Self-organised criticality on the stock market

joint with Marco Formentin, Vít Peržina, and Jana Plačková

Jan M. Swart

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Drifted random walk

Consider a continuous-time Markov chain $(N_t)_{t\geq 0}$ taking values in $\mathbb{N}$ and starting in $N_0 = 0$, with the following description:

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This model exhibits a *phase transition* as the parameter λ passes the *critical point* $\lambda_c := 1$.

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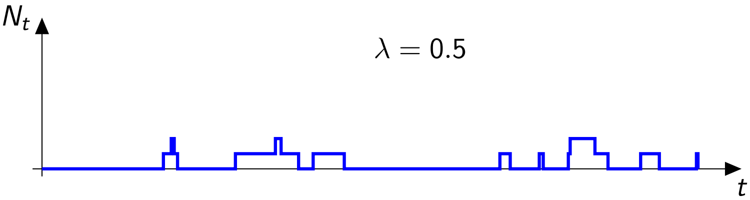

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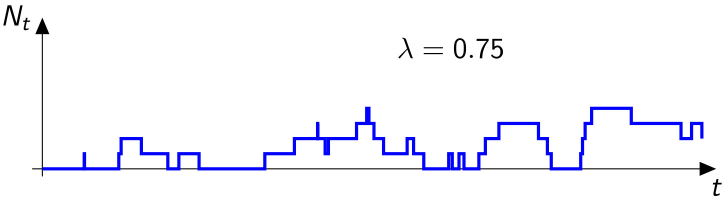

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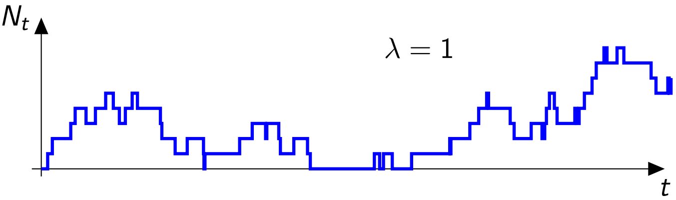

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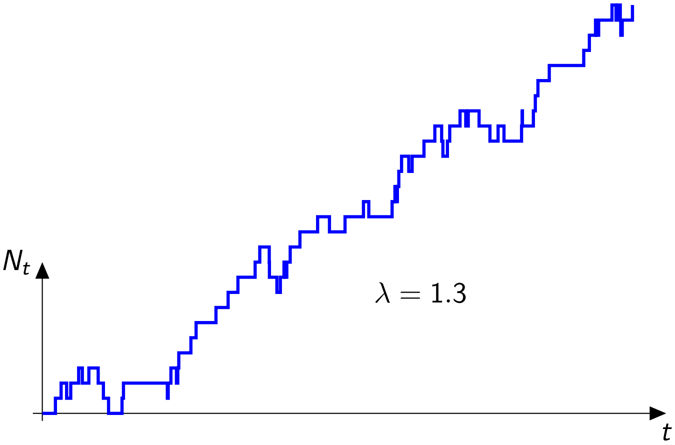

Drifted random walk

The Markov chain N_t is:

- *positive recurrent* for $\lambda < 1$,
- *null recurrent* for $\lambda = 1$,
- *transient* for $\lambda > 1$.

Let $\mathbb{P}^n$ denote the law of the process started in $N_0 = n$ and let $\tau_0 := \inf\{t \geq 0 : N_t = 0\}$ be the first hitting time of 0.

- If $\lambda < 1$, then $\mathbb{P}^1[\tau_0 > t] = e^{-r_\lambda t + o(t)}$ as $t \to \infty$.
- If $\lambda = 1$, then $\mathbb{P}^1[\tau_0 > t] \propto t^{-1/2}$ as $t \to \infty$.
- If $\lambda > 1$, then $\mathbb{P}^0[N_t = 0] = e^{-r_\lambda t + o(t)}$ as $t \to \infty$.

Here $r_\lambda := (\sqrt{\lambda} - 1)^2$.

Drifted random walk

The critical process with $\lambda = 1$ has a scaling limit:

$$\big(\varepsilon N_{\varepsilon^{-2}t}\big) \underset{\varepsilon\downarrow 0}{\Longrightarrow} \big(B_t\big)_{t\geq 0},$$

where $(B_t)_{t\geq 0}$ is Brownian motion with reflection at the origin.

The sub- and supercritical processes with $\lambda < 1$ and $\lambda > 1$ only have trivial (deterministic) scaling limits.

Critical behaviour

The sort of phenomena we have just seen are typical for a large class of processes, including the *contact process*, *percolation*, and the *Ising model*.
All these processes exhibit a *phase transition* as the parameter λ governing the process passes a *critical point* $\lambda_{\rm c}$.

For the process with $\lambda < \lambda_{\rm c}$ or $\lambda > \lambda_{\rm c}$:

- Correlations in space and time decay exponentially fast.
- There are only trivial scaling limits.

On the other hand, at the critical point $\lambda = \lambda_{\rm c}$:

- Correlations decay as a a power law t^{-c}, where c is a *critical exponent*.
- The process has a nontrivial scaling limit.

This is called *critical behaviour*.

Self-organised criticality

One might think that in the world around us, critical phenomena are quite rare, since they require a parameter of the process to be tuned to exactly the right value.

However, there exist processes that by some mechanism tune themselves to exactly the right critical value. This is called *self-organised criticality*.

Per Bak promoted this idea in his 1996 book *How Nature Works: the science of self-organized criticality*.

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Priorities are i.i.d. with some atomless law. Without loss of generality we can take the uniform distribution on $[-\lambda_{\mathrm{in}}, 0]$.

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Rescaling time, we can assume that $\lambda_{\mathrm{out}} = 1$.
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We can read off $N^\lambda(t)$ from the Poisson processes describing the arrivals of new emails and answering times.

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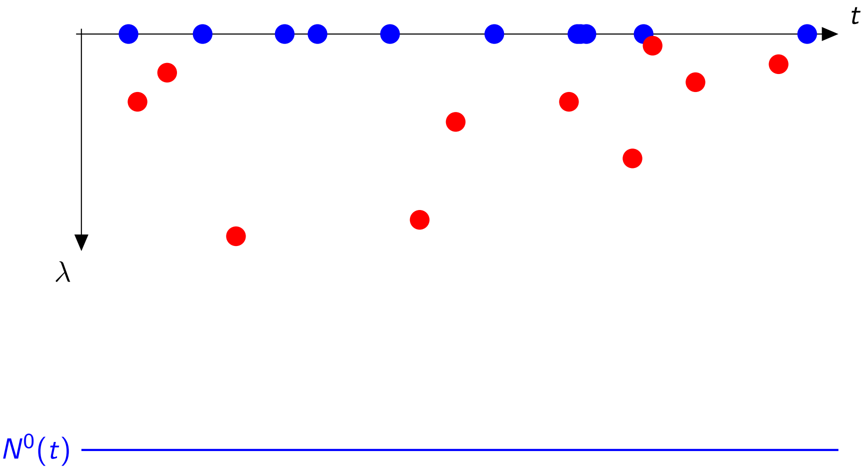

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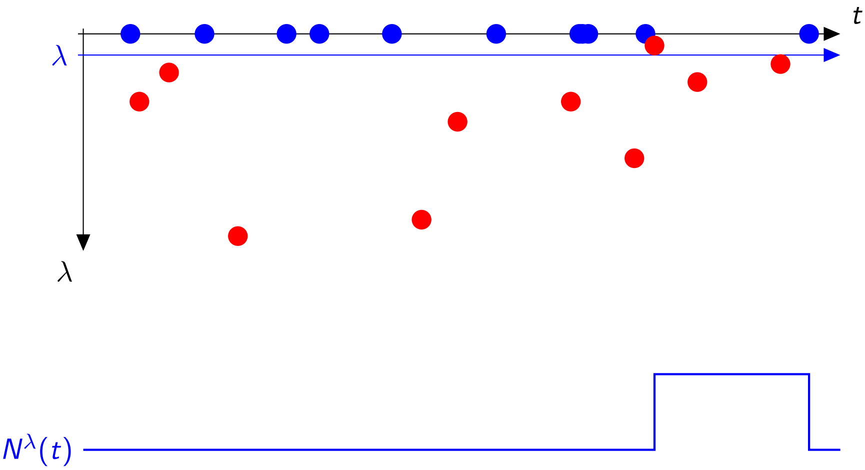

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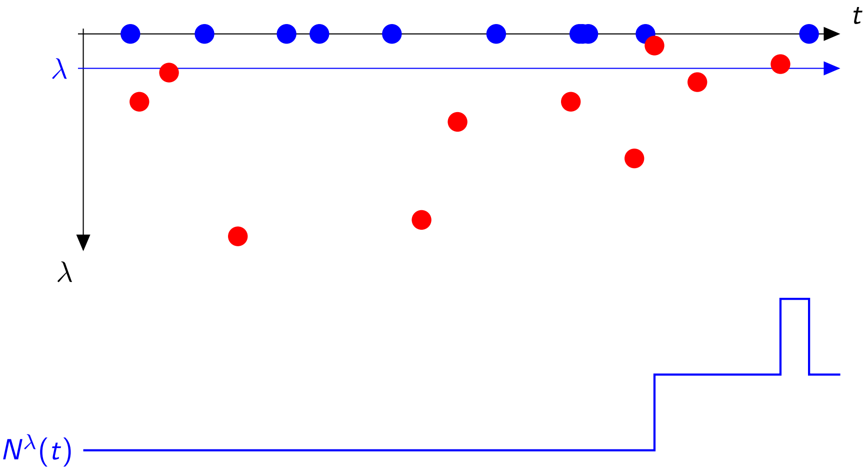

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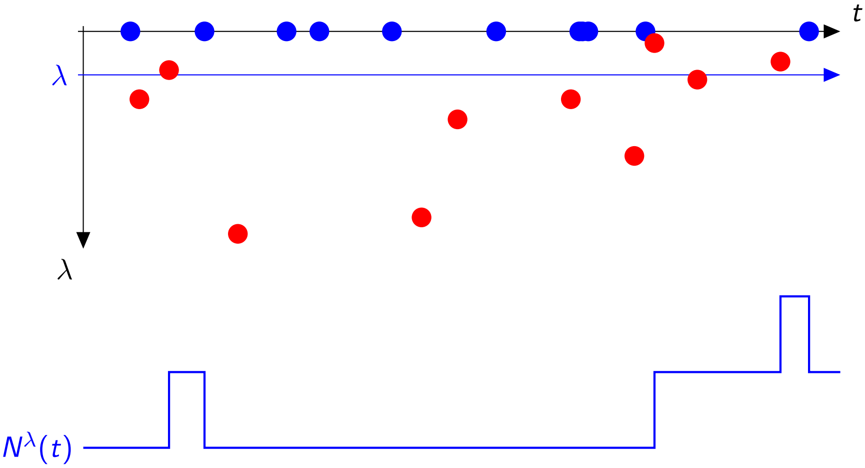

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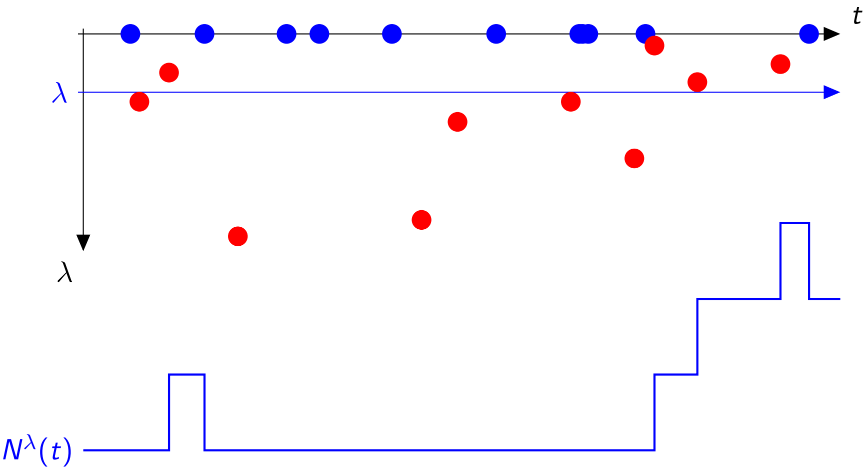

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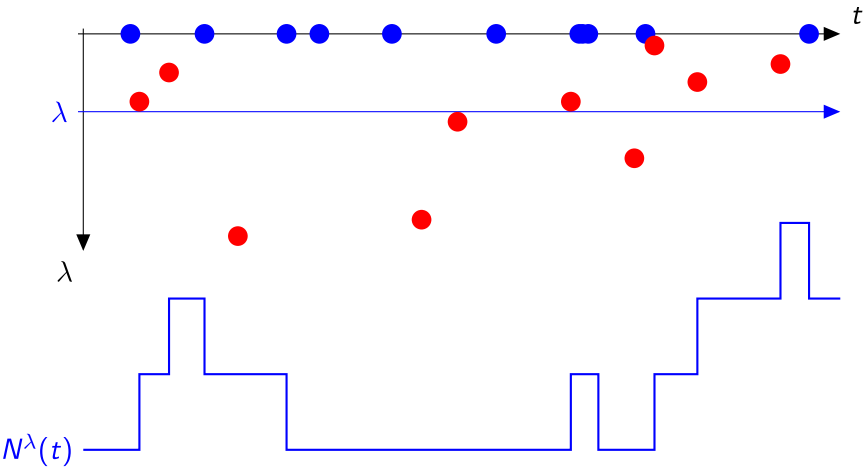

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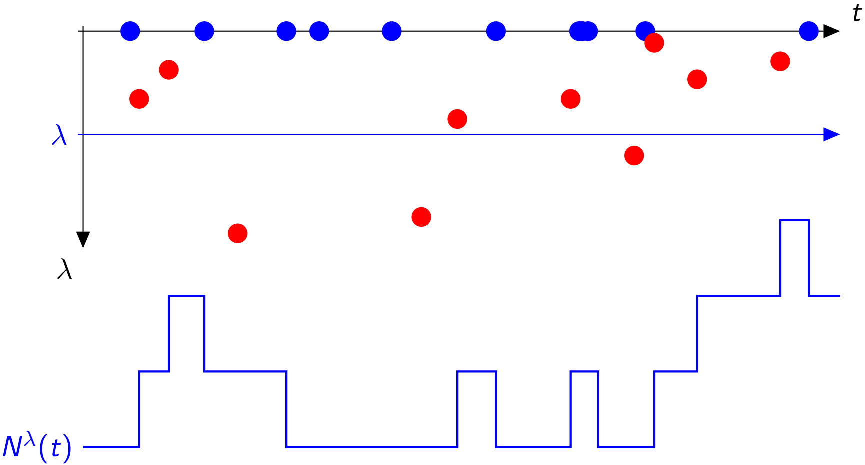

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Let τ be the answering time of one email, drawn at random from all emails that are eventually answered. Then

$$\mathbb{P}^1[\tau > t] \propto t^{-1/2} \quad \text{as} \quad t \to \infty.$$

This is quite different from the exponential decay of waiting times more commonly observed in queueing models.

We observe critical behaviour associated with the transition between positive recurrence and transience of drifted random walk, *without the need to tune a parameter to exactly the right value*.

The predicted power law decay with the exponent $1/2$ has even been observed in real data, provided time is measured in units proportional to the activity of the owner of the inbox (as judged from the number of emails sent). [Formentin, Lovison, Maritan, Zanzotto, J. Stat. Mech. 2015].

