Lecture 3
Monotone duality

Jan M. Swart

Rouen

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Monotone maps

Let S be a finite lattice. A map $m : S \to S$ is *monotone* if

$$x \leq y \quad \Rightarrow \quad m(x) \leq m(y)$$

and *additive* if

$$m(0) = 0 \quad \text{and} \quad m(x \vee y) = m(x) \vee m(y).$$

Note If m is monotone, then

$$\begin{aligned} x \leq x \vee y \quad &\Rightarrow \quad m(x) \leq m(x \vee y) \\ y \leq x \vee y \quad &\Rightarrow \quad m(y) \leq m(x \vee y) \\ &\Rightarrow \quad m(x \vee y) \geq m(x) \vee m(y), \end{aligned}$$

so every monotone map is "superadditive".

Monotone maps

A typical example of a monotone map that is not additive is the *cooperative branching map* $\mathrm{coop}_{ijk} : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$

$$\mathrm{coop}_{ijk}(x)(l) := \begin{cases} \big(x(i) \wedge x(j)\big) \vee x(k) & \text{if } l = k, \\ x(l) & \text{otherwise.} \end{cases}$$

Let $\Lambda = \{1,2,3\}$. Then

$$x = (1,0,0) \xrightarrow{\mathrm{coop}_{123}} (1,0,0),$$
$$y = (0,1,0) \xrightarrow{\mathrm{coop}_{123}} (0,1,0),$$
$$x \vee y = (1,1,0) \xrightarrow{\mathrm{coop}_{123}} (1,1,1).$$

And $\mathrm{coop}_{123}(x \vee y) > \mathrm{coop}_{123}(x) \vee \mathrm{coop}_{123}(y)$.

Monotone particle systems

Let S be a finite partially ordered set with least element 0. Equip $\mathbf{S} := S^\Lambda$ with the product order and let $\underline{0}(i) := 0$ $(i \in \Lambda)$.

Let $(\mathbb{X}_{s,u})_{s \le u}$ be the stochastic flow of a particle system with generator

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}.$$

Assume that each $m \in \mathcal{G}$ is monotone with $m(\underline{0}) = \underline{0}$.

Let $T := \{0, 1\}$ and let $\mathcal{C}_+(\mathbf{S}, T)$ be the space of continuous monotone functions $f : S \to T$ with $f(\underline{0}) = 0$. Then

$$f \in \mathcal{C}_+(\mathbf{S}, T) \quad \Rightarrow \quad \mathbb{F}_{u,s}(f) = f \circ \mathbb{X}_{s,u} \in \mathcal{C}_+(\mathbf{S}, T),$$

so $\mathcal{C}_+(\mathbf{S}, T)$ is invariant under the backward stochastic flow.

Can we make a pathwise duality out of this?

Lower semi-continuous monotone functions

We need a way to characterise elements of $\mathcal{C}_+(\mathbf{S}, T)$.

Let $\mathcal{L}_+(\mathbf{S}, T)$ be the space of *lower semi-continuous* monotone functions $f : S \to T$ with $f(\underline{0}) = 0$. Then

$$f \in \mathcal{L}_+(\mathbf{S}, T) \quad \Leftrightarrow \quad f = 1_A \text{ with } A \subset \mathbf{S} \text{ open increasing, } \underline{0} \notin A.$$

Moreover

$$f \in \mathcal{L}_+(\mathbf{S}, T) \quad \Rightarrow \quad \mathbb{F}_{u,s}(f) = f \circ \mathbb{X}_{s,u} \in \mathcal{L}_+(\mathbf{S}, T),$$

so also $\mathcal{L}_+(\mathbf{S}, T)$ is invariant under the backward stochastic flow.

Lower semi-continuous monotone functions

Recall A increasing $\Leftrightarrow$ $A = A^{\uparrow}$ with

$$A^{\uparrow} := \big\{x \in \mathbf{S} : \exists y \in A \text{ s.t. } y \leq x\big\}.$$

A *minimal element* of A is an $y \in A$ s.t. there is no $y \neq y' \in A$ with $y' \leq y$. Let

$$A^{\circ} := \big\{y \in A : y \text{ minimal}\big\}.$$

Let

$$\begin{aligned}
\mathbf{S}_{\rm fin} &:= \Big\{x \in \mathbf{S} : 0 < \sum_{i \in \Lambda} 1_{\{x(i) \neq 0\}} < \infty\Big\}, \\
\mathcal{I}(\mathbf{S}) &:= \big\{A \subset \mathbf{S} : A \text{ is open and increasing, } \underline{0} \notin A\big\}, \\
\mathcal{H}(\mathbf{S}) &:= \big\{Y \subset \mathbf{S}_{\rm fin} : Y^{\circ} = Y\big\}, \\
\mathcal{H}_{\rm fin}(\mathbf{S}) &:= \big\{Y \in \mathcal{H}(\mathbf{S}) : |Y| < \infty\big\}.
\end{aligned}$$

Encoding of open increasing sets

Recall that $1_A \in \mathcal{L}_+(\mathbf{S}, T) \quad \Leftrightarrow \quad A \in \mathcal{I}(\mathbf{S})$.

