u \geq s}$ of the backtracking process if

$$f \in \mathcal{E} \quad \Rightarrow \quad \mathbb{F}_{u,s}(f) \in \mathcal{E} \qquad (u \geq s).$$

Claim *Useful pathwise dualities are associated with invariant subspaces of the backward stochastic flow of the backtracking process.*

A bit of order theory

Let S be a partially ordered set.

Def A *dual* of a partially ordered set S is a partially ordered set S' together with a bijection $S \ni x \mapsto x' \in S'$ such that

$$x \leq y \quad \Leftrightarrow \quad x' \geq y'.$$

Example 1 $S' := S$ equipped with the *reverse order* $x \leq' y \;\Leftrightarrow\; x \geq y$ and $x \mapsto x'$ is the identity map.

Example 2 If $S \subset \mathcal{P}(A) := \{x : x \subset A\}$, equipped with the order of set inclusion, then we may take $x' := A \backslash x$ the complement and $S' := \{x' : x \in S\}$.

Naturally $S'' = S$.

A bit of order theory

Define the *upset* $A^{\uparrow}$ and *downset* $A^{\downarrow}$ of $A \subset S$ as

$$A^{\uparrow} := \{y \in S : \exists x \in A \text{ s.t. } x \leq y\},$$
$$A^{\downarrow} := \{y \in S : \exists x \in A \text{ s.t. } x \geq y\}.$$

Then A is *increasing* if $A = A^{\uparrow}$ and *decreasing* if $A = A^{\downarrow}$.

S is a *lattice* if $\forall x, y \in S \ \exists! x \vee y, x \wedge y \in S$ s.t.

$$\{x\}^{\uparrow} \cap \{y\}^{\uparrow} = \{x \vee y\}^{\uparrow} \quad \text{and} \quad \{x\}^{\downarrow} \cap \{y\}^{\downarrow} = \{x \wedge y\}^{\downarrow}.$$

A finite lattice has a unique least element 0 and greatest element 1.
A map $m : S \to T$ is *additive* if

$$m(0) = 0 \quad \text{and} \quad m(x \vee y) = m(x) \vee m(y) \qquad (x, y \in S).$$

Additive duality

Let S, T be finite lattices and let Λ be countable.
Equip $\mathbf{S} := S^\Lambda$ with the product order and let $\underline{0}(i) := 0$ $(i \in \Lambda)$.
Let $\mathcal{C}_{\rm add}(\mathbf{S}, T) := \big\{ m \in \mathcal{C}(\mathbf{S}, T) : m \text{ is additive} \big\}$.

Lemma If $m : \mathbf{S} \to \mathbf{S}$ additive for all $m \in \mathcal{G}$, then $\mathcal{C}_{\rm add}(\mathbf{S}, T)$ is invariant under the backward stochastic flow $(\mathbb{F}_{u,s})_{u \geq s}$ of the backtracking process,

Proof Since the concatenation of additive maps is additive, $\mathbb{F}_{u,s}(f) = f \circ \mathbb{X}_{s,u}$ is additive for each $f \in \mathcal{C}_{\rm add}(\mathbf{S}, T)$. ∎

Additive duality

Let S be a finite lattice, let R be its dual, and let $T := \{0,1\}$. For each $y \in \mathbf{R}_{\rm fin}$, define

$$f_y(x) := 1_{\{x \not\leq y'\}} \qquad (x \in \mathbf{S}).$$

Lemma $f_y \in \mathcal{C}_{\rm add}(\mathbf{S}, T)$ and for each $f \in \mathcal{C}_{\rm add}(\mathbf{S}, T)$ there exists a unique $y \in \mathbf{R}_{\rm fin}$ such that $f = f_y$.