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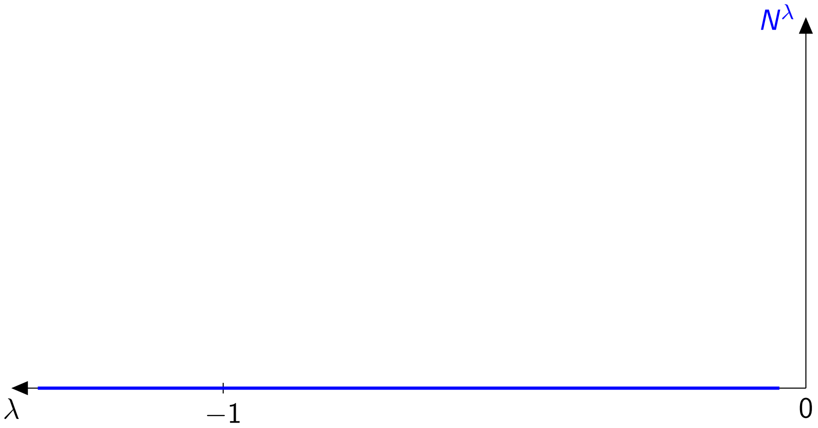

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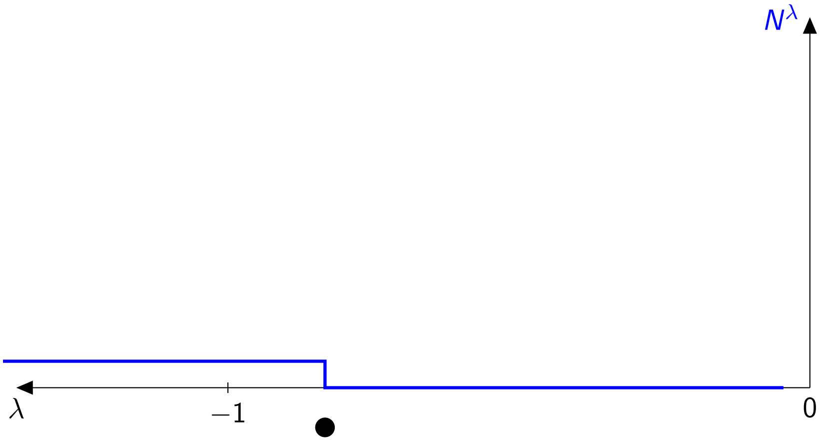

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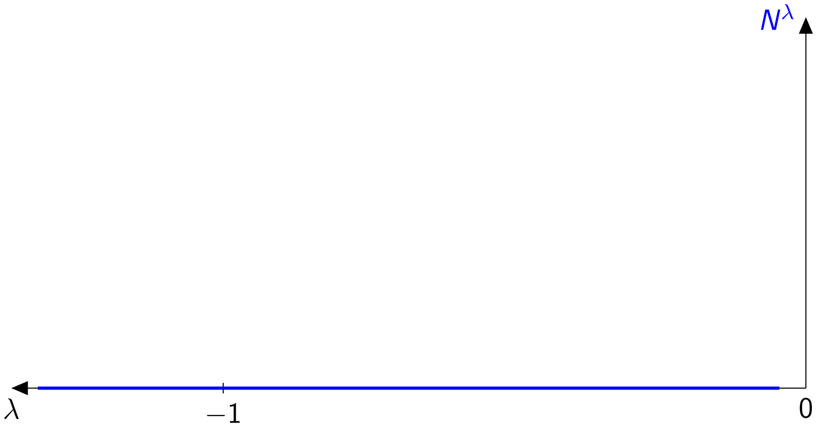

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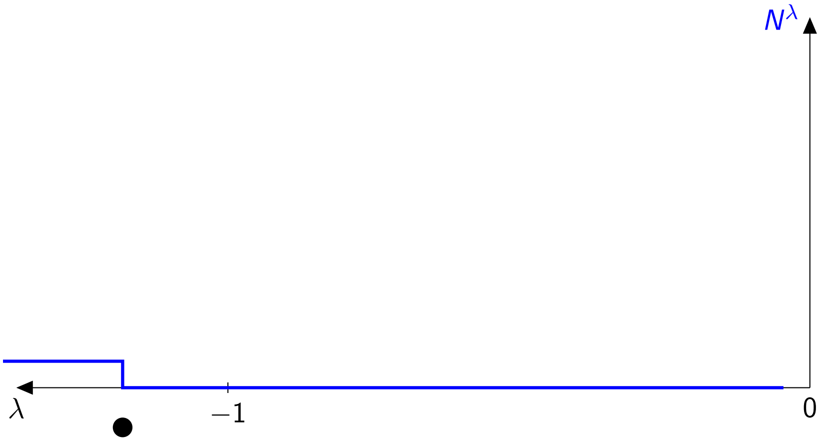

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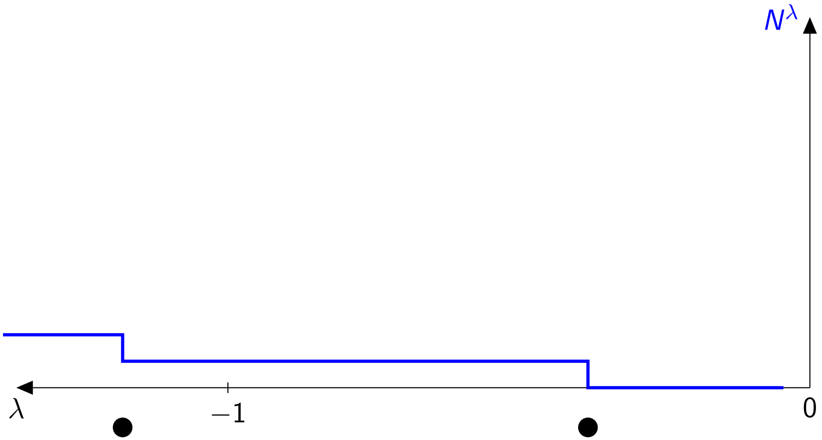

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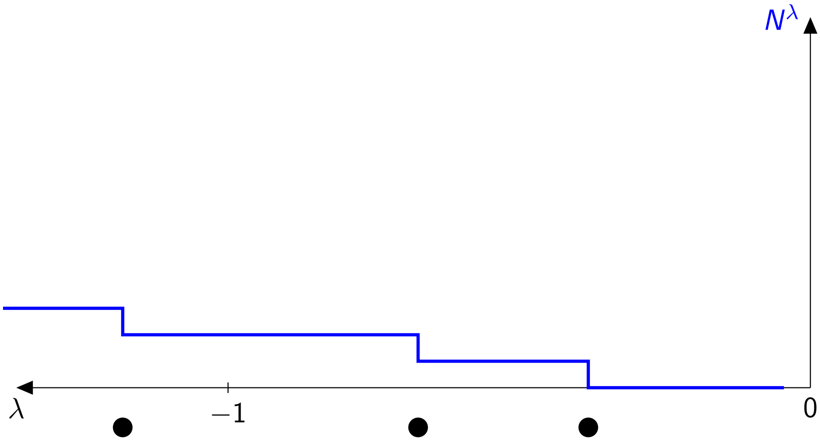

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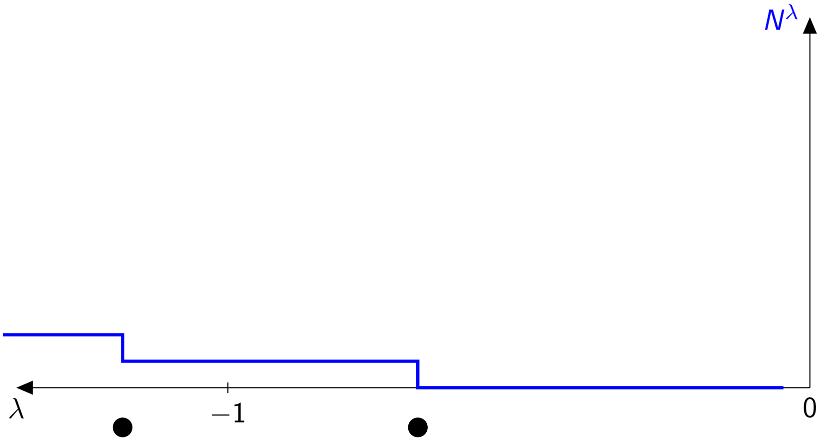

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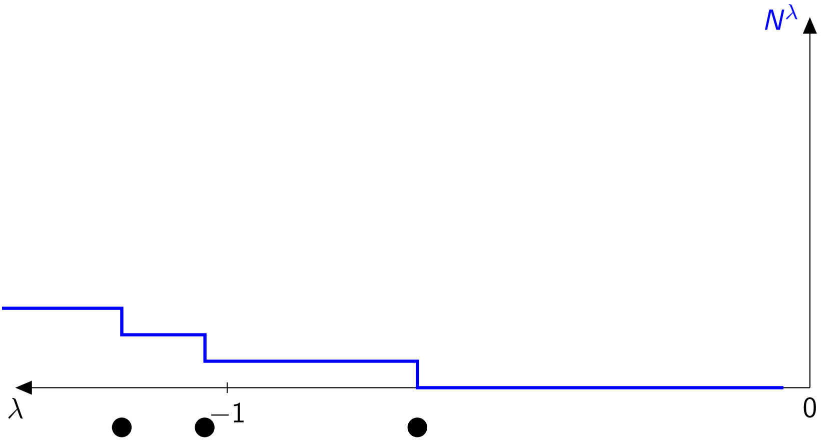

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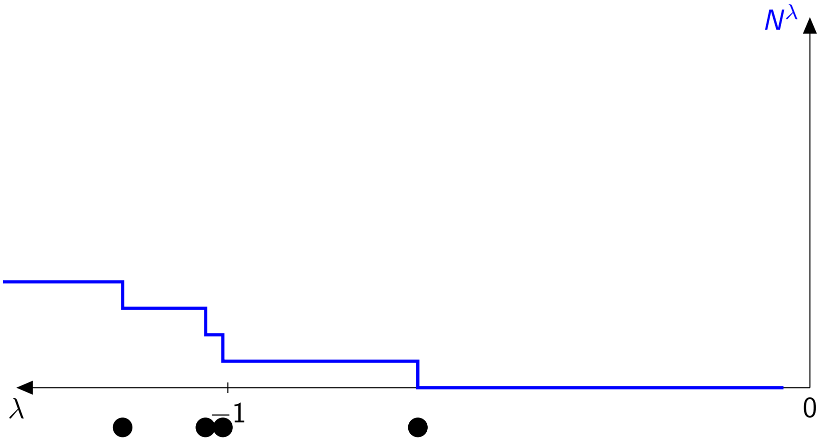

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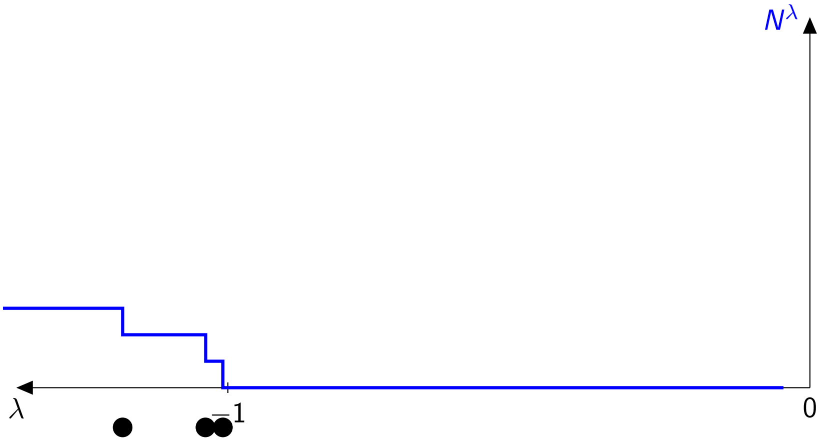

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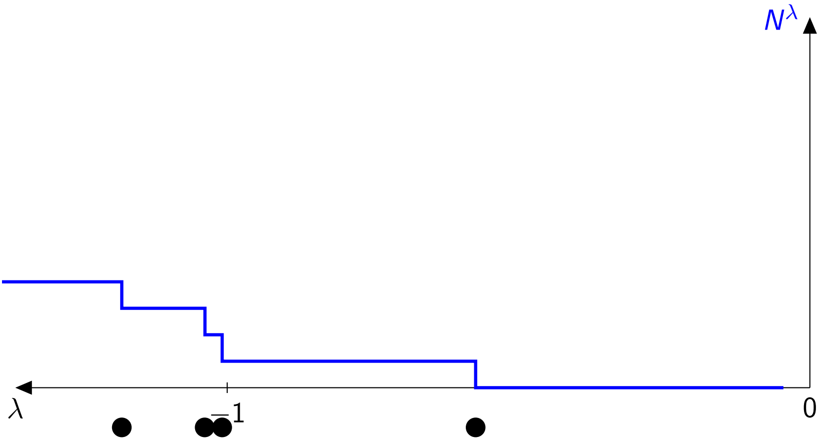

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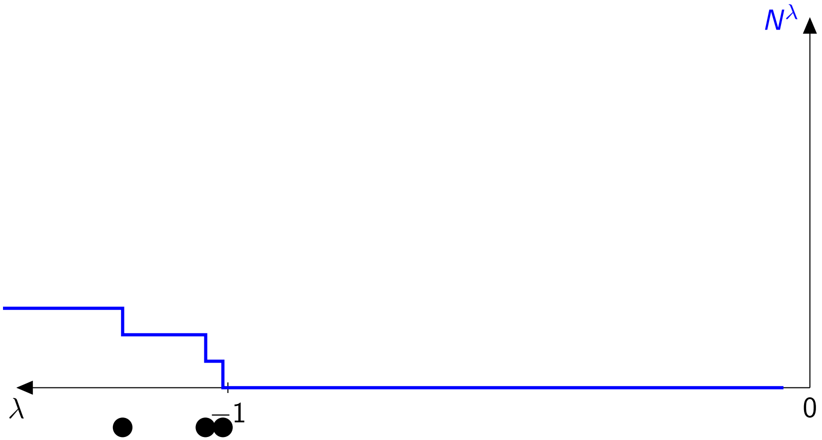

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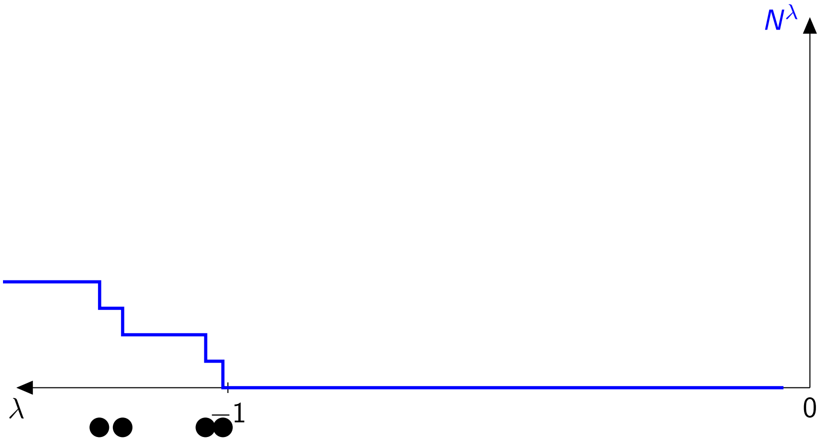

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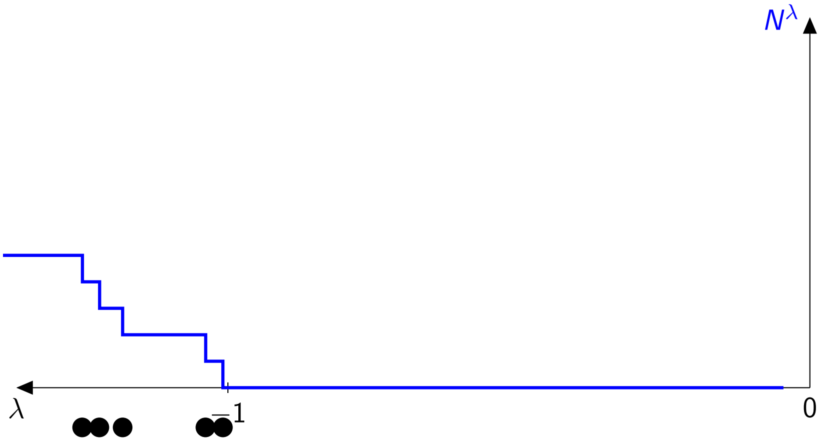

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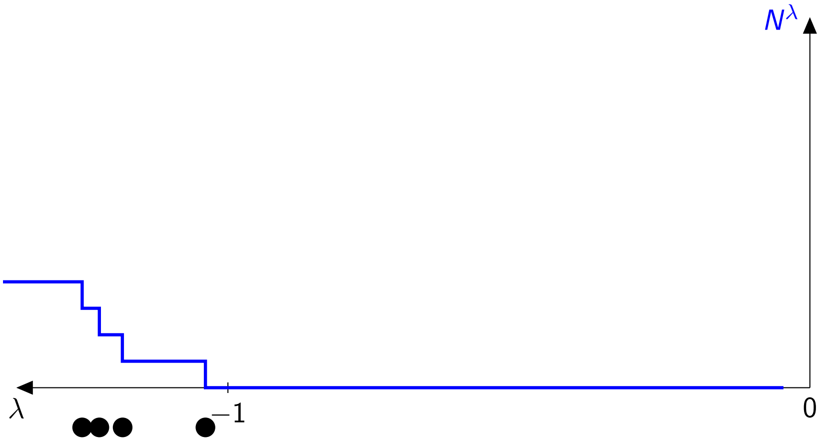