Lemma The map $Y \mapsto Y^\uparrow$ is a bijection from $\mathcal{H}(\mathbf{S})$ to $\mathcal{I}(\mathbf{S})$ and the map $A \mapsto A^\circ$ is its inverse.
Moreover $1_{Y^\uparrow} \in \mathcal{C}_+(\mathbf{S}, T) \quad \Leftrightarrow \quad Y \in \mathcal{H}_{\rm fin}(\mathbf{S})$.

Recall that $(\mathbb{X}_{s,u})_{s\leq u}$ is dual to $(\mathbb{F}_{u,s})_{u\geq s}$ with duality function $\psi_{\rm back}(x,f) = f(x)$. Through the bijection $\mathcal{H}(\mathbf{S}) \ni Y \mapsto 1_{Y^\uparrow} \in \mathcal{L}_+(\mathbf{S}, T)$, this abstract duality function now takes the concrete form

$$\psi_{\rm mon}(x, Y) := 1_{Y^\uparrow}(x) = 1_{\{\exists y \in Y \text{ s.t. } x \geq y\}}$$

$\big(x \in \mathbf{S},\ \ Y \in \mathcal{H}(\mathbf{S})\big)$.

Monotone duality

Proposition There exists a backward stochastic flow $(\mathbb{Y}_{u,s})_{u \geq s}$ on $\mathcal{H}(\mathbf{S})$ such that

$$\psi_{\rm mon}\big(\mathbb{X}_{s,u}(x), Y\big) = \psi_{\rm mon}\big(x, \mathbb{Y}_{u,s}(Y)\big)$$

$\big(x \in \mathbf{S},\ \ Y \in \mathcal{H}(\mathbf{S})\big)$. One has

$$Y \in \mathcal{H}_{\rm fin}(\mathbf{S}) \quad \Rightarrow \quad \mathbb{Y}_{u,s}(Y) \in \mathcal{H}_{\rm fin}(\mathbf{S}).$$

Remark The law of an $\mathbf{S}$-valued random variable X is uniquely determined by

$$\Big(\mathbb{E}\big[\psi_{\rm mon}(X, Y)\big]\Big)_{Y \in \mathcal{H}_{\rm fin}(\mathbf{S})}.$$

The law of an $\mathcal{H}(\mathbf{S})$-valued random variable Y is uniquely determined by

$$\Big(\mathbb{E}\big[\prod_{i=1}^{n} \psi_{\rm mon}(x_i, Y)\big]\Big)_{n \geq 1,\ x_1,\ldots,x_n \in \mathbf{S}_{\rm fin}}.$$

A cooperative contact process

Example

For $i \in \Lambda$, let $p_1(i,\cdot)$ and $p_2(i,\cdot)$ be probability laws on Λ and on $\{(j,k) \in \Lambda^2 : j \neq k\}$, respectively.

Consider the interacting particle system with generator

$$\begin{aligned} Gf(x) := (1-\alpha) &\sum_{i\in\Lambda}\sum_{j\in\Lambda} p_1(i,j)\big\{f\big(\mathtt{bra}_{ji}\big) - f\big(x\big)\big\} \\ +\alpha &\sum_{i\in\Lambda}\sum_{(j,k)\in\Lambda} p_2(i,j,k)\big\{f\big(\mathtt{coop}_{jki}\big) - f\big(x\big)\big\} \\ + &\sum_{i\in\Lambda}\big\{f\big(\mathtt{dth}_i\big) - f\big(x\big)\big\}. \end{aligned}$$

For $\alpha = 0$ this is a contact process, which is additive.

A cooperative contact process

For $Y \in \mathcal{H}(\mathbf{S}) = \{Y \subset \mathbf{S}_{\mathrm{fin}} : Y^\circ = Y\}$, define

$$\widehat{\mathtt{bra}}_{ji}(Y) := \big(Y \cup \{\mathtt{dth}_i(y) \vee \delta_j : y \in \mathbf{S}_{\mathrm{fin}},\ y(i) = 1\}\big)^\circ,$$
$$\widehat{\mathtt{coop}}_{jki}(Y) := \big(Y \cup \{\mathtt{dth}_i(y) \vee \delta_j \vee \delta_k : y \in \mathbf{S}_{\mathrm{fin}},\ y(i) = 1\}\big)^\circ,$$
$$\widehat{\mathtt{dth}}_i(Y) := \{y \in Y : y(i) = 0\}.$$

Then

$$\psi_{\mathrm{mon}}\big(\mathtt{bra}_{ji}(x), Y\big) = \psi_{\mathrm{mon}}\big(x, \widehat{\mathtt{bra}}_{ji}(Y)\big) \qquad \big(x \in \mathbf{S},\ Y \in \mathcal{H}(\mathbf{S})\big),$$

etcetera. Indeed

$$\exists y \in Y \text{ s.t. } \mathtt{bra}_{ji}(x) \geq y \quad \Leftrightarrow \quad \exists y \in \widehat{\mathtt{bra}}_{ji}(Y) \text{ s.t. } x \geq y.$$