Partial proof $f_y(\underline{0}) = 1_{\{0 \not\leq y'\}} = 0$ and

$$\begin{aligned} f_y(x_1 \vee x_2) &= 1_{\{x_1 \vee x_2 \not\leq y'\}} \\ &= 1_{\{x_1 \not\leq y'\}} \vee 1_{\{x_2 \not\leq y'\}} = f_y(x_1) \vee f_y(x_2). \end{aligned} \qquad \blacksquare$$

Additive duality

The abstract duality function

$$\psi_{\mathrm{back}}(x,f) := f(x) \qquad \big(x \in \mathbf{S},\ f \in \mathcal{C}_{\mathrm{add}}(\mathbf{S},T)\big)$$

now takes the concrete form

$$\psi_{\mathrm{add}}(x,y) := \psi_{\mathrm{back}}(x,f_y) = 1_{\{x \not\leq y'\}} \qquad (x \in \mathbf{S},\ y \in \mathbf{R}_{\mathrm{fin}}).$$

Our arguments so far show that for each continuous additive map $m : \mathbf{S} \to \mathbf{S}$, there exists a unique map $\hat{m} : \mathbf{R}_{\mathrm{fin}} \to \mathbf{R}_{\mathrm{fin}}$ such that

$$\psi_{\mathrm{add}}\big(m(x),y\big) = \psi_{\mathrm{add}}\big(x,\hat{m}(y)\big) \qquad (x \in \mathbf{S},\ y \in \mathbf{R}_{\mathrm{fin}}).$$

Remarkable fact $\hat{m}$ is also additive. If m is local, then so is $\hat{m}$.

Additive duality

Theorem Let $(\mathbb{X}_{s,u})_{s\leq u}$ be the cadlag stochastic flow of an interacting particle system with state space $\mathbf{S} = S^\Lambda$ and generator

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big\{f\big(m(x)\big) - f\big(x\big)\big\}.$$

Assume that S is a finite lattice with dual R and that all local maps $m \in \mathcal{G}$ are additive. Then there exists a cadlag backward stochastic flow $(\mathbb{Y}_{u,s})_{u\geq s}$ on $\mathbf{R}_{\rm fin}$ such that

$$\psi_{\rm add}\big(\mathbb{X}_{s,u}(x), y\big) = \psi_{\rm add}\big(x, \mathbb{Y}_{u,s}(y)\big) \qquad (s \leq u,\ x \in \mathbf{S},\ y \in \mathbf{R}_{\rm fin}).$$

The generator of the associated Markov process takes the form

$$Hf(y) = \sum_{m\in\mathcal{G}} r_m\big\{f\big(\hat{m}(y)\big) - f\big(y\big)\big\}.$$

Additive duality

Remark If the rates satisfy

$$\sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(\hat m)\ni i}} r_m\big(|\mathcal{R}(\hat m([i])| + 1\big) < \infty,$$

then $(\mathbb{Y}_{u,s})_{u\geq s}$ can be extended to $\mathbf{R}$, and

$$\psi_{\rm add}\big(\mathbb{X}_{s,u}(x), y\big) = \psi_{\rm add}\big(x, \mathbb{Y}_{u,s}(y)\big) \qquad (s\leq u,\ x\in\mathbf{S},\ y\in\mathbf{R}).$$

The contact process

For the contact process we set $S = R = \{0,1\}$ and define $S \ni x \mapsto x' \in R$ by $x' := 1 - x$. Then

$$\psi_{\rm add}(x,y) = 1_{\{x \not\leq y'\}} = 1_{\{x \wedge y \neq \underline{0}\}} \qquad (x \in \mathbf{S},\ y \in \mathbf{R}).$$