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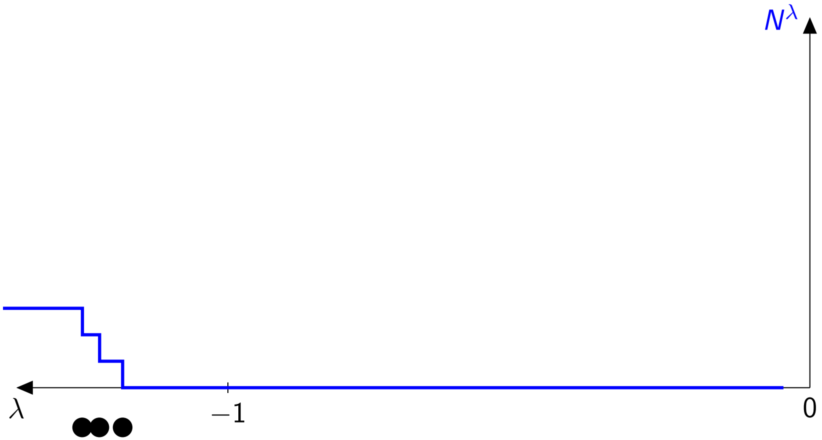

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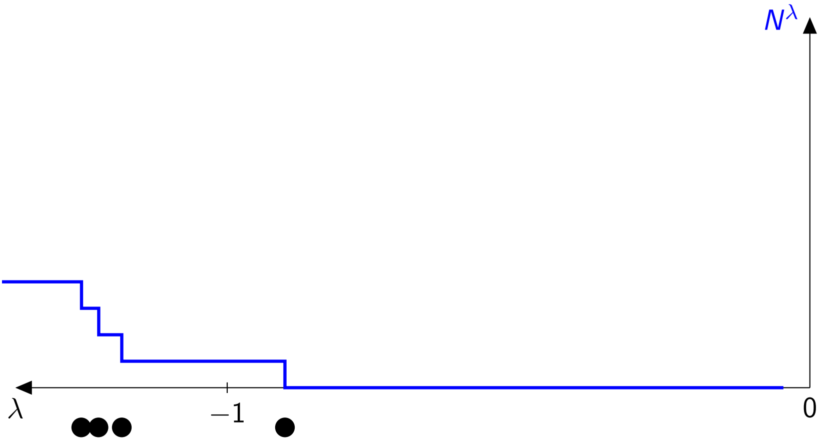

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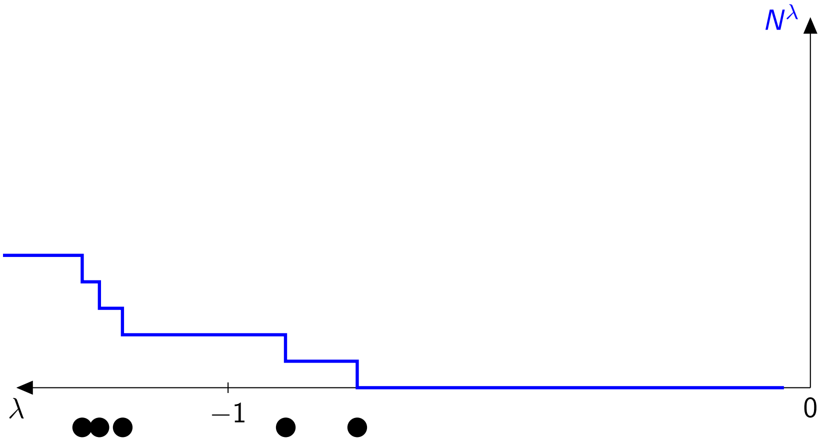

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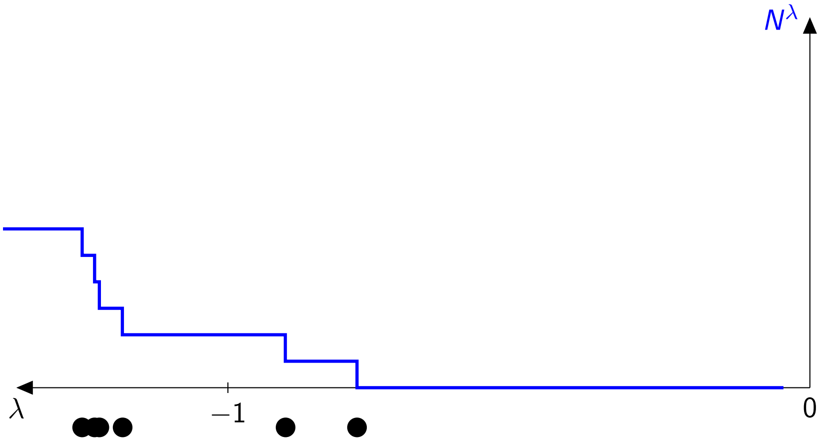

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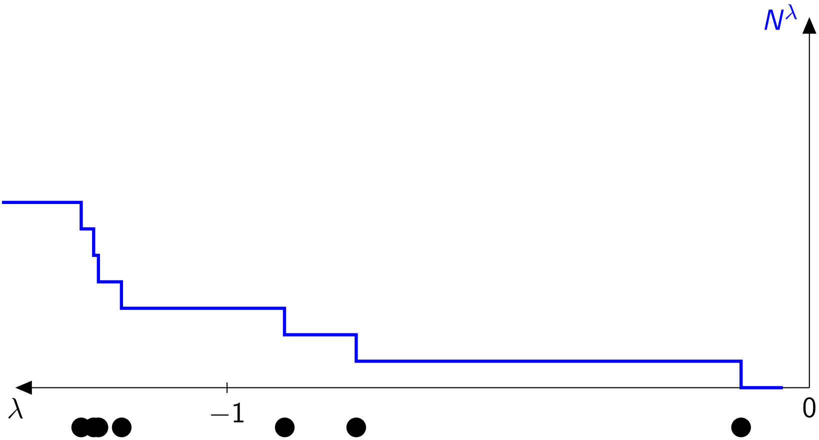

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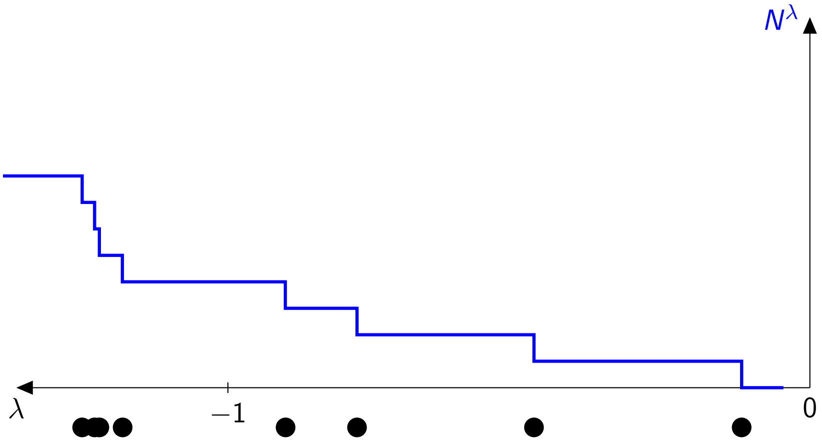

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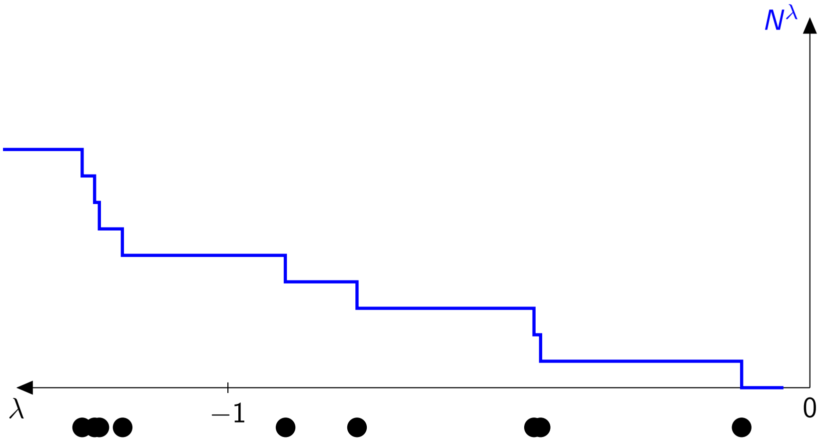

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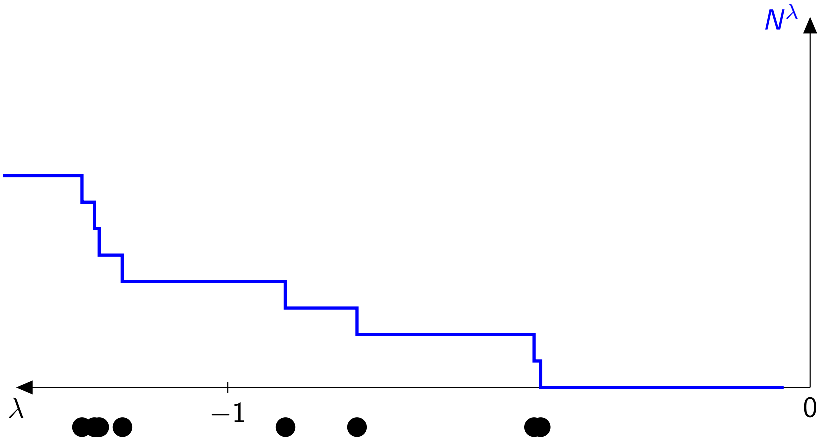

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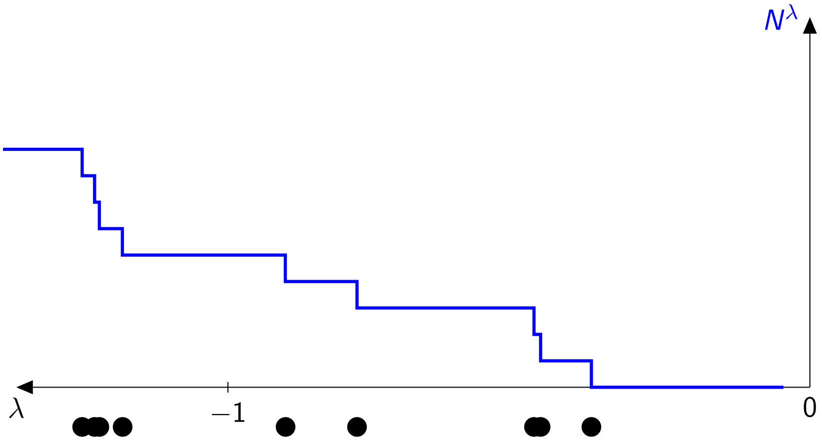

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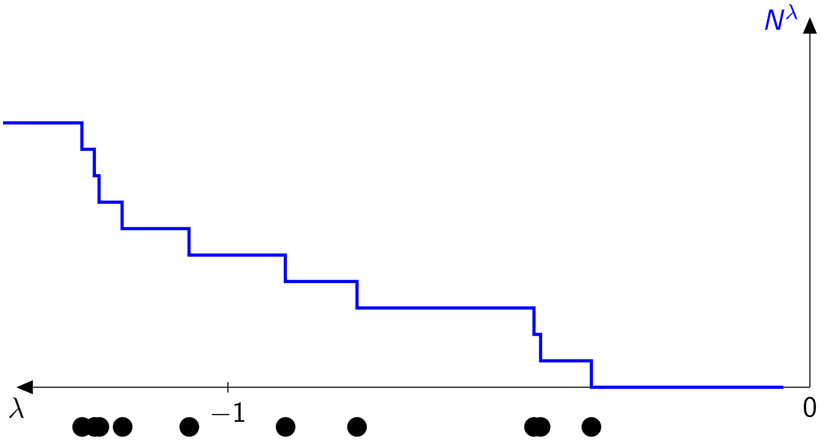

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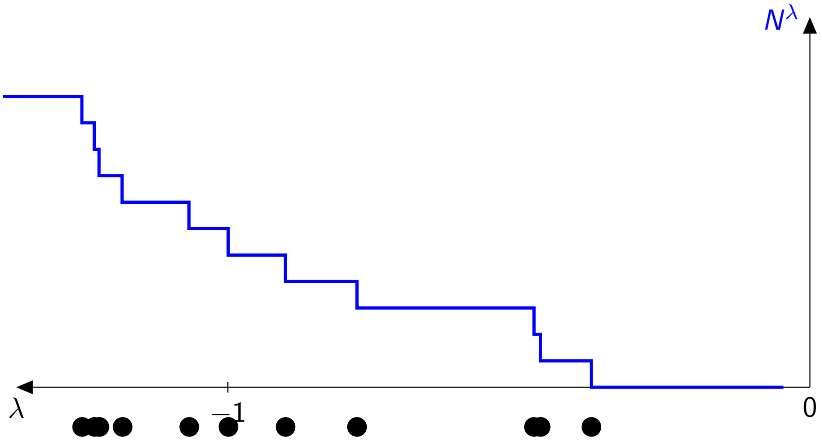

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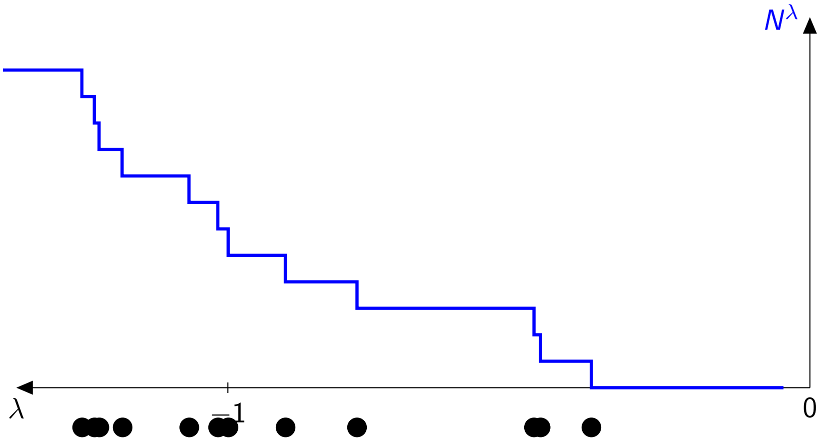

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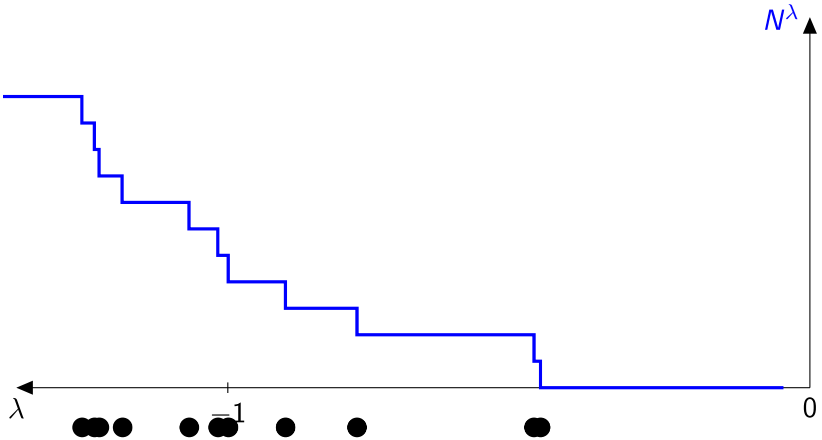

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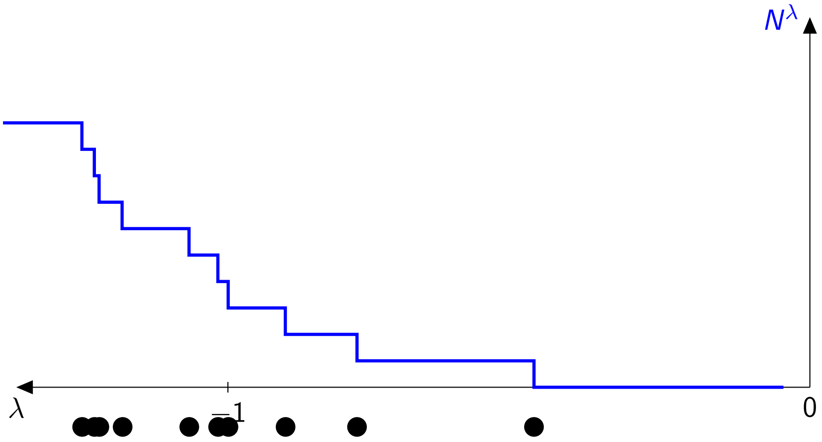