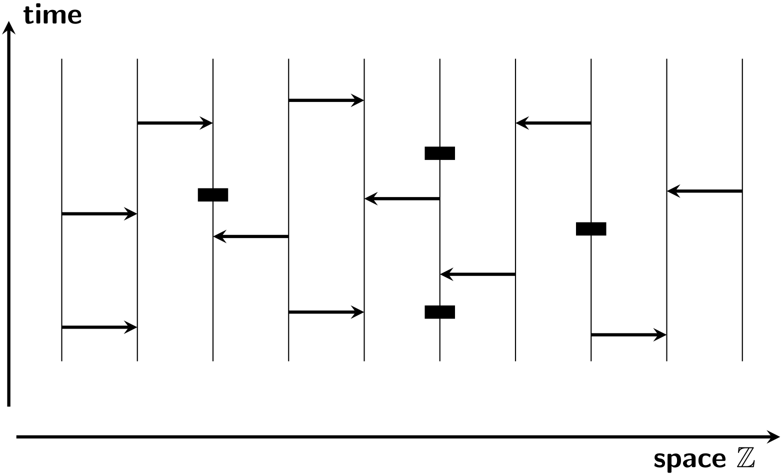
A cooperative contact process

The maps $\widehat{\texttt{bra}}_{ji}$ and $\widehat{\texttt{dth}}_i$ preserve the subspace

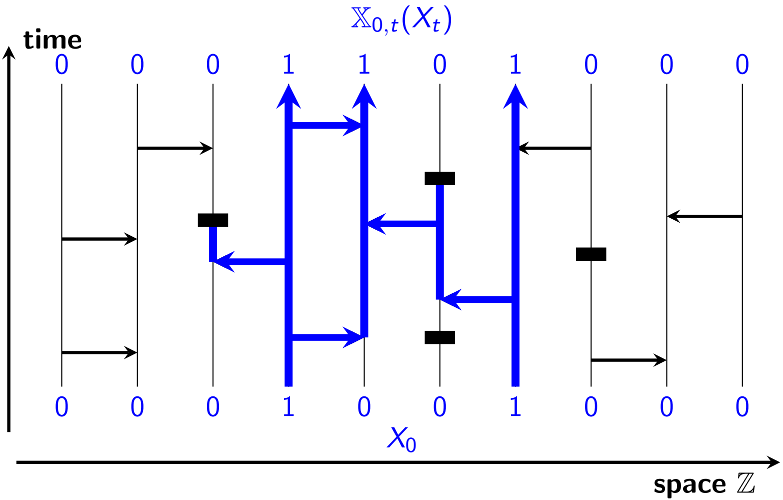
$$\mathcal{H}_{\rm add}(\mathbf{S}) := \big\{ Y \in \mathcal{H}(\mathbf{S}) : |y| = 1 \ \forall y \in Y \big\}.$$

This reflects the fact that the contact process is additive.