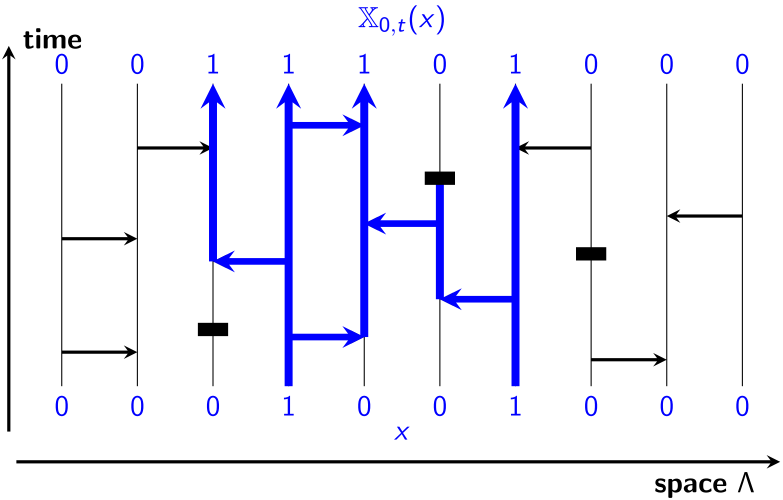
We observe that

$$\begin{aligned}\psi_{\rm add}\big(\mathtt{bra}_{ji}(x), y\big) &= \psi_{\rm add}\big(x, \mathtt{bra}_{ij}(y)\big),\\ \psi_{\rm add}\big(\mathtt{dth}_{i}(x), y\big) &= \psi_{\rm add}\big(x, \mathtt{dth}_{i}(y)\big).\end{aligned}$$

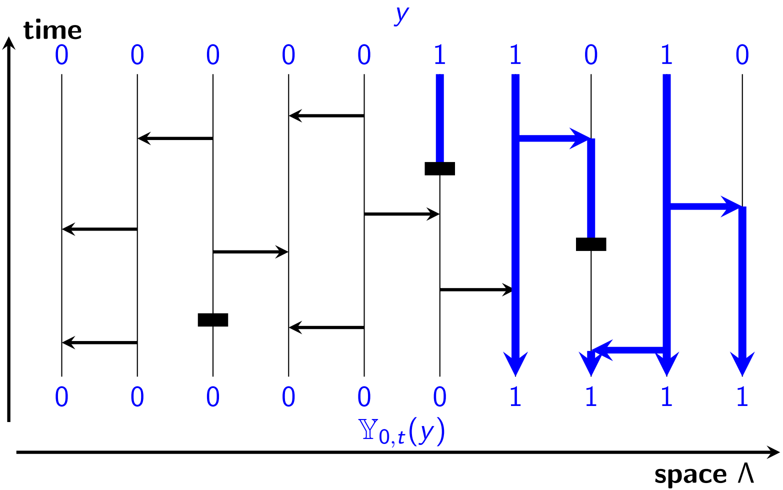
The generators of the forward process X and dual process Y are

$$\begin{aligned}Gf(x) &= \sum_{i,j\in\Lambda} \lambda(j,i)\big\{f\big(\mathtt{bra}_{ji}(x)\big) - f\big(x\big)\big\} + \sum_{i\in\Lambda}\big\{f\big(\mathtt{dth}_{i}(x)\big) - f\big(x\big)\big\},\\ Gf(y) &= \sum_{i,j\in\Lambda} \lambda(j,i)\big\{f\big(\mathtt{bra}_{ij}(y)\big) - f\big(y\big)\big\} + \sum_{i\in\Lambda}\big\{f\big(\mathtt{dth}_{i}(y)\big) - f\big(y\big)\big\}.\end{aligned}$$

The contact process



The contact process



The contact process

$$
\begin{aligned}
\psi_{\rm add}\big(\mathbb{X}_{0,t}(x),y\big) &= 1_{\{\mathbb{X}_{0,t}(x)\wedge y\neq \underline{0}\}}\\
&= 1_{\{\text{there is an open path from } x \text{ to } y\}}\\
&= 1_{\{x\wedge \mathbb{Y}_{t,0}(y)\neq \underline{0}\}} = \psi_{\rm add}\big(x,\mathbb{Y}_{t,0}(y)\big).
\end{aligned}
$$