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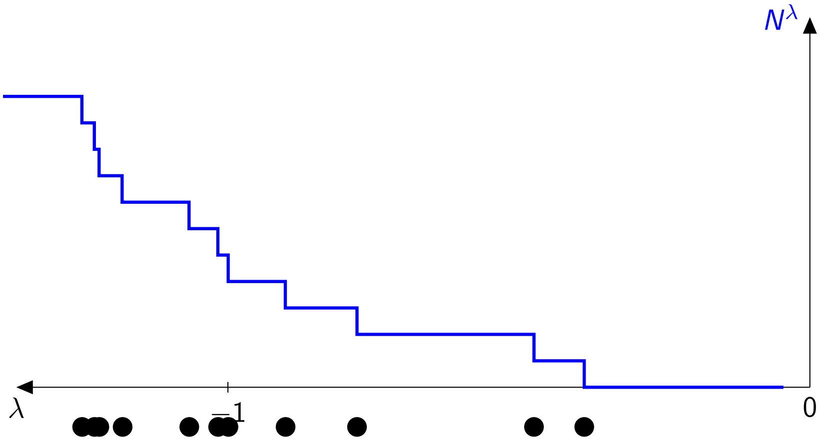

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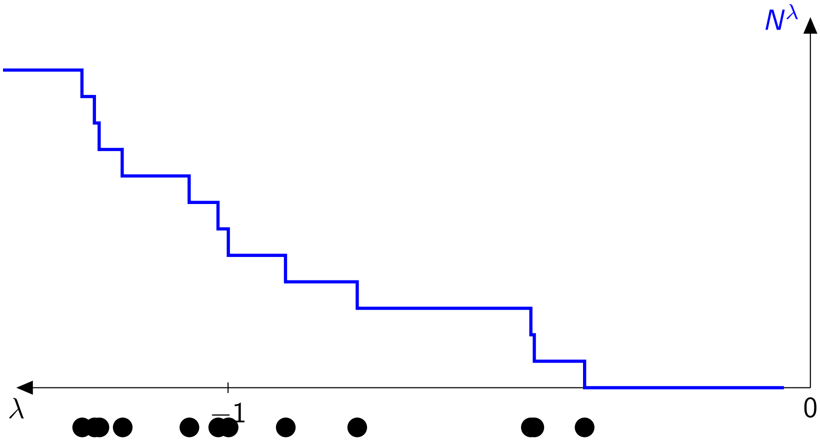

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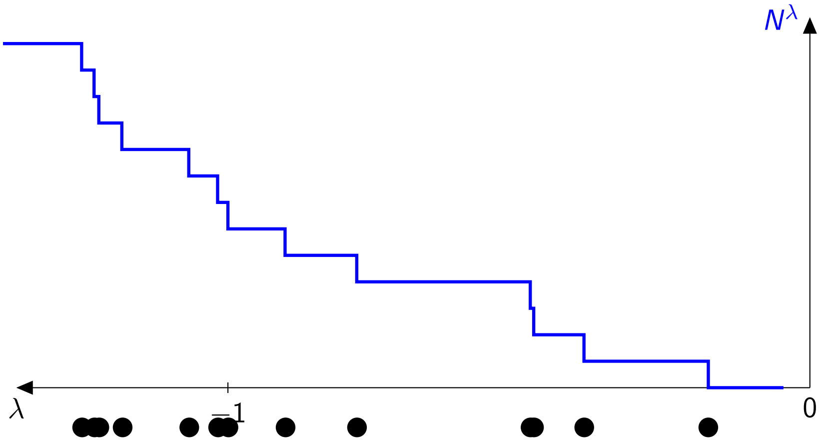

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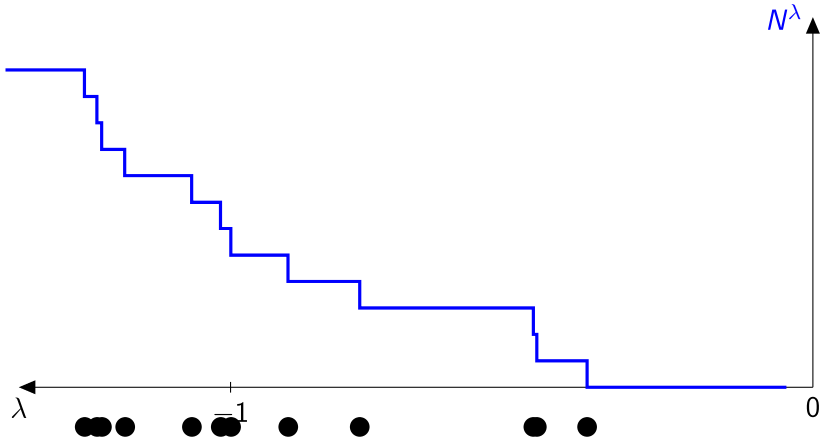

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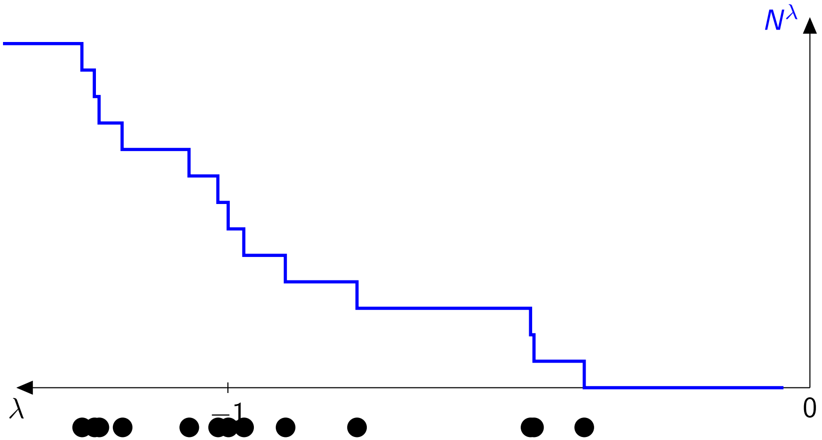

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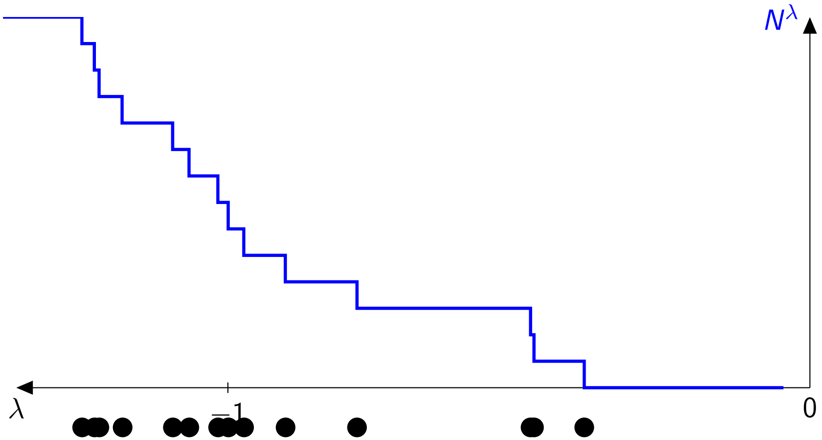

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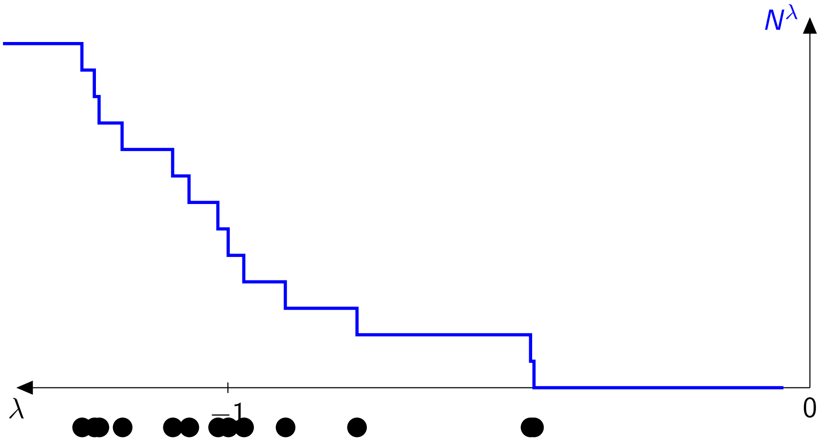

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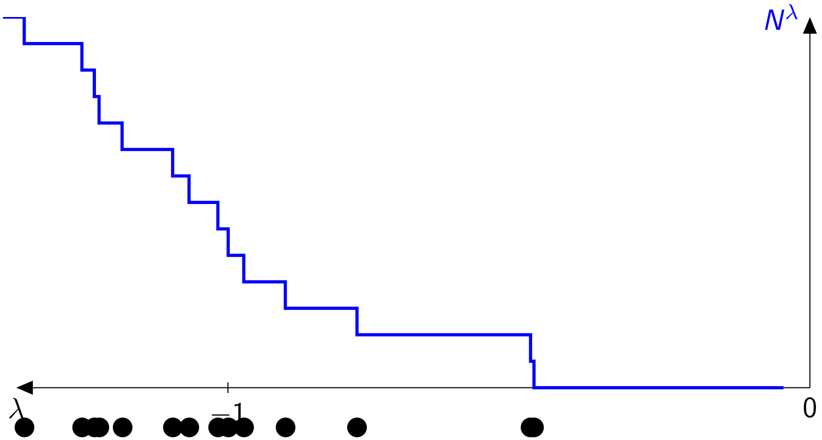

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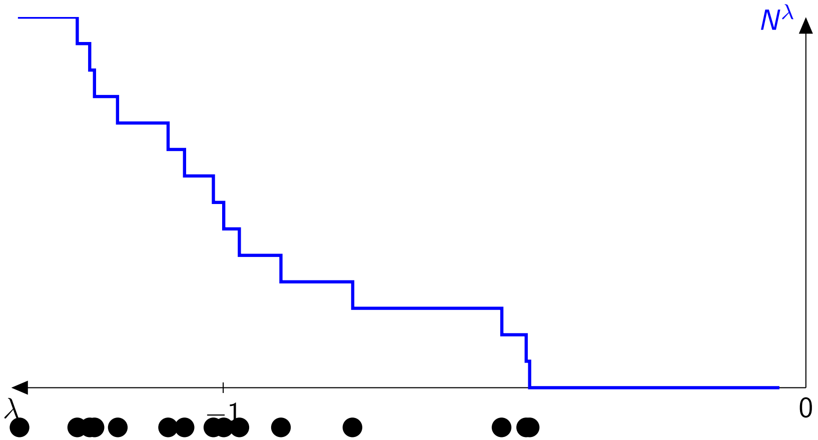

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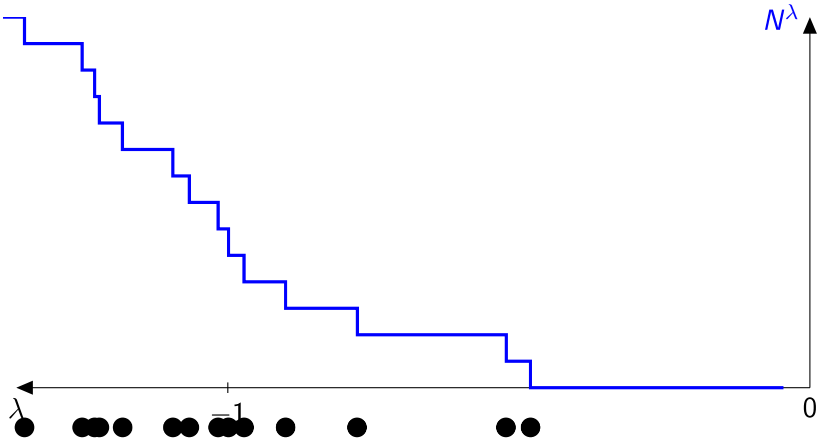

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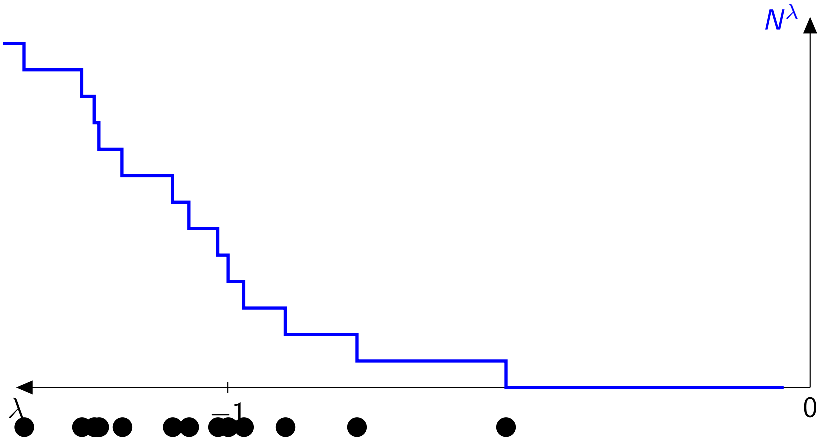

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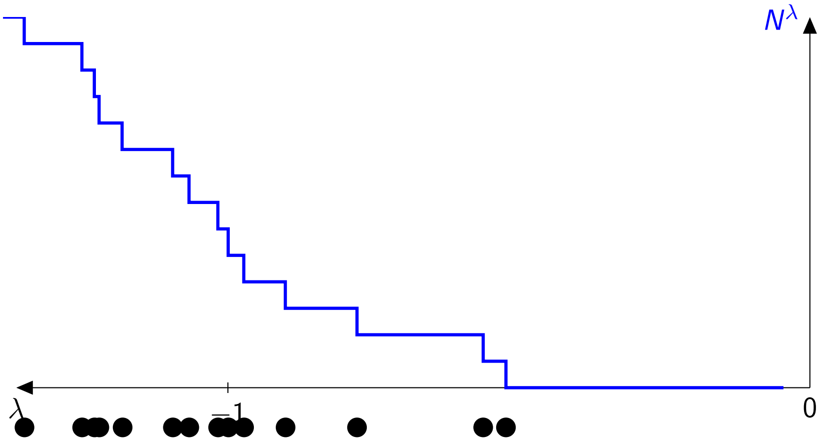

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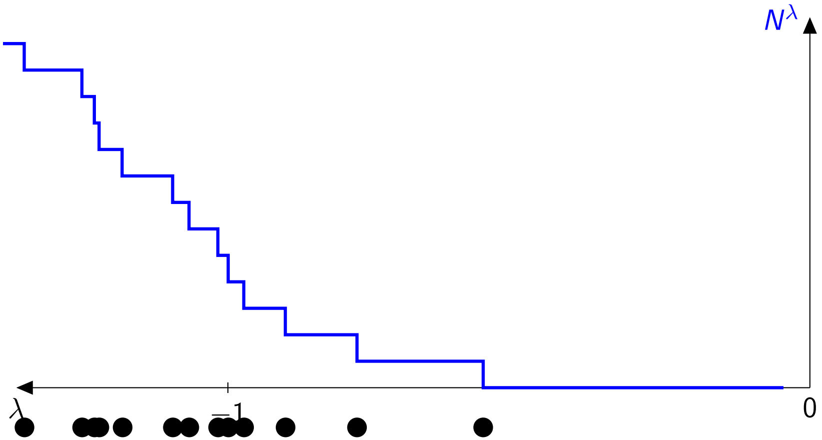