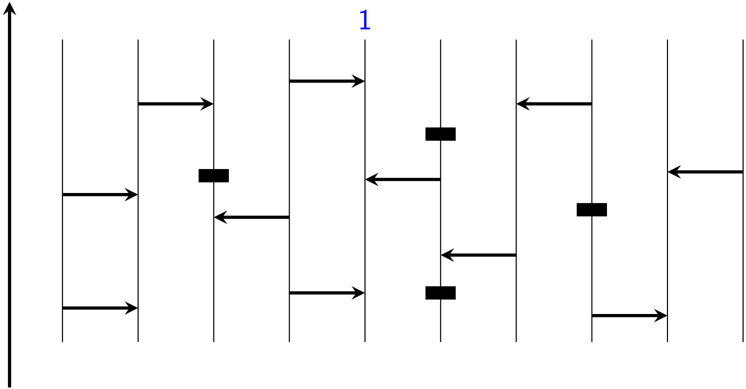
The contact process



The contact process

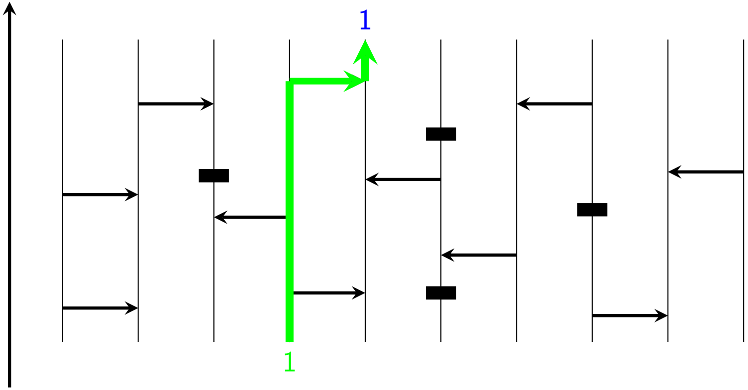


Open paths



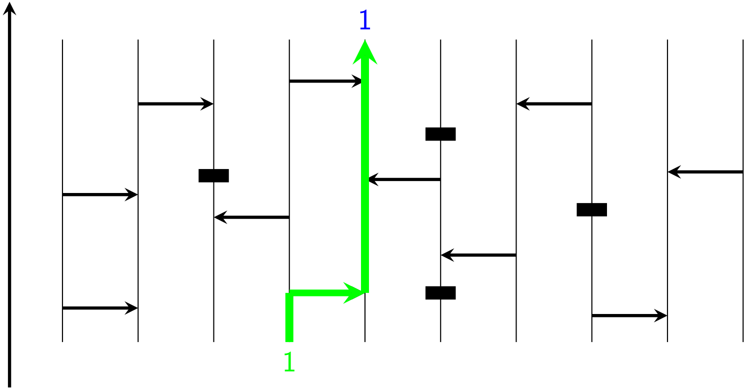
space $\mathbb{Z}$

Open paths



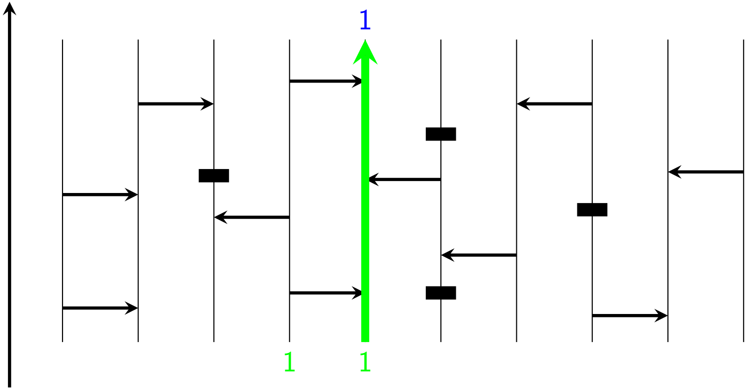
space $\mathbb{Z}$

Open paths



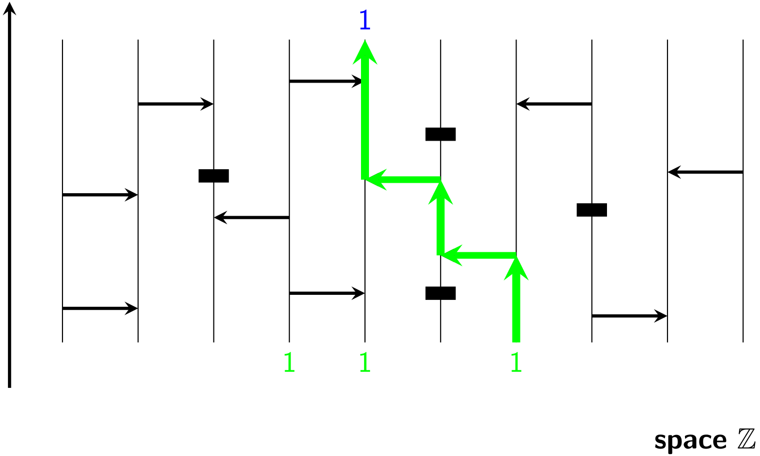
space $\mathbb{Z}$

Open paths

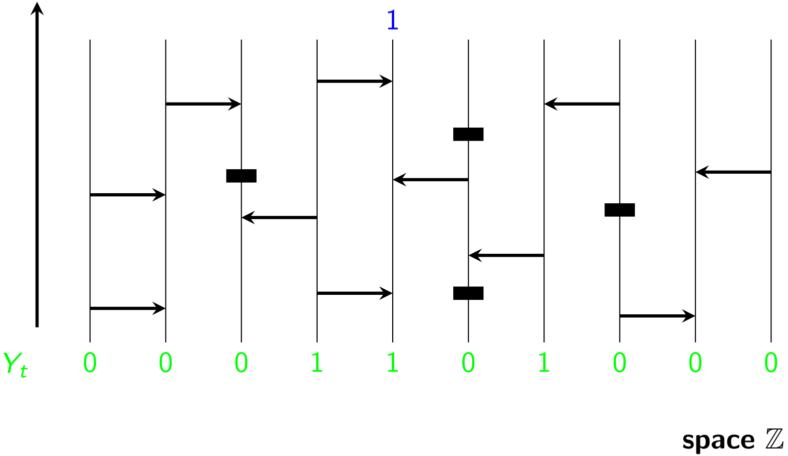