The contact process

Lemma The contact process $X = (X_t)_{t \geq 0}$ started in $X_0 = \underline{1}$ with infection rates $\lambda(j, i)$ satisfies

$$\mathbb{P}^{\underline{1}}\big[X_t \in \cdot\,\big] \Rightarrow \overline{\nu},$$

where $\overline{\nu}$ is an invariant law that is uniquely characterised by

$$\int \overline{\nu}(\mathrm{d}x) 1_{\{x \wedge y \neq \underline{0}\}} = \mathbb{P}^y\big[Y_t \neq \underline{0}\ \forall t \geq 0\big] \qquad (y \in \mathbf{S}_{\rm fin}),$$

where $Y = (Y_t)_{t \geq 0}$ is the contact process with reversed infection rates $\lambda^\dagger(j, i) := \lambda(i, j)$.

Proof

$\mathbb{P}^{\underline{1}}\big[X_t \wedge y \neq \underline{0}\big] = \mathbb{P}^y\big[\underline{1} \wedge Y_t \neq \underline{0}\big] \underset{t \to \infty}{\longrightarrow} \mathbb{P}^y\big[Y_t \neq \underline{0}\ \forall t \geq 0\big].$

Since this holds for all $y \in \mathbf{S}_{\rm fin}$, the claim follows. ∎

The contact process

Theorem Assume that $\Lambda = \mathbb{Z}^d$.
Assume that $\lambda(j,i) = \lambda(j-i)$ depends only on the difference $j-i$.
Assume that $\mathbb{P}\big[X_0 \in \cdot\,\big]$ is translation invariant and $\mathbb{P}\big[X_0 = \underline{0}\big] = 0$. Then

$$\mathbb{P}\big[X_t \in \cdot\,\big] \Rightarrow \overline{\nu}.$$

Proof idea Need to show

$$\mathbb{P}\big[X_t \wedge y \neq 0\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}^y\big[Y_t \neq \underline{0}\ \forall t \geq 0\big] =: \rho(y) \quad (y \in \mathbf{S}_{\rm fin}).$$

Fix $N > 0$. Then

$$\mathbb{P}\big[X_t \wedge y \neq 0\big] = \mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \mathbb{Y}_{t,1}(y) \neq \underline{0}\big]$$

$$= \mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \mathbb{Y}_{t,1}(y) \neq \underline{0} \,\big|\, |\mathbb{Y}_{t,1}(y)| = 0\big]\, \mathbb{P}\big[|\mathbb{Y}_{t,1}(y)| = 0\big]$$
$$+\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \mathbb{Y}_{t,1}(y) \neq \underline{0} \,\big|\, 0 < |\mathbb{Y}_{t,1}(y)| < N\big]\, \mathbb{P}\big[0 < |\mathbb{Y}_{t,1}(y)| < N\big]$$
$$+\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \mathbb{Y}_{t,1}(y) \neq \underline{0} \,\big|\, N \leq |\mathbb{Y}_{t,1}(y)|\big]\, \mathbb{P}\big[N \leq |\mathbb{Y}_{t,1}(y)|\big].$$

The contact process

Almost surely

$$\exists t < \infty \text{ s.t. } \mathbb{Y}_{t,1}(y) \neq \underline{0} \quad \text{or} \quad |\mathbb{Y}_{t,1}(y)| \underset{t\to\infty}{\longrightarrow} \infty.$$

As a consequence

$$\begin{aligned}
&\underbrace{\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \mathbb{Y}_{t,1}(y) \neq \underline{0} \,\big|\, |\mathbb{Y}_{t,1}(y)| = 0\big]}_{=0} \mathbb{P}\big[|\mathbb{Y}_{t,1}(y)| = 0\big] \\
&+\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \mathbb{Y}_{t,1}(y) \neq \underline{0} \,\big|\, 0 < |\mathbb{Y}_{t,1}(y)| < N\big] \underbrace{\mathbb{P}\big[0 < |\mathbb{Y}_{t,1}(y)| < N\big]}_{\underset{t\to\infty}{\longrightarrow} 0} \\
&+ \underbrace{\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \mathbb{Y}_{t,1}(y) \neq \underline{0} \,\big|\, N \leq |\mathbb{Y}_{t,1}(y)|\big]}_{\approx 1} \underbrace{\mathbb{P}\big[N \leq |\mathbb{Y}_{t,1}(y)|\big]}_{\underset{t\to\infty}{\longrightarrow} \rho(y)}.
\end{aligned}$$