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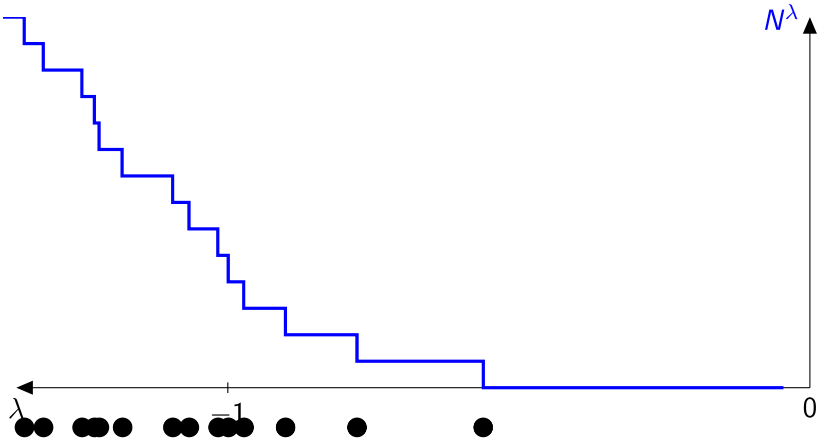

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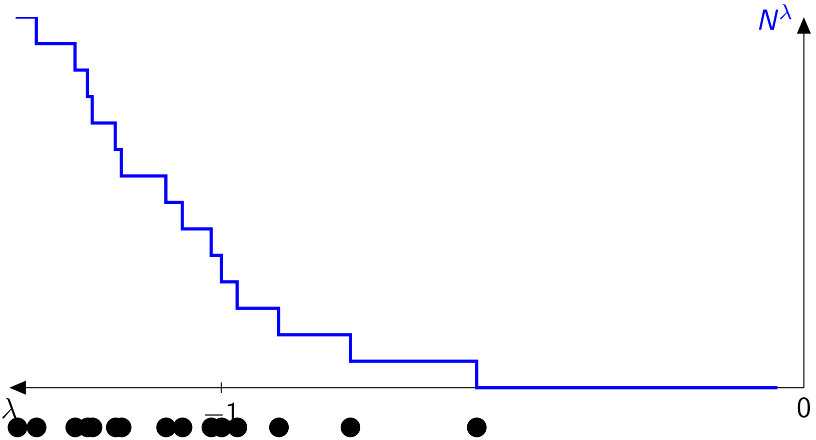

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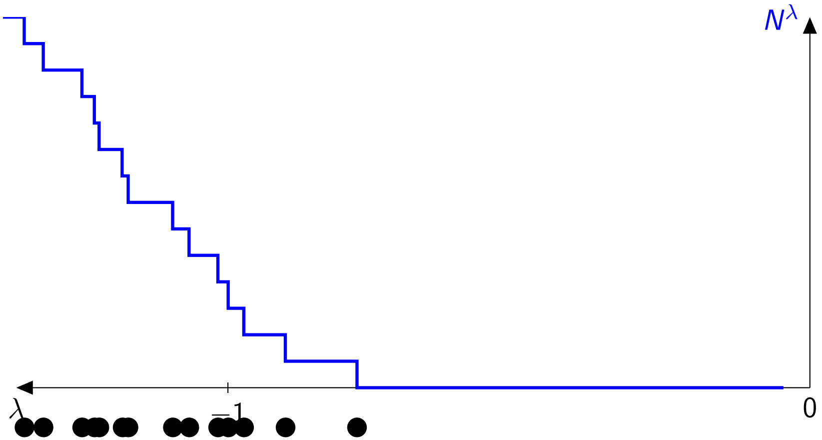

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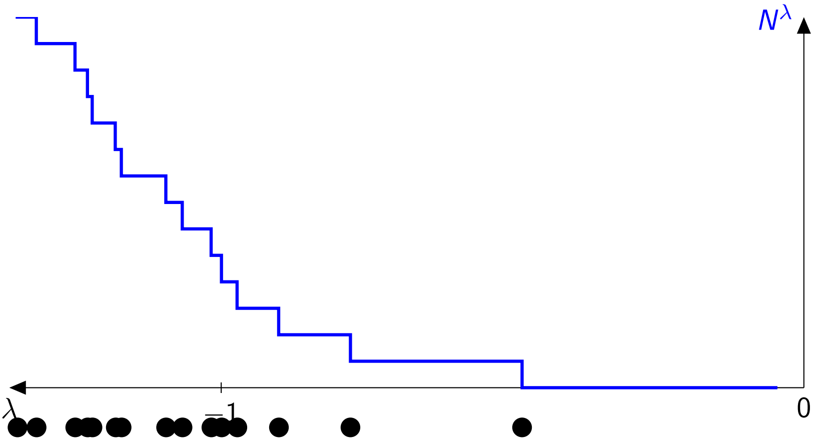

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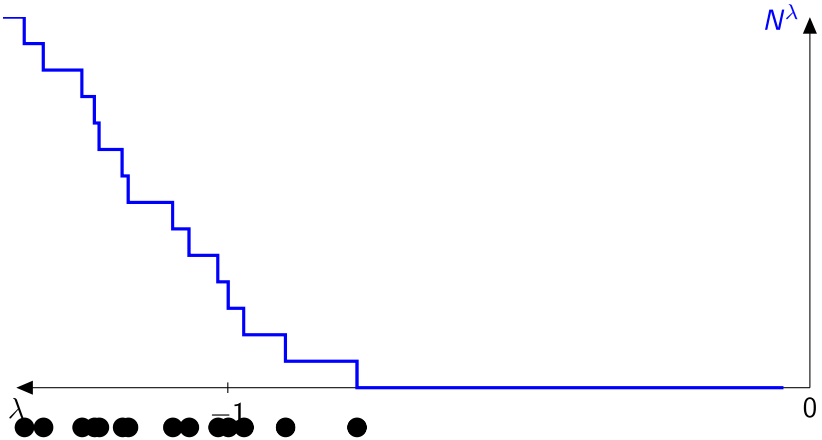

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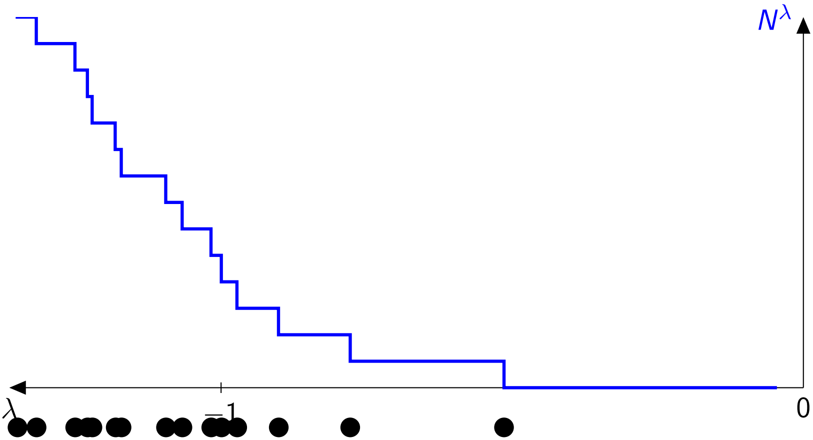

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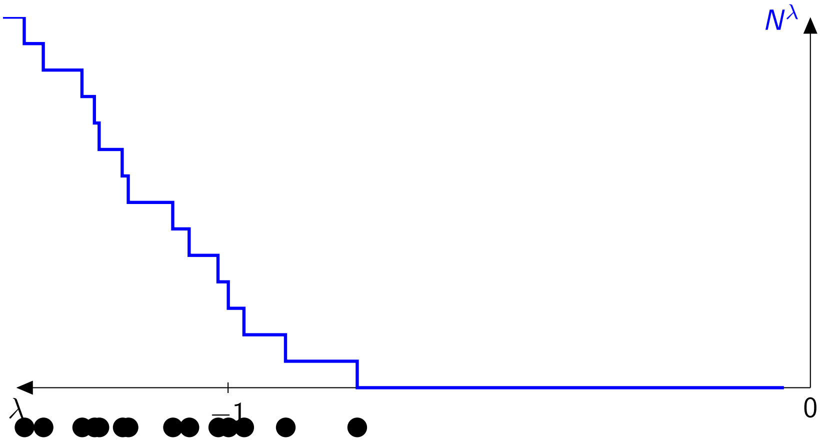

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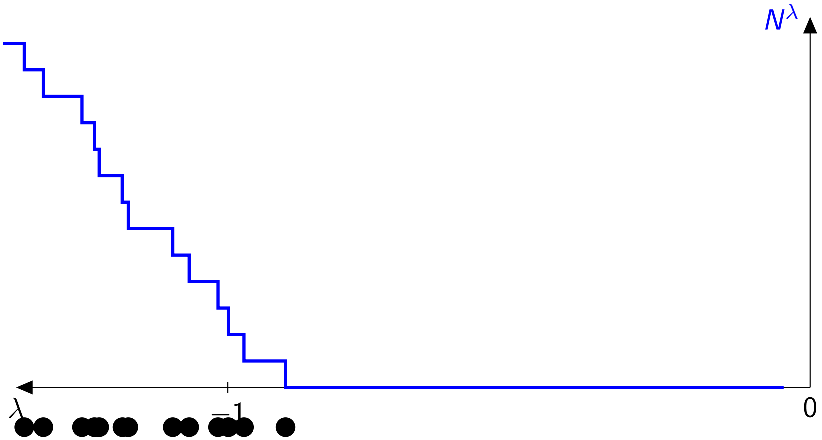

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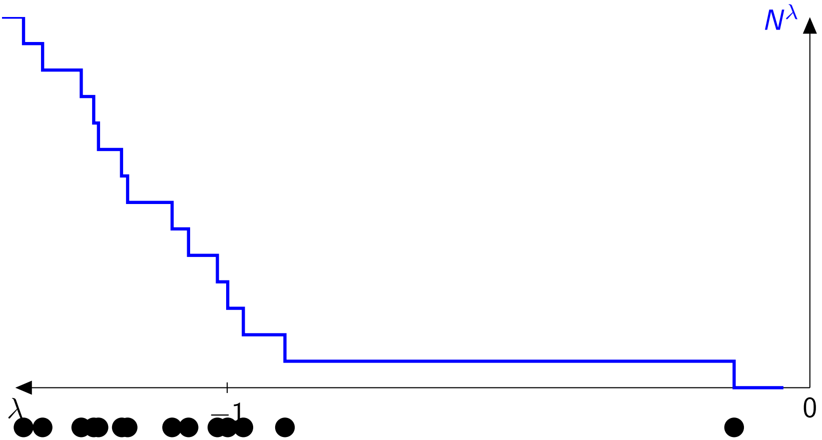

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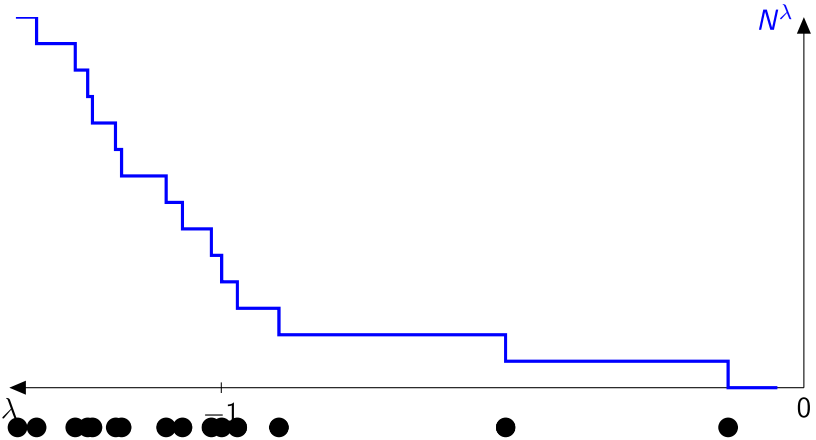

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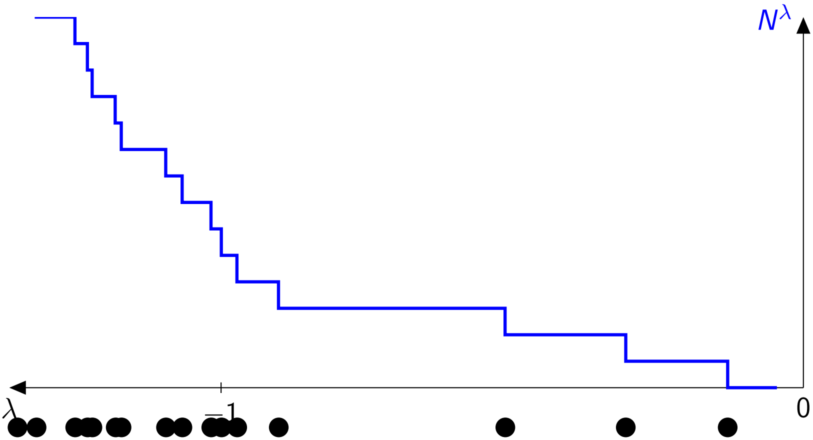

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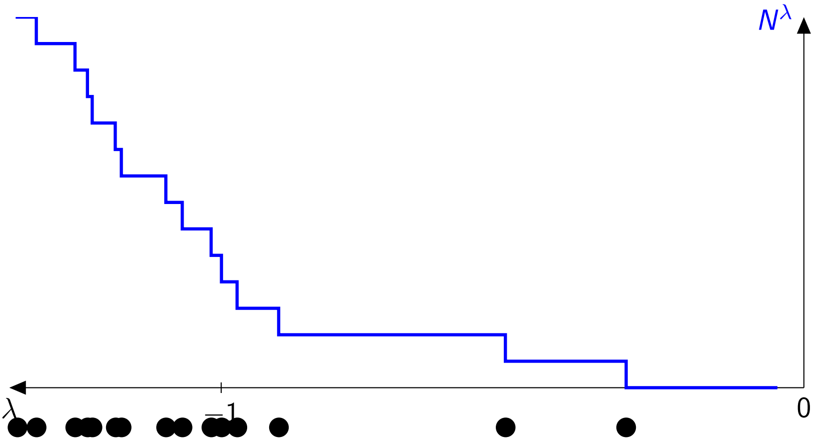

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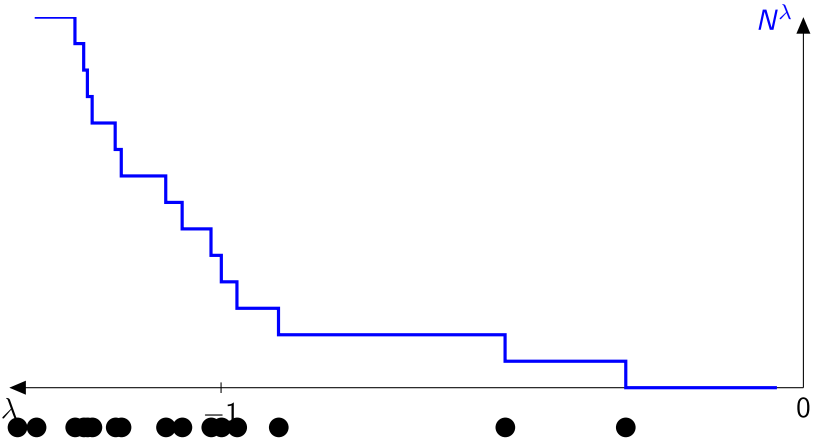

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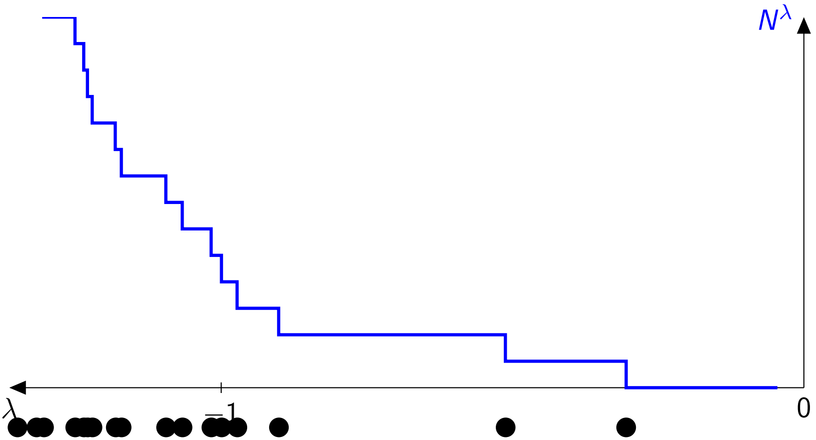