space $\mathbb{Z}$

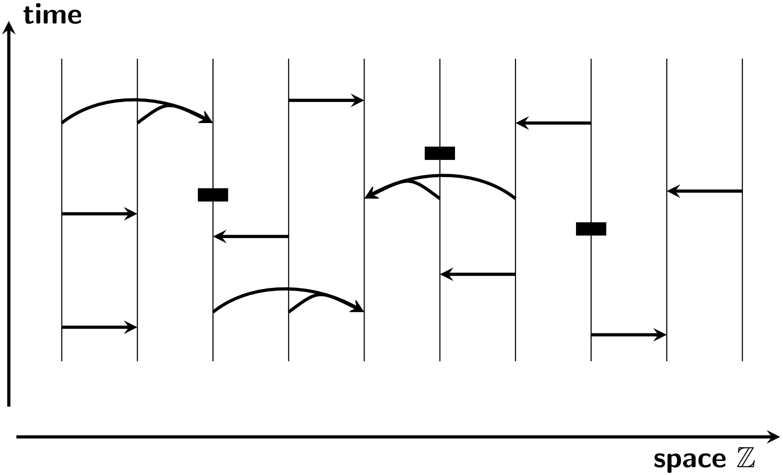
Open paths



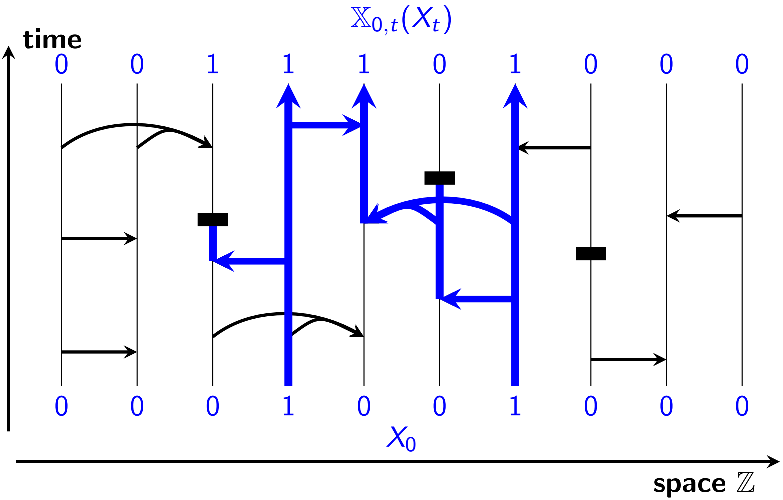
Open paths



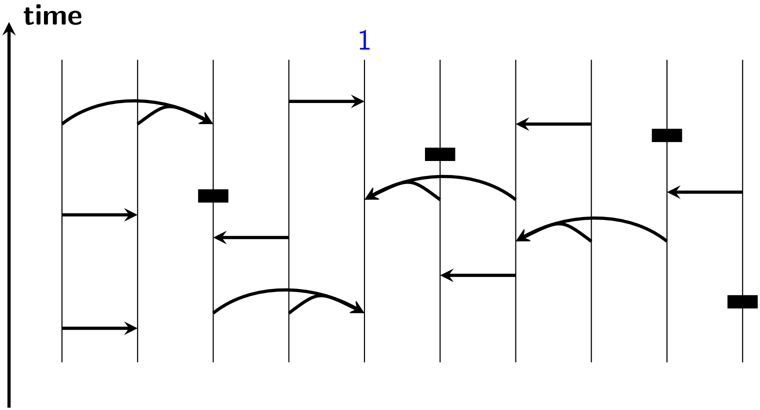
A cooperative contact process



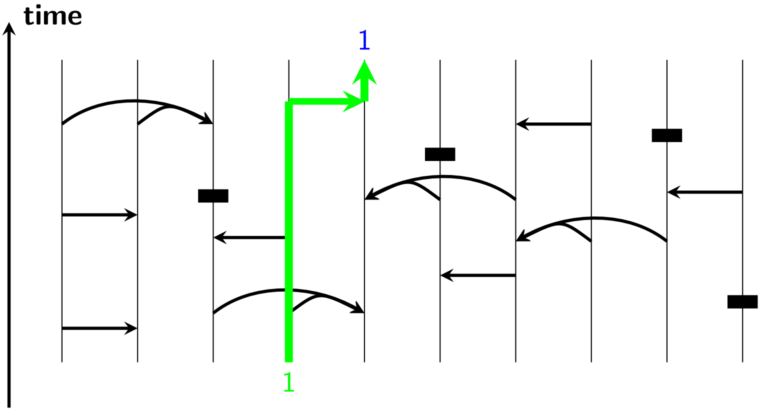
A cooperative contact process



The dual process



The dual process



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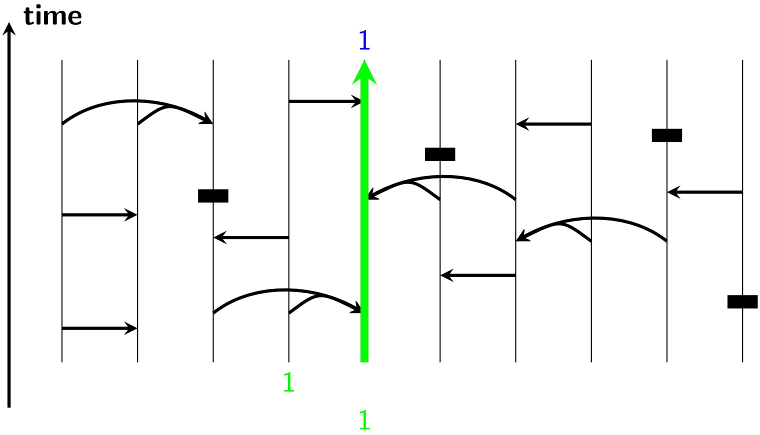