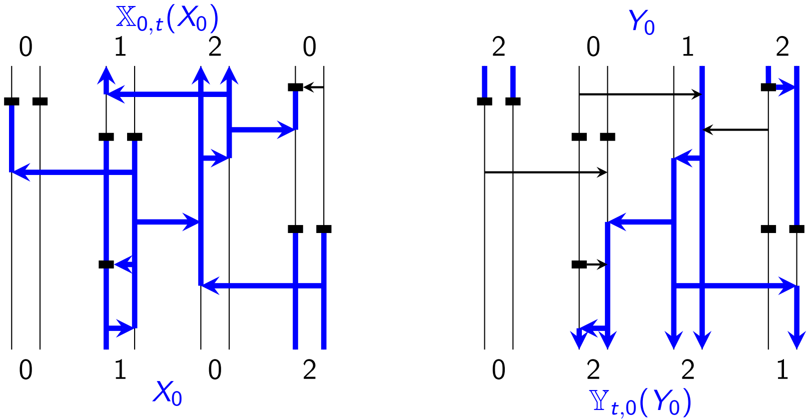
The two-stage contact process

Krone (1999) has studied a *two-stage contact process*.
Here $S = R = \{0,1,2\}$ and $x' := 2 - x$. The duality function is

$$\psi_{\rm Krone}(x,y) := 1_{\{x(i)+y(i)>2 \text{ for some } i\in\Lambda\}} \qquad (x,y\in S^\Lambda).$$

Let U be a partially ordered set and $\mathcal{P}_{\rm dec}(U) := \{A \subset U : A^\downarrow = A\}$.
Then $S := \big(\mathcal{P}_{\rm dec}(U), \subset\big)$ is a distributive lattice
Birkhoff's representation theorem says that each distributive lattice S is of this form.
In Krone's example, we can take $U = \{0,1\}$.
Each additive duality on a distributive lattice has an interpretation in terms of open paths.

The two-stage contact process



Cancellative duality

Let $\oplus$ denote (componentwise) addition modulo 2.
A map $m : \{0,1\}^\Lambda \to \{0,1\}^{\Lambda'}$ is *cancellative* if

$$m(\underline{0}) = \underline{0} \quad \text{and} \quad m(x \oplus y) = m(x) \oplus m(y) \qquad \big(x, y \in \{0,1\}^\Lambda\big).$$

Let $S = T = \{0,1\}$, let Λ be countable, and let $\mathbf{S} = S^\Lambda$.
Let $\mathcal{C}_{\rm canc}(\mathbf{S}, T) := \big\{m \in \mathcal{C}(\mathbf{S}, T) : m \text{ cancellative}\big\}$.

Lemma If $m : \mathbf{S} \to \mathbf{S}$ cancellative for all $m \in \mathcal{G}$, then $\mathcal{C}_{\rm canc}(\mathbf{S}, T)$ is invariant under the backward stochastic flow $(\mathbb{F}_{u,s})_{u \geq s}$ of the backtracking process,

Proof Since the concatenation of cancellative maps is cancellative, $\mathbb{F}_{u,s}(f) = f \circ \mathbb{X}_{s,u}$ is cancellative for each $f \in \mathcal{C}_{\rm canc}(\mathbf{S}, T)$. ∎

Cancellative duality

For each $y \in \mathbf{S}_{\rm fin}$, define

$$f_y(x) := \bigoplus_{i \in \Lambda} x(i)y(i) \qquad (x \in \mathbf{S}).$$

Lemma $f_y \in \mathcal{C}_{\rm canc}(\mathbf{S}, T)$ and for each $f \in \mathcal{C}_{\rm canc}(\mathbf{S}, T)$ there exists a unique $y \in \mathbf{S}_{\rm fin}$ such that $f = f_y$.