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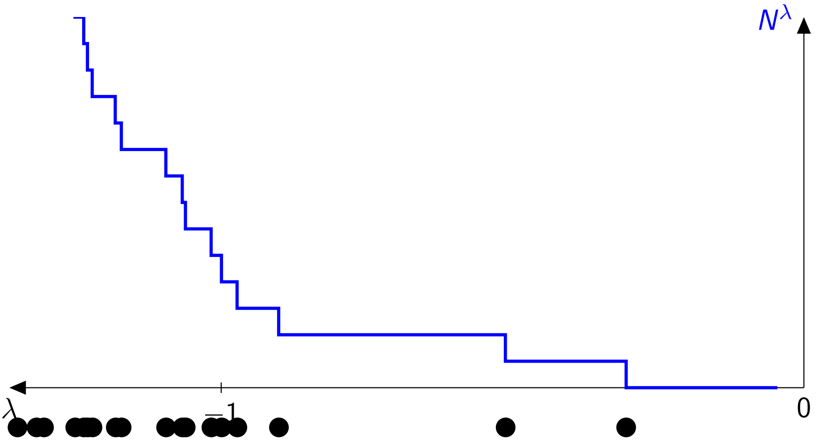

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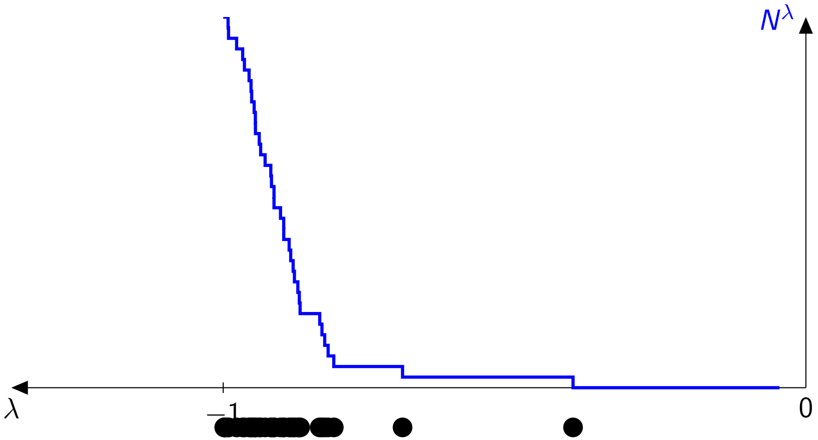

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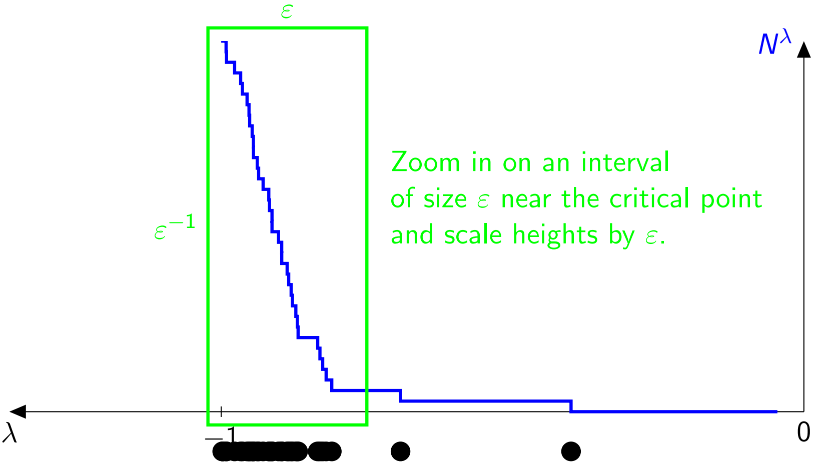


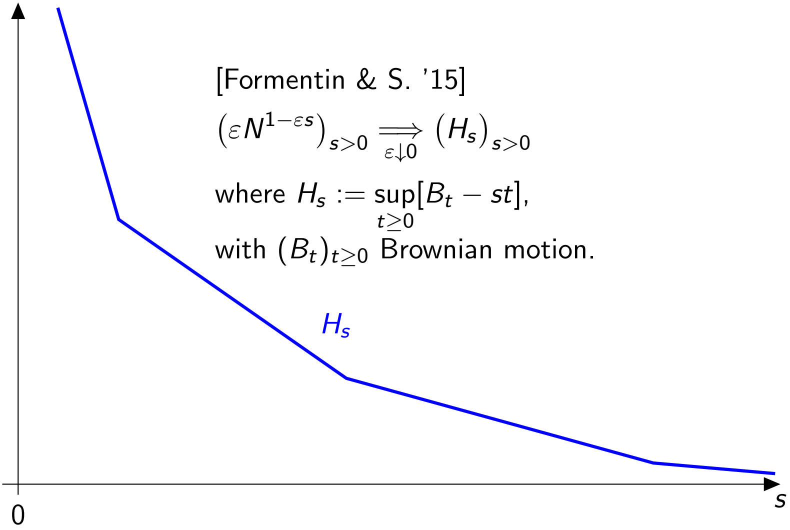
A model for email communication

[Formentin & S. '15]

$$\left(\varepsilon N^{1-\varepsilon s}\right)_{s>0} \underset{\varepsilon\downarrow 0}{\Longrightarrow} \left(H_s\right)_{s>0}$$

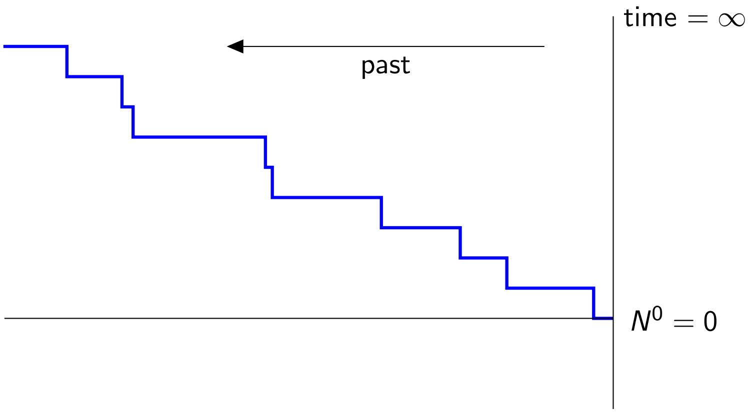
where $H_s := \sup_{t\geq 0}[B_t - st]$,

with $(B_t)_{t\geq 0}$ Brownian motion.



Groeneboom (1983): the concave majorant of Brownian motion.

Coupling from the past



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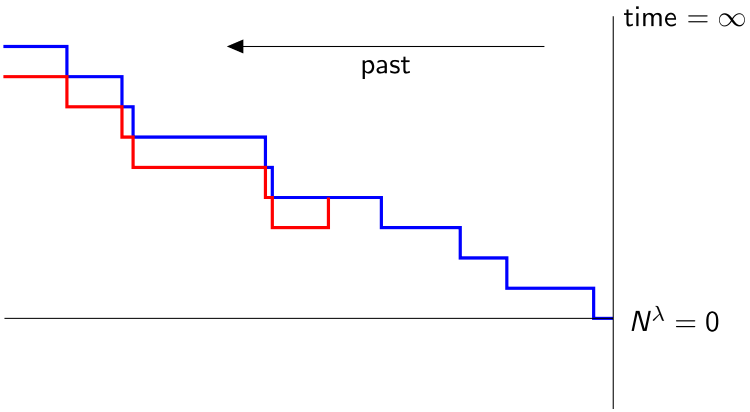

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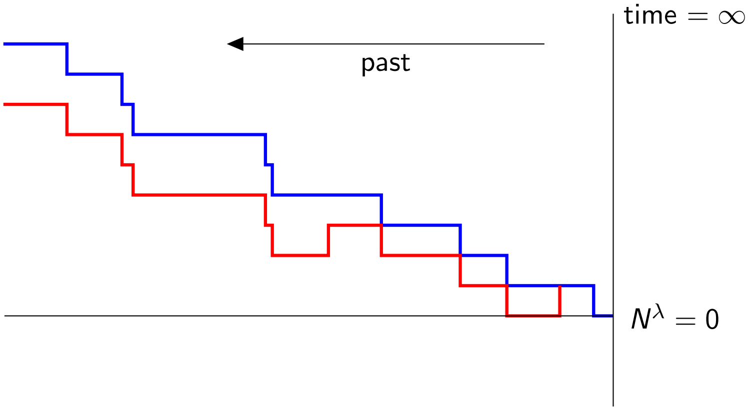

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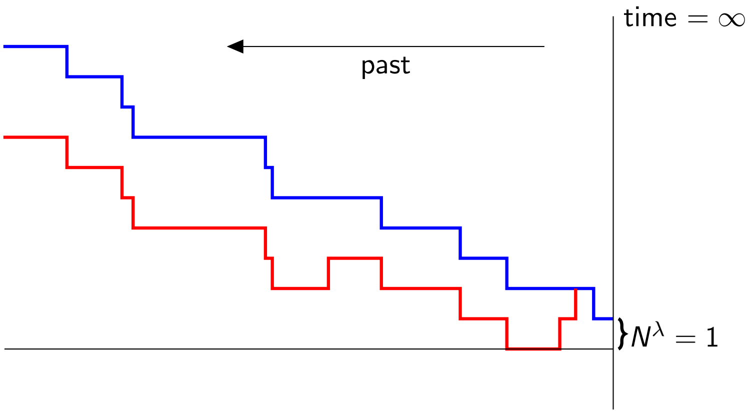

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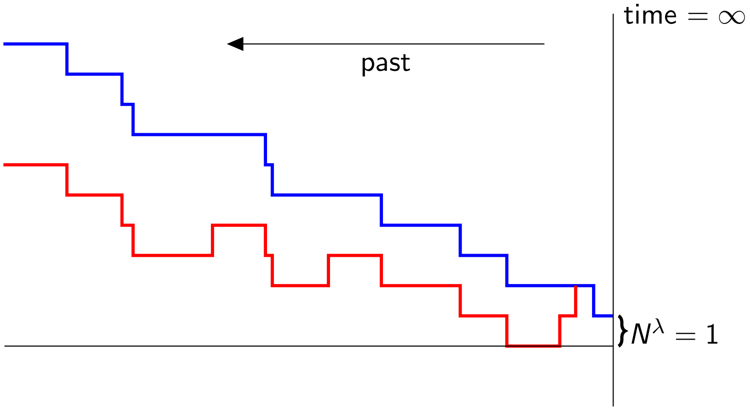

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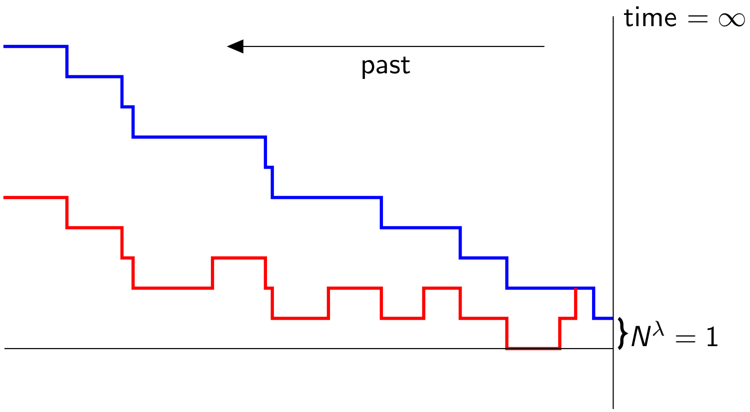

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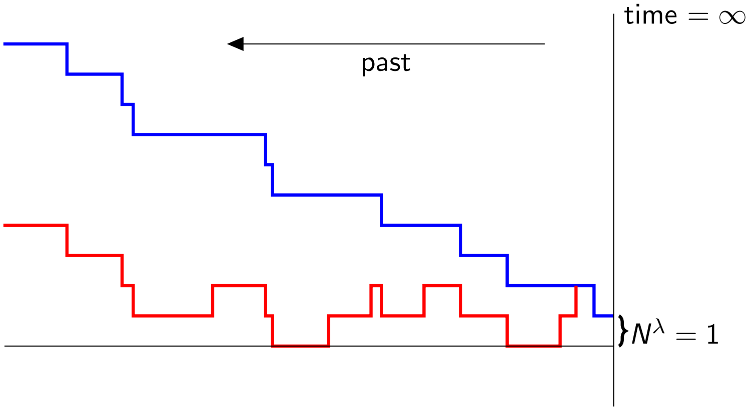

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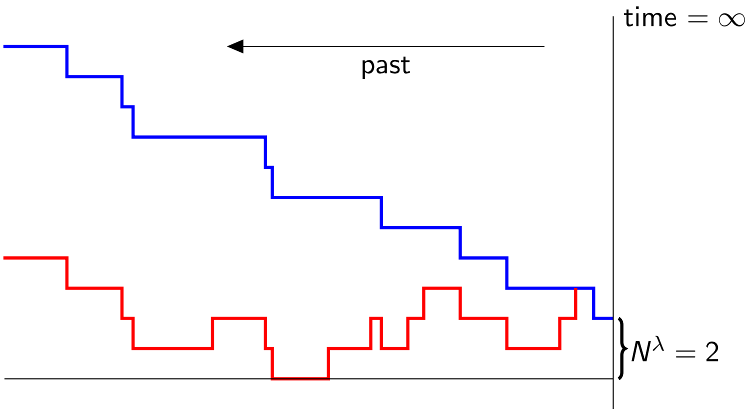

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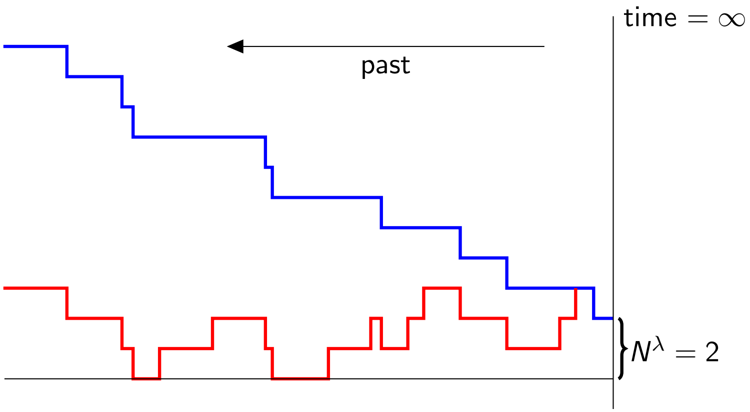

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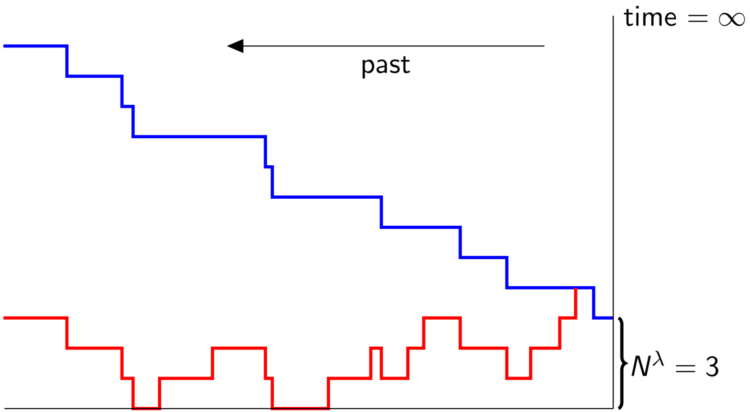


Some classical ecomomic theory

In classical economic theory (Walras,[1] 1874), the *price* of a commodity is determined by *demand* and *supply*.

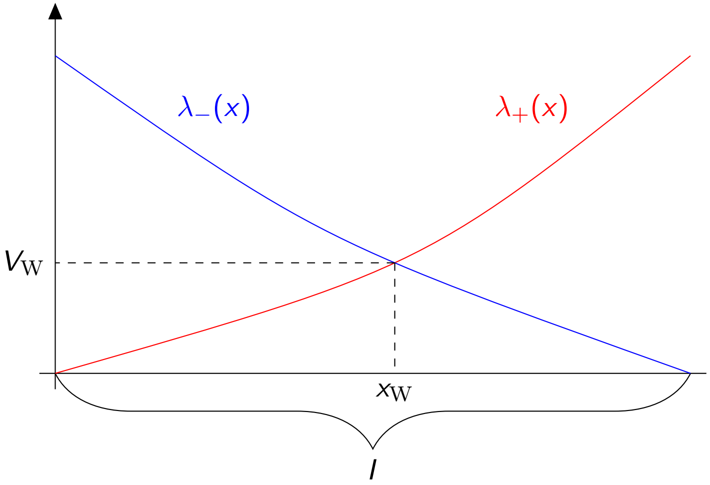
Let $\lambda_-(x)$ (resp. $\lambda_+(x)$) be the total *demand* (resp. *supply*) for a commodity at price level p, i.e., the total amount that people are willing to buy (resp. sell), per unit of time, for a price of at most (resp. at least) p per unit.