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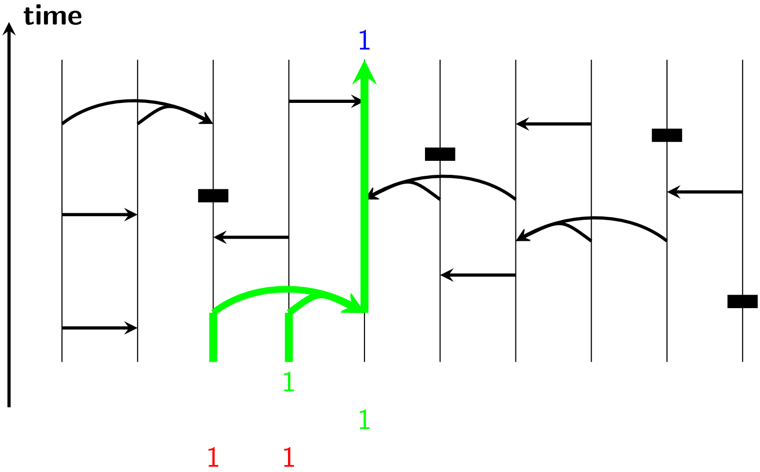

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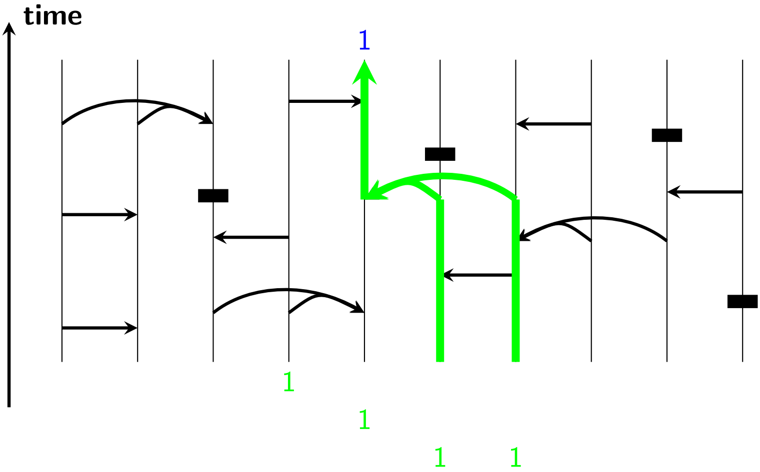

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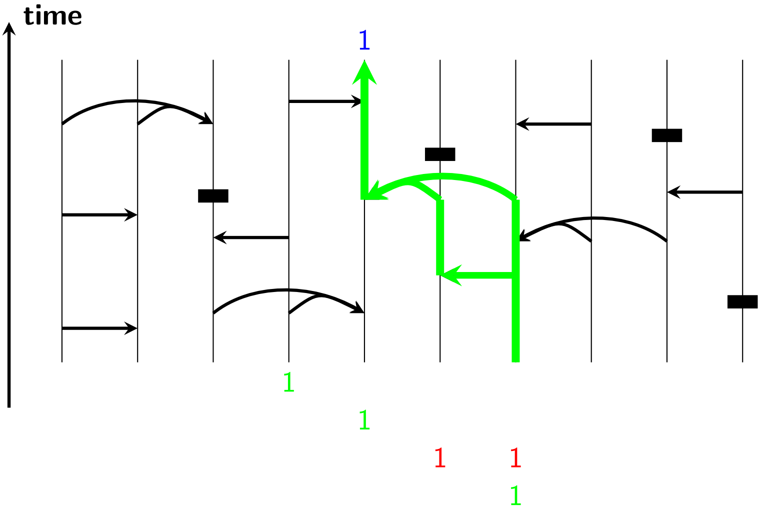

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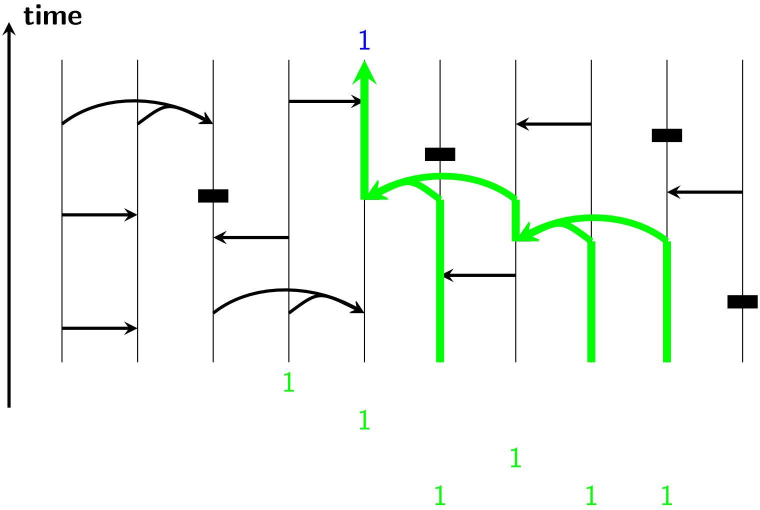