Cancellative duality

The abstract duality function

$$\psi_{\rm back}(x,f) := f(x) \qquad \big(x\in\mathbf{S},\ f\in\mathcal{C}_{\rm add}(\mathbf{S},T)\big)$$

now takes the concrete form

$$\psi_{\rm canc}(x,y) := \psi_{\rm back}(x,f_y) = \bigoplus_{i\in\Lambda} x(i)y(i) \qquad (x\in\mathbf{S},\ y\in\mathbf{S}_{\rm fin}).$$

The abstract theory now implies that for each continuous cancellative map $m:\mathbf{S}\to\mathbf{S}$, there exists a unique map $\hat m:\mathbf{S}_{\rm fin}\to\mathbf{S}_{\rm fin}$ such that

$$\psi_{\rm canc}\big(m(x),y\big) = \psi_{\rm canc}\big(x,\hat m(y)\big) \qquad (x\in\mathbf{S},\ y\in\mathbf{S}_{\rm fin}).$$

Remarkable fact $\hat m$ is also cancellative. If m is local, then so is $\hat m$.

Cancellative duality

Theorem Let $(\mathbb{X}_{s,u})_{s\leq u}$ be the cadlag stochastic flow of an interacting particle system with state space $\mathbf{S} = S^\Lambda$ and generator

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big\{f\big(m(x)\big) - f\big(x\big)\big\}.$$

Assume that $S = \{0,1\}$ and that all local maps $m \in \mathcal{G}$ are cancellative. Then there exists a cadlag backward stochastic flow $(\mathbb{Y}_{u,s})_{u\geq s}$ on $\mathbf{S}_{\rm fin}$ such that

$$\psi_{\rm canc}\big(\mathbb{X}_{s,u}(x), y\big) = \psi_{\rm canc}\big(x, \mathbb{Y}_{u,s}(y)\big) \qquad (s \leq u,\ x \in \mathbf{S},\ y \in \mathbf{S}_{\rm fin}).$$

The generator of the associated Markov process takes the form

$$Hf(y) = \sum_{m\in\mathcal{G}} r_m\big\{f\big(\hat{m}(y)\big) - f\big(y\big)\big\}.$$

Cancellative duality

Remark Under suitable assumptions on the rates, the backward stochastic flow $(\mathbb{Y}_{u,s})_{u \geq s}$ on $\mathbf{S}_{\mathrm{fin}}$ can be extended to $\mathbf{S}$.

However, the duality function

$$\psi_{\mathrm{canc}}(x, y) := \bigoplus_{i \in \Lambda} x(i)y(i) = 1_{\{|xy| \text{ is odd}\}}$$

may fail to be defined unless at least one of x and y lies in $\mathbf{S}_{\mathrm{fin}}$.

The cancellative contact process

Lemma The cancellative contact process $X = (X_t)_{t\geq 0}$ started in a product law with intensity $1/2$ satisfies

$$\mathbb{P}\big[X_t \in \cdot\,\big] \Rightarrow \nu_{1/2},$$

where $\nu_{1/2}$ is an invariant law that is uniquely characterised by

$$\int \overline{\nu}(\mathrm{d}x)\psi_{\rm canc}(x,y) = \tfrac{1}{2}\mathbb{P}^y\big[Y_t \neq \underline{0}\ \forall t \geq 0\big] \qquad (y \in \mathbf{S}_{\rm fin}),$$

where $Y = (Y_t)_{t\geq 0}$ is the cancellative contact process with reversed infection rates $\lambda^\dagger(j,i) := \lambda(i,j)$.