Assume that λ_- and λ_+ are continuous, strictly de/increasing, and zero at the endpoints of the price interval I.

Postulate In an equilibrium market, the commodity is traded at the *equilibrium prize* $x_{\rm W}$ defined by $\lambda_-(x_{\rm W}) = \lambda_+(x_{\rm W})$ and the total volume of trade is given by $V_{\rm W} := \sup_{x\in I} \lambda_-(x) \wedge \lambda_+(x)$.

[1] Walras developed the theory of equilibrium markets in his book *Éléments d' économie politique pure.*

Some classical ecomomic theory



Stock & Commodity Exchanges & the Order Book

On stock & commodity exchanges, goods are traded using an *order book*.

The order book for a given asset contains a list of offers to buy or sell a given amount for a given price. Traders arriving at the market have two options.

- Place a **market order**, i.e., either *buy* (*buy market order*) or *sell* (*sell market order*) n units of the asset at the best price available in the order book.

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Market orders are matched to existing limit orders according to a mechanism that depends on the trading system.

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Stigler's model

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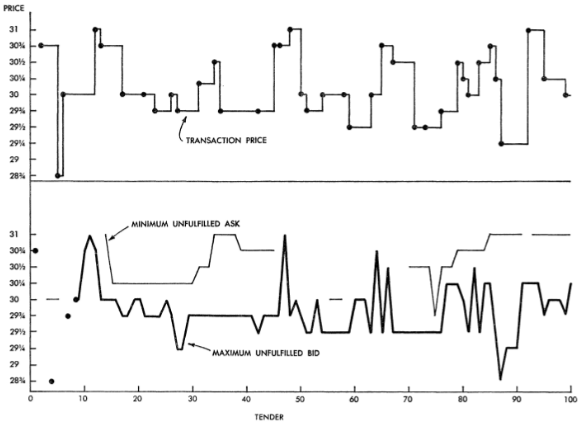


Fig. 1.—Hypothetical sequence of transaction prices, generated by sequence of random numbers, and maximum unfulfilled bid and minimum unfulfilled ask prices (equilibrium price of 29¾ or 30).

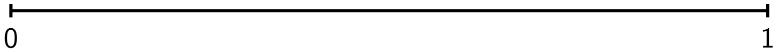
Stigler (1964) simulated the model with $\mu_\pm$ the uniform distributions on a set of 10 possible prices.

The uniform model

In the *uniform model*, the price interval is $I = [0, 1]$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = x$. For this model, $\mu_- = \mu_+$ is the Lebesgue measure on $[0, 1]$, and $x_{\mathrm{W}} = 0.5$, $V_{\mathrm{W}} = 0.5$.

Every model for which $\mu_\pm$ are atomless and $\mu_- = \mu_+$ can be transformed into the uniform model. In general, we can transform any model into a model for which $\mu_- + \mu_+$ is the Lebesgue measure.

Numerical simulation



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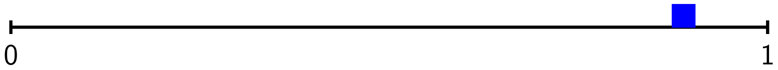

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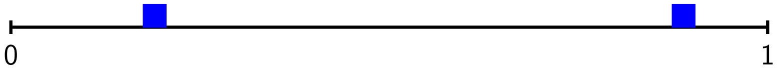

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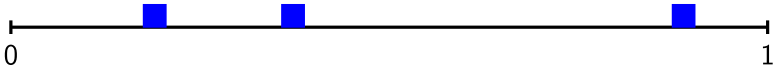

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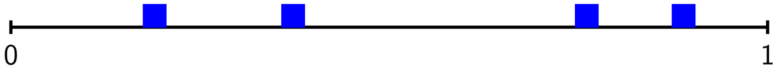

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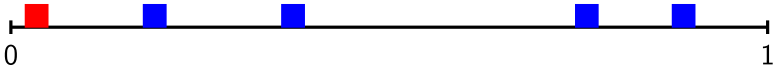

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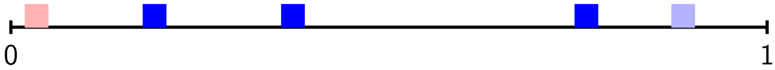

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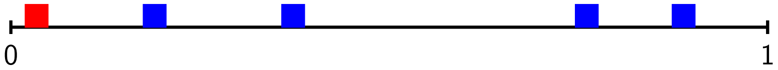

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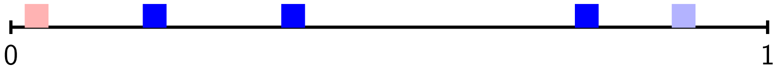

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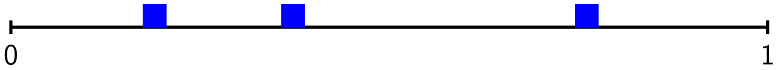

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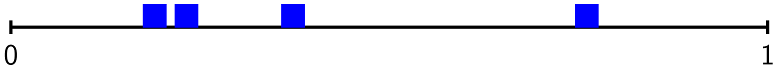

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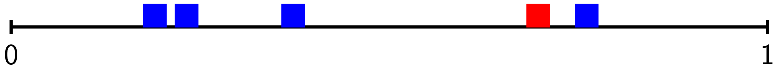

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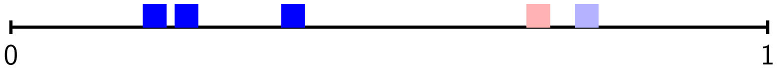

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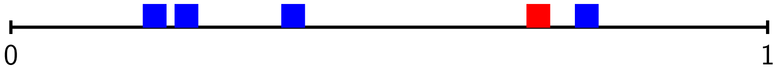

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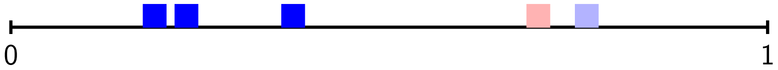

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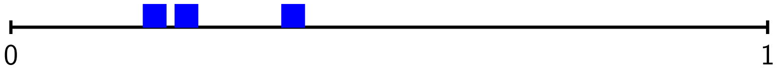

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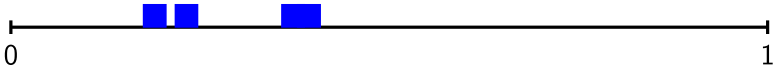

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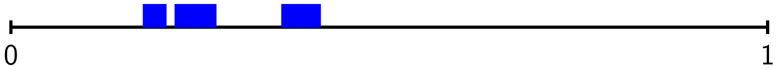

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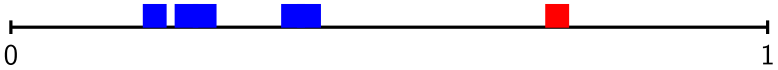

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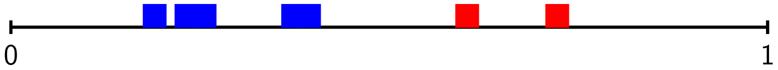

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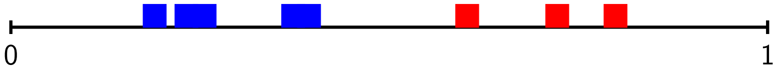

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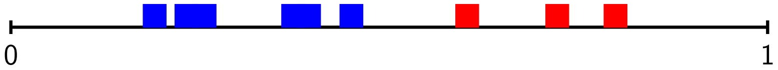

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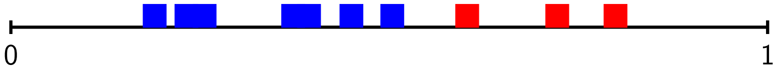

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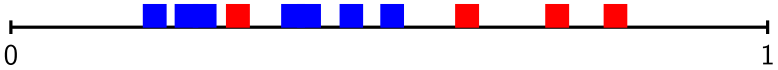

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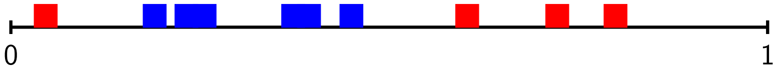

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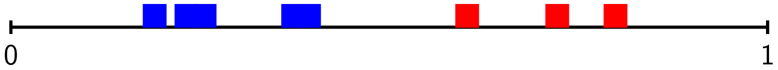

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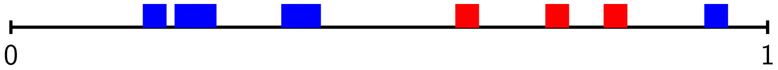

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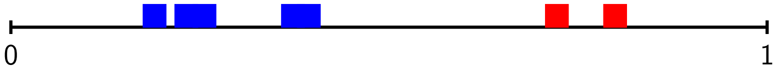

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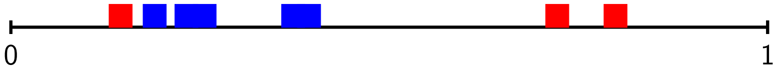

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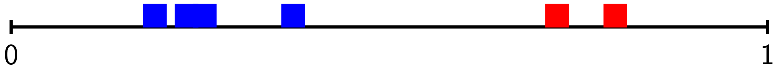

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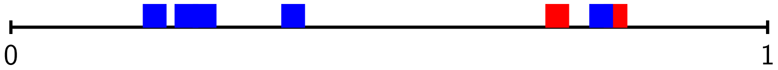

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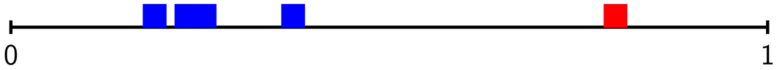

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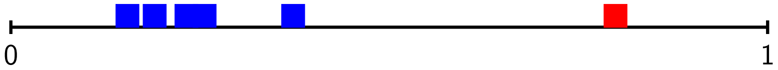

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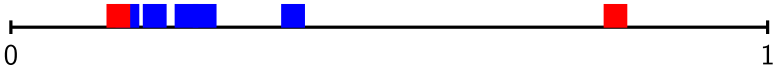

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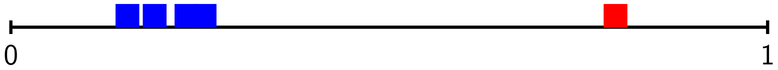

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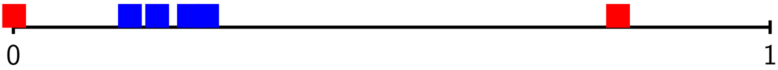

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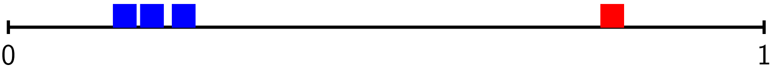

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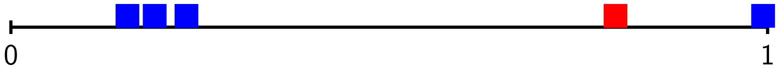

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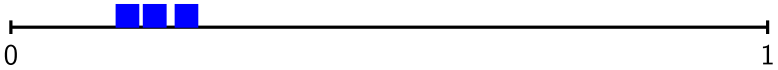

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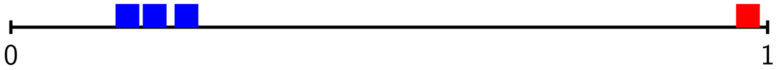

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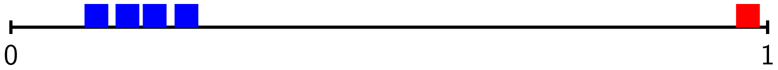

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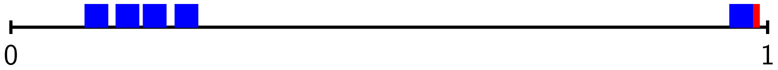

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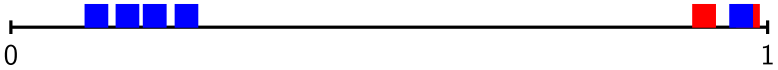

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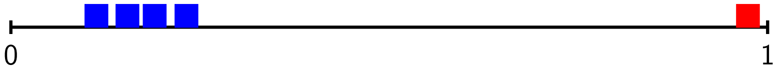

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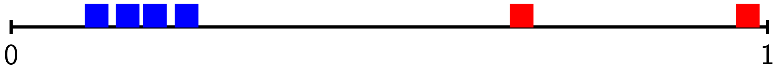

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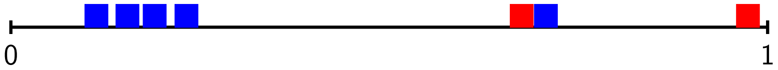

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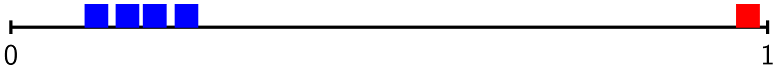

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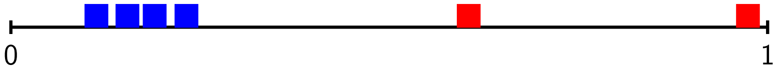

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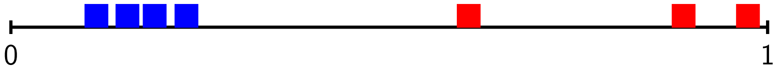

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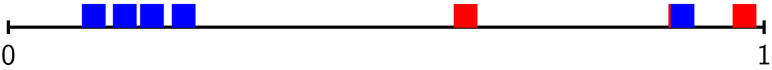

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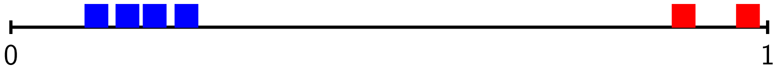

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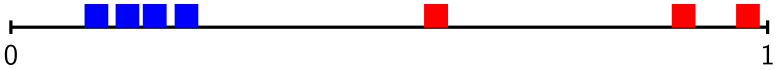

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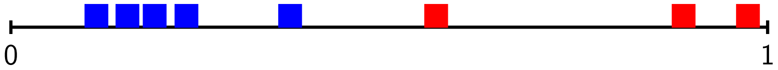

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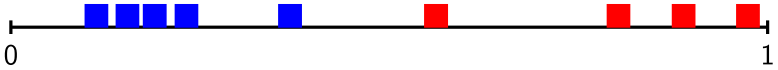

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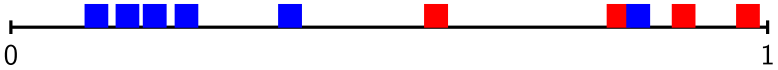

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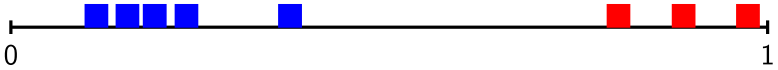

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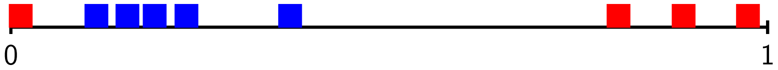

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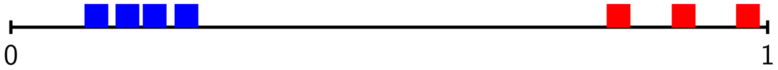

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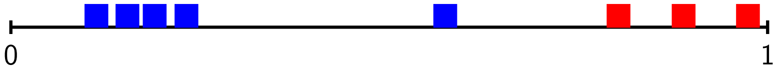

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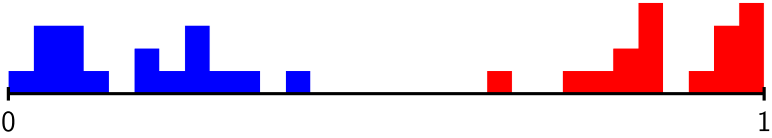
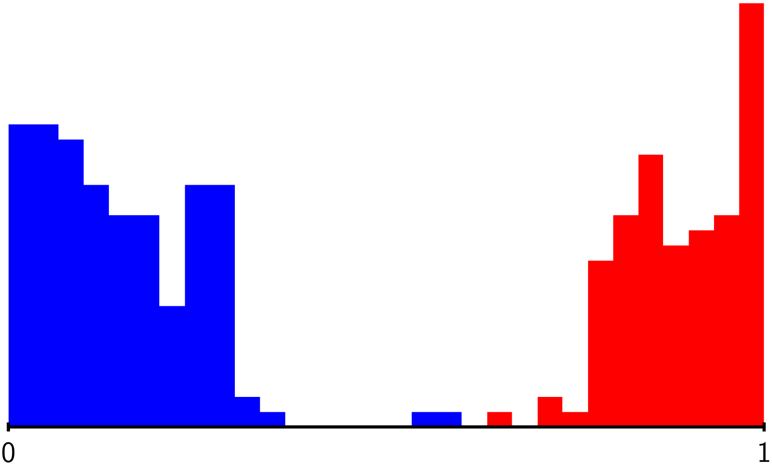


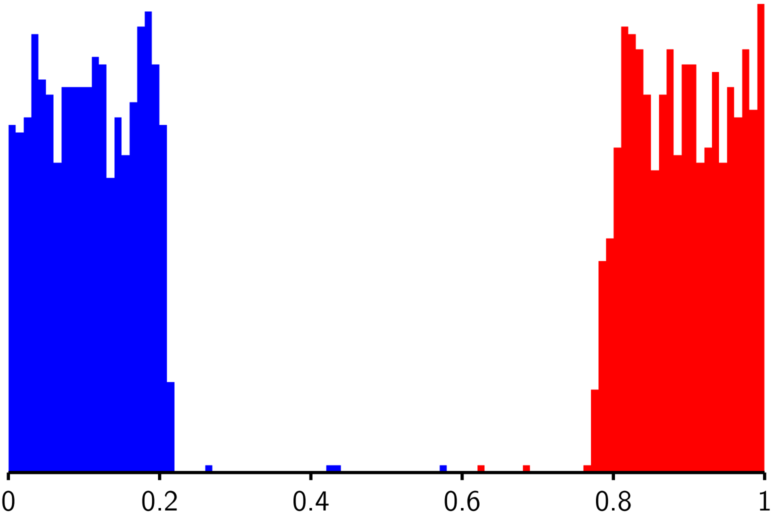
The order book after the arrival of 100 traders.