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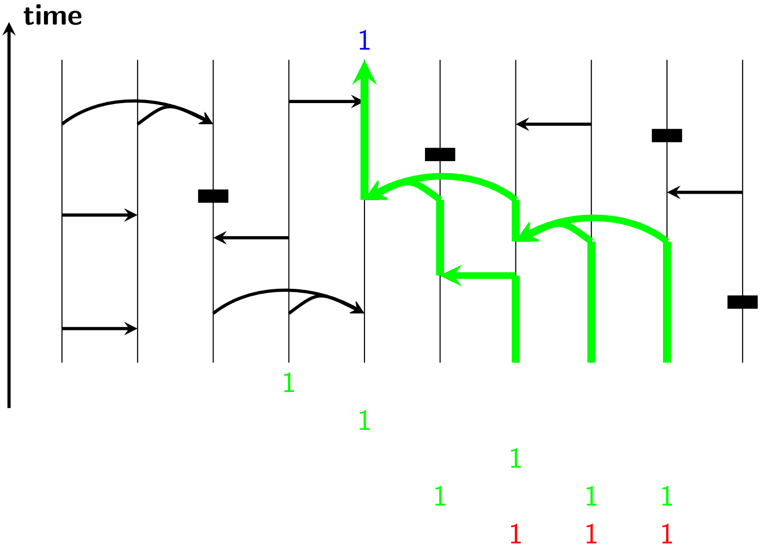
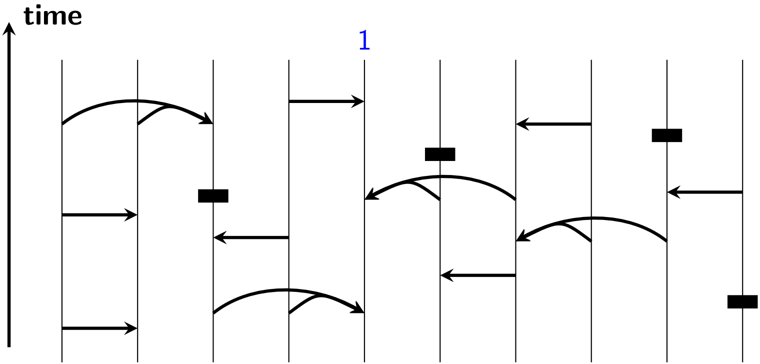


The dual process



y_1	0	0	0	1	0	0	0	0	0	0
y_2	0	0	0	0	1	0	0	0	0	0
y_3	0	0	0	0	0	0	1	0	0	0
y_4	0	0	0	0	0	1	0	1	1	0

$$Y_t = \{y_1, y_2, y_3, y_4\}$$

The dual state space

Lemma It is possible to equip $\mathcal{H}(\mathbf{S})$ with a metric such that

$$Y_n \to Y \quad \Leftrightarrow \quad \psi_{\mathrm{mon}}(x, Y_n) \to \psi_{\mathrm{mon}}(x, Y) \quad \forall x \in \mathbf{S}_{\mathrm{fin}}$$

and $\mathcal{H}(\mathbf{S})$ is compact in this topology.

We equip $\mathcal{H}(\mathbf{S})$ with a partial order by setting

$$Y_1 \leq Y_2 \quad \Leftrightarrow \quad Y_1^{\uparrow} \subset Y_2^{\uparrow}.$$

Then $\mathcal{H}(\mathbf{S})$ has a least element $\emptyset$ and a greatest element

$$\top := \{\delta_i : i \in \Lambda\}.$$

The upper invariant laws

Lemma The process $X = (X_t)_{t \geq 0}$ started in $X_0 = \underline{1}$ satisfies

$$\mathbb{P}^{\underline{1}}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu},$$

where $\overline{\nu}$ is an invariant law, called the *upper invariant law*. One has

$$\begin{aligned} &\mathbb{P}^{\underline{1}}\big[X_t(i) = 1\big] = \mathbb{P}^{\underline{1}}\big[X_t \geq \delta_i\big] = \mathbb{P}^{\{\delta_i\}}\big[\exists y \in Y_t \text{ s.t. } \underline{1} \geq y\big] \\ &= \mathbb{P}^{\{\delta_i\}}\big[Y_t \neq \emptyset\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}^{\{\delta_i\}}\big[Y_t \neq \emptyset \ \forall t \geq 0\big]. \end{aligned}$$

So $\overline{\nu}$ is nontrivial iff the dual process $Y = (Y_t)_{t \geq 0}$ started from an initial state of the form $\{\delta_i\}$ survives with positive probability.