Proof

$$\begin{aligned}\mathbb{P}\big[|X_t y| \text{ is odd}\big] &= \mathbb{P}^y\big[|X_0 Y_t| \text{ is odd}\big]\\ &= \tfrac{1}{2}\mathbb{P}^y\big[Y_t \neq \underline{0}\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}^y\big[Y_t \neq \underline{0}\ \forall t \geq 0\big].\end{aligned}$$

Since this holds for all $y \in \mathbf{S}_{\rm fin}$, the claim follows. ∎

Theorem Assume that $\Lambda = \mathbb{Z}^d$.
Assume that $\lambda(j,i) = \lambda(j-i)$ depends only on the difference $j-i$.
Assume that $\mathbb{P}\big[X_0 \in \cdot\,\big]$ is translation invariant and
$\mathbb{P}\big[X_0 = \underline{0}\big] = 0$. Then

$$\mathbb{P}\big[X_t \in \cdot\,\big] \Rightarrow \nu_{1/2}.$$

Proof idea Similar to what we did before:

$$\begin{aligned}
&\mathbb{P}\big[|X_t y| \text{ is odd}\big] = \mathbb{P}\big[|\mathbb{X}_{0,1}(X_0)\mathbb{Y}_{t,1}(y)| \text{ is odd}\big] \\
&= \mathbb{P}\big[|\mathbb{X}_{0,1}(X_0)\mathbb{Y}_{t,1}(y)| \text{ is odd}\,\big|\,|\mathbb{Y}_{t,1}(y)| = 0\big]\,\mathbb{P}\big[|\mathbb{Y}_{t,1}(y)| = 0\big] \\
&+\mathbb{P}\big[|\mathbb{X}_{0,1}(X_0)\mathbb{Y}_{t,1}(y)| \text{ is odd}\,\big|\,0 < |\mathbb{Y}_{t,1}(y)| < N\big]\,\mathbb{P}\big[0 < |\mathbb{Y}_{t,1}(y)| < N\big] \\
&+\underbrace{\mathbb{P}\big[|\mathbb{X}_{0,1}(X_0)\mathbb{Y}_{t,1}(y)| \text{ is odd}\,\big|\,N \le |\mathbb{Y}_{t,1}(y)|\big]}_{\approx 1/2}\,\mathbb{P}\big[N \le |\mathbb{Y}_{t,1}(y)|\big].
\end{aligned}$$

The threshold voter model

The threshold voter model has a random mapping representation in terms of monotone maps

$$\begin{aligned} Gf(x) = & \sum_{i\in\Lambda} \big\{ f\big(\mathtt{min}_i(x)\big) - f\big(x\big) \big\} \\ & + \sum_{i\in\Lambda} \big\{ f\big(\mathtt{max}_i(x)\big) - f\big(x\big) \big\} \qquad \big(x \in \{0,1\}^\Lambda\big). \end{aligned}$$

and another random mapping representation in terms of cancellative maps

$$Gf(x) = \sum_{i\in\Lambda} 2^{-|\mathcal{N}_i|} \sum_{\substack{\Delta\subset\mathcal{N}_i \\ |\Delta| \text{ is odd}}} \big\{ f\big(\mathtt{flip}_{i,\Delta}(x)\big) - f\big(x\big) \big\}.$$

Remarkably, the latter is more useful for proving ergodic statements.

Parity preservation

Except in the 1D nearest-neighbour case, it is known that the threshold voter model $(X_t)_{t\geq 0}$ has three extremal invariant laws, and *complete convergence* holds: for arbitrary initial laws,

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} p_0\delta_{\underline{0}} + p_1\delta_{\underline{1}} + (1 - p_0 - p_1)\nu_{1/2},$$

where $p_s = \mathbb{P}[\exists t \geq 0 \text{ s.t. } X_t = \underline{s}]$.

The cancellative dual $(Y_t)_{t\geq 0}$ is *parity preserving*:

$$|Y_0| \bmod(2) = |Y_t| \bmod(2) \qquad (t \geq 0).$$

If $\mathbb{P}[X_0\,\cdot\,]$ is product law with intensity $1/2$, then

$$\begin{aligned}\mathbb{P}\big[X_t(i) \neq X_t(j)\big] &= \mathbb{P}\big[|X_t(\delta_i + \delta_j)| \text{ is odd}\big] \\ &= \mathbb{P}^{\delta_i+\delta_j}\big[|X_0 Y_t| \text{ is odd}\big] = \tfrac{1}{2}\mathbb{P}^{\delta_i+\delta_j}\big[Y_t \neq \underline{0}\big] \\ &\underset{t\to\infty}{\longrightarrow} \tfrac{1}{2}\mathbb{P}^{\delta_i+\delta_j}\big[Y_t \neq \underline{0}\ \forall t \geq 0\big].\end{aligned}$$

Monoid duality

A *semigroup* is a set S equipped with an associative operation $+$.
A *monoid is a semigroup* S with a neutral element 0.
If S, T are monoids, then we let $\mathcal{H}(S,T)$ denote the set of *homomorphisms* $h : S \to T$, which satisfy

$$h(0) = 0 \quad \text{and} \quad h(x+y) = h(x) + h(y) \qquad (x, y \in S).$$

Def Let S, R, T be *commutative* monoids.
Then S is T*-dual* to R with *duality function* $\psi : S \times R \to T$ if:

1. $\psi(x_1, y) = \psi(x_2, y)$ for all $y \in R$ implies $x_1 = x_2$ $(x_1, x_2 \in S)$,
2. $\mathcal{H}(S,T) = \{\psi(\,\cdot\,, y) : y \in R\}$,
3. $\psi(x, y_1) = \psi(x, y_2)$ for all $x \in S$ implies $y_1 = y_2$ $(y_1, y_2 \in R)$,
4. $\mathcal{H}(R,T) = \{\psi(x, \,\cdot\,) : x \in S\}$.

Monoid duality

Example Let S be a finite lattice, let R be its dual, and let $T = \{0, 1\}$.
We view S, R, and T as commutative monoids with neutral element 0 and sum $\vee$.
Then S^Λ is T-dual to R^Λ with duality function $\psi_{\rm add}(x, y) = 1_{\{x \not\leq y'\}}$.

Example Let $S = T = \{0, 1\}$, equipped with addition modulo 2.
Then S^Λ is T-dual to S^Λ with duality function $\psi_{\rm canc}(x, y) = \bigoplus_{i \in \Lambda} x(i)y(i)$.

Example Let $S = \{0, 1\}^2$, equipped with the operation

$$(x_1, x_2) + (y_1, y_2) := (x_1 \vee x_2, y_1 \oplus y_2).$$

Let $T = \{-1, 0, 1\}$ equipped with the usual product. Then S^Λ is T-dual to S^Λ with duality function
$$\psi\big((x_1, x_2), (y_1, y_2)\big) = \prod_{i \in \Lambda} \big(1 - x_1(i)y_1(i)\big)(-1)^{x_2(i)y_2(i)}.$$

Monoid duality

Theorem Assume that $\Lambda = \mathbb{Z}^d$.
Let (X, Y) be a combination of a contact process and a cancellative contact process, constructed using the same graphical representation.
Assume that $\lambda(j, i) = \lambda(j - i)$ depends only on the difference $j - i$.
There exists an invariant law ν_* such that for each initial law $\mathbb{P}\big[(X_0, Y_0) \in \cdot\,\big]$ that is translation invariant and satisfies $\mathbb{P}\big[X_0 = \underline{0} \text{ or } Y_0 = \underline{0}\big] = 0$, one has

$$\mathbb{P}\big[(X_t, Y_t) \in \cdot\,\big] \Rightarrow \nu_*.$$

Note The marginals of ν_* are $\overline{\nu}$ and $\nu_{1/2}$.
The law ν_* is concentrated on $\big\{(x, y) : x \geq y\big\}$.

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