The upper invariant laws

Lemma The process $Y = (Y_t)_{t\geq 0}$ started in $Y_0 = \top$ satisfies

$$\mathbb{P}^{\top}\big[Y_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\mu},$$

where $\overline{\mu}$ is an invariant law. For each $x \in \mathbf{S}_{\rm fin}$, one has

$$\begin{aligned}
&\overline{\mu}\big(\{Y \in \mathcal{H}(\mathbf{S}) : \exists y \in Y \text{ s.t. } x \geq y\}\big)\\
&\quad = \lim_{t\to\infty} \mathbb{P}^{\top}\big[\exists y \in Y_t \text{ s.t. } x \geq y\big] = \lim_{t\to\infty} \mathbb{P}^{\underline{1}}\big[\exists y \in \top \text{ s.t. } X_t \geq y\big]\\
&\quad = \lim_{t\to\infty} \mathbb{P}^{x}\big[X_t \neq \underline{0}\big] = \mathbb{P}^{x}\big[X_t \neq \underline{0}\ \forall t \geq 0\big].
\end{aligned}$$

So $\overline{\mu} = \delta_{\emptyset}$ iff the forward process X dies out started from any finite initial state x.

Some simulations

$i^{\uparrow}$	
i	$i^{\rightarrow}$

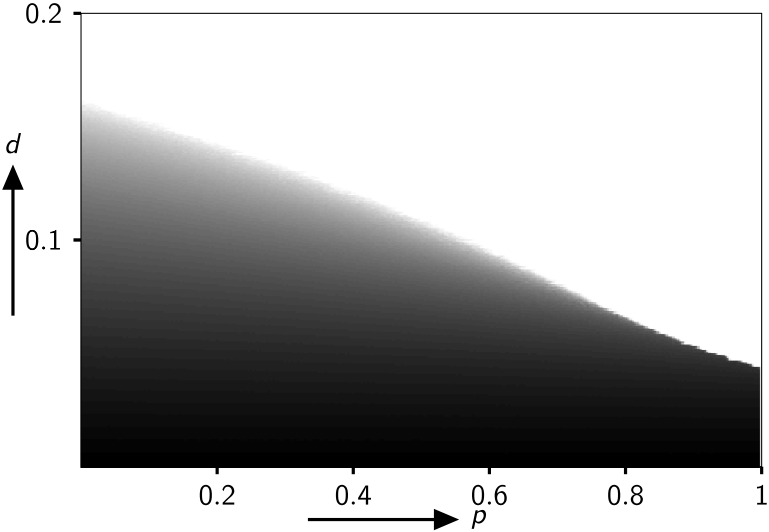
For each $i = (i_1, i_2) \in \mathbb{Z}^2$, let
$i^{\rightarrow} := (i_1 + 1, i_2)$ and $i^{\uparrow} := (i_1, i_2 + 1)$.

Let $p, d \in [0, 1]$ and let $X = (X_t)_{t \in \mathbb{N}}$ be a Markov chain with values in $\{0, 1\}^{\mathbb{Z}^2}$ such that independently for each i and t,

$$
\begin{array}{ll}
X_{t+1}(i) = X_t(i) \vee \big(X_t(i^{\rightarrow}) \wedge X_t(i^{\uparrow})\big) & \text{w. prob. } p(1-d), \\
X_{t+1}(i) = X_t(i) \vee X_t(i^{\rightarrow}) & \text{w. prob. } \frac{1}{2}(1-p)(1-d), \\
X_{t+1}(i) = X_t(i) \vee X_t(i^{\uparrow}) & \text{w. prob. } \frac{1}{2}(1-p)(1-d), \\
X_{t+1}(i) = 0 & \text{w. prob. } d.
\end{array}
$$

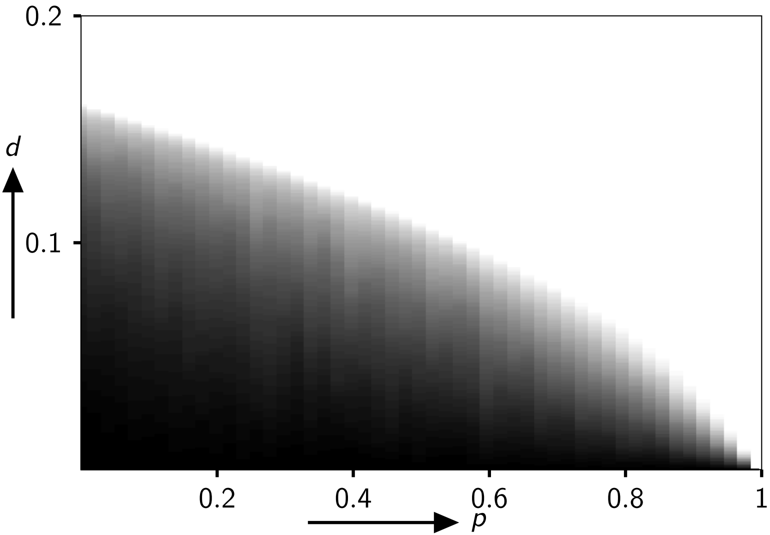
For $p = 0$ this model is additive.
For $p = 1$, it does not survive for any $d > 0$.

Some simulations



Density of the upper invariant law.

Some simulations



Survival probability started from a single one.

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