Numerical simulation



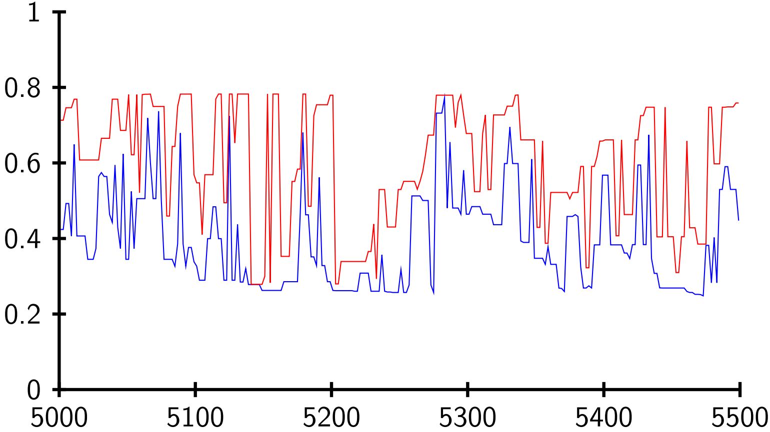
The order book after the arrival of 1000 traders.

Numerical simulation



The order book after the arrival of 10,000 traders.

Numerical simulation



Evolution of the highest bid and lowest ask prices between the arrivals of the 5000th and 5500th trader.

Numerical simulation

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- Sell limit orders at a price above $J_{+} := V_{\mathrm{L}}$ are never matched.
- The bid and ask prices keep fluctuating between J_{-} and J_{+}.
- The spread is huge, most of the time.

Restricted models

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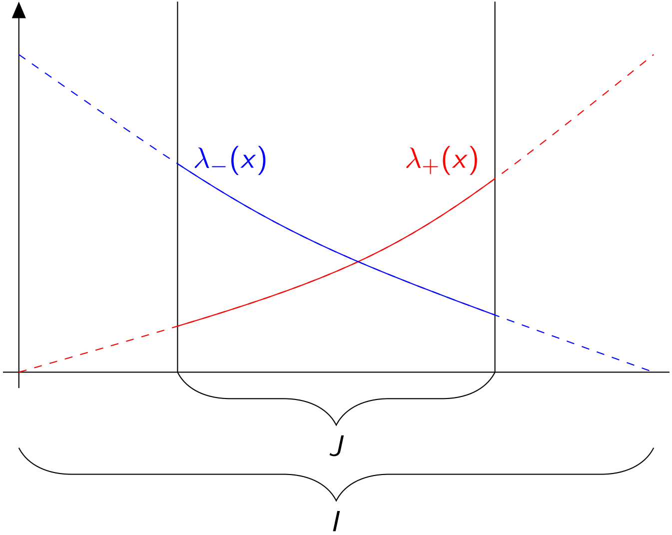
- For the restricted model, it is no longer true that $\lambda_\pm$ are zero at the endpoints of the interval.

Restricted models



Restricted models

We model the state of the order book by a signed counting measure of the form

$$\mathcal{X} = \sum_i \sigma_i \delta_{x_i} \quad \text{with} \quad x_i \in J,\ \sigma_i \in \{-1,+1\}.$$

We interpret $-\delta_x$ as a buy and $+\delta_x$ as a sell limit order at the price x and write $\mathcal{X} = \mathcal{X}^+ - \mathcal{X}^-$ with $\mathcal{X}^\pm$ disjoint nonnegative measures.
The state space $\mathcal{S}_{\rm ord}$ consists of all $\mathcal{X}$ such that

(i) there are no $x, y \in J$ such that $x < y$, $\mathcal{X}(\{x\}) > 0$, $\mathcal{X}(\{y\}) < 0$.
(ii) $\mathcal{X}^-\big([J_- + \varepsilon, J_+)\big) < \infty$ for all $\varepsilon > 0$.
(iii) $\mathcal{X}^+\big((J_-, J_+ - \varepsilon]\big) < \infty$ for all $\varepsilon > 0$.

We set $\mathcal{S}^{\rm fin}_{\rm ord} := \{\mathcal{X} \in \mathcal{S}_{\rm ord} : \mathcal{X}^\pm(J) \text{ are finite}\}$.

Restricted models

Assume $\lambda_-(J_+), \lambda_+(J_-) > 0$. Then the following statements are equivalent:

(i) The model is positive recurrent, i.e., starting with an empty order book, we return to the empty order book in finite expected time.

(ii) There exists an invariant law on $\mathcal{S}_{\rm ord}^{\rm fin}$.

(iii) There exists an invariant law ν on $\mathcal{S}_{\rm ord}^{\rm fin}$ and the process is ergodic, in the sense that the law at time t of the process started in any finite initial state converges in total variation norm to ν as $t \to \infty$.

However, even if the process is not positive recurrent, presumably, it can sometimes have an invariant law on $\mathcal{S}_{\rm ord}$.

Luckock's equation

[Luckock '03] Let $M^{\pm}$ denote the price of the best buy/sell offer. Assume that the process is in equilibrium. Then

$$f_-(x) := \mathbb{P}[M^- < x] \quad \text{and} \quad f_+(x) := \mathbb{P}[M^+ > x]$$

solve the differential equation

$$\begin{aligned} \text{(i)} \quad & f_- \mathrm{d}\lambda_+ = -\lambda_- \mathrm{d}f_+, \\ \text{(ii)} \quad & f_+ \mathrm{d}\lambda_- = -\lambda_+ \mathrm{d}f_- \\ \text{(iii)} \quad & f_+(J_-) = 1 = f_-(J_+). \end{aligned}$$

Proof: Since buy orders are added to $A \subset J$ at the same rate as they are removed,

$$\int_A \mathbb{P}[M^- < x]\, \mathrm{d}\lambda_+(\mathrm{d}x) = \int_A \lambda_-(x)\, \mathbb{P}[M^+ \in \mathrm{d}x].$$

Luckock's equation

Assume $\lambda_-(J_+), \lambda_+(J_-) > 0$. Then Luckock's equation has a unique solution (f_-, f_+). In general, however, $f_\pm$ need not take values in $[0,1]$. In such a case, by Luckock's result, no invariant law is possible.

Theorem [S. '16] Assume $\lambda_-(J_+), \lambda_+(J_-) > 0$. Then the Stigler-Luckock model on J is positive recurrent if and only if the unique solution to Luckock's equation satisfies $f_-(J_-) > 0$ and $f_+(J_+) > 0$.

Note F. Kelly and E. Yudovina have a similar, but somewhat less complete result.

Linear functionals

Let $\mathcal{X}_t^{\pm}$ denote the sets of buy and sell limit orders in the order book at time t and consider a weighted sum over the limit orders of the form

$$F(\mathcal{X}_t) := \int_J w_-(x)\mathcal{X}^-(\mathrm{d}x) + \int_J w_+(x)\mathcal{X}^+(\mathrm{d}x),$$

where $w_\pm : \overline{I} \to \mathbb{R}$ are "weight" functions.
Lemma One has

$$\tfrac{\partial}{\partial t}\mathbb{E}[F(\mathcal{X}_t)] = q_-(M_t^-) + q_+(M_t^+),$$

where $q_- : [J_-, J_+) \to \mathbb{R}$ and $q_+ : (J_+, J_-] \to \mathbb{R}$ are given by

$$\begin{aligned} q_-(x) &:= \int_x^{J_+} w_+ \mathrm{d}\lambda_+ - w_-(x)\lambda_+(x) \qquad \big(x \in [J_-, J_+)\big), \\ q_+(x) &:= -\int_{J_-}^{x} w_- \mathrm{d}\lambda_- - w_+(x)\lambda_-(x) \qquad \big(x \in (J_-, J_+]\big). \end{aligned}$$

Linear functionals

Proof of the theorem Assume $\lambda_-(J_+), \lambda_+(J_-) > 0$. For each $z \in \overline{J}$, there exist a unique pair of weight functions (w_-, w_+) such that

$$\frac{\partial}{\partial t}\mathbb{E}[F(\mathcal{X}_t)] = 1_{\{M_t^- \leq z\}} - f_-(z),$$

where (f_-, f_+) is the unique solution to Luckock's equation. Likewise, there exist a unique pair of weight functions (w_-, w_+) such that

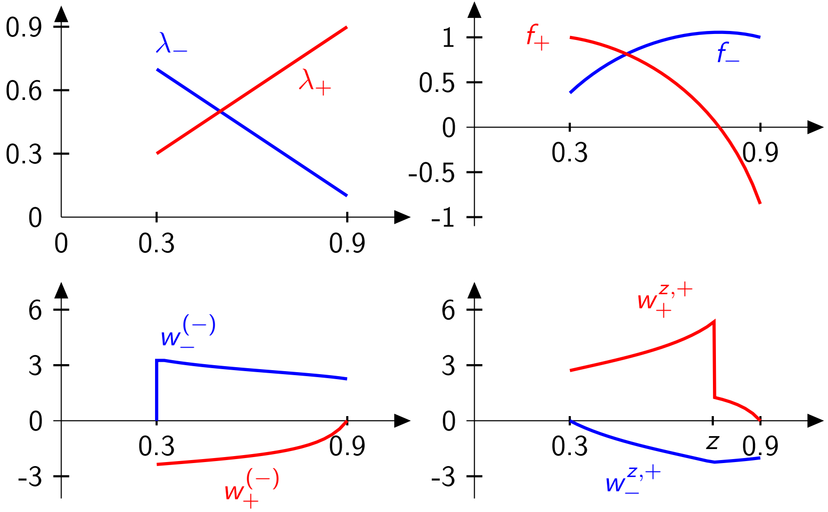
$$\frac{\partial}{\partial t}\mathbb{E}[F(\mathcal{X}_t)] = 1_{\{M_t^+ \geq z\}} - f_+(z).$$

In particular, there exist linear functionals $F^{(\pm)}$ such that

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{E}[F^{(-)}(\mathcal{X}_t)] &= 1_{\{M_t^- \leq J_-\}} - f_-(J_-),\\ \frac{\partial}{\partial t}\mathbb{E}[F^{(+)}(\mathcal{X}_t)] &= 1_{\{M_t^+ \geq J_+\}} - f_+(J_+).\end{aligned}$$

If $f_-(J_-), f_+(J_+) > 0$, then it is possible to construct a Lyapunov function from $F^{(-)}$ and $F^{(+)}$, proving positive recurrence. ∎

Restricted models



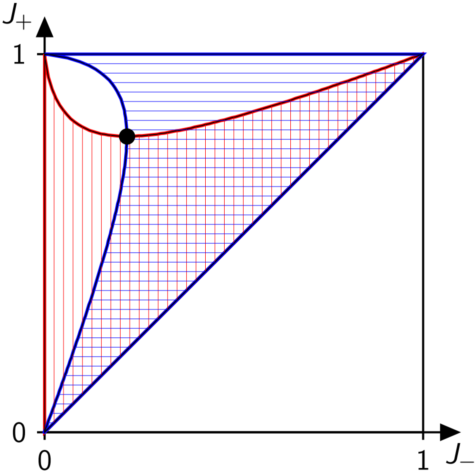
Define

$$\Lambda_-(J_-,J_+) := \frac{1}{\lambda_-(J_-)\lambda_-(J_+)} - \int_{J_-}^{J_+} \frac{1}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big),$$
$$\Lambda_+(J_-,J_+)+ := \frac{1}{\lambda_+(J_-)\lambda_+(J_+)} + \int_{J_-}^{J_+} \frac{1}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big).$$

Then, for the restricted model on J,

$$f_-(J_-) > 0 \quad \Leftrightarrow \quad \Lambda_-(J_-,J_+) > 0,$$
$$f_+(J_+) > 0 \quad \Leftrightarrow \quad \Lambda_+(J_-,J_+) > 0.$$

Restricted models



Restrictions of the uniform model: subintervals (J_-, J_+) for which $f_-(J_-) > 0$ resp. $f_+(J_+) > 0$.

The critical window

The dot in the previous picture indicates the *critical window*, i.e., the unique subinterval J such that the solution of Luckock's equation on J satisfies $f_-(I_-) = 0 = f_+(I_+)$.

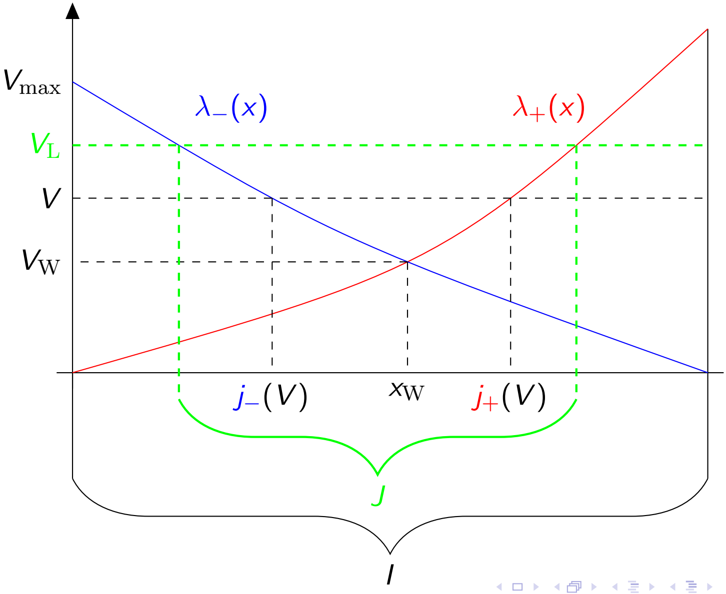
Recall that $V_{\rm W}$ is the Walrasian volume or trade. Let $V_{\max} := \lambda_-(I_-) \wedge \lambda_+(I_+)$. For $V \in [V_{\rm W}, V_{\max}]$, define

$$\begin{aligned} j_-(V) &= \sup\big\{x \in \overline{I} : \lambda_-(x) \geq V\big\},\\ j_+(V) &= \inf\big\{x \in \overline{I} : \lambda_+(x) \geq V\big\}, \end{aligned}$$

and define $\Psi : [V_{\rm W}, V_{\max}] \to [0, \infty]$ by

$$\Psi(V) := -\int_{V_{\rm W}}^{V} \Big\{\frac{1}{\lambda_+\big(j_-(U)\big)} + \frac{1}{\lambda_-\big(j_+(U)\big)}\Big\} \mathrm{d}\Big(\frac{1}{U}\Big).$$

The critical window



The critical window

Luckock's volume of trade is given by

$$V_{\rm L} := \sup\big\{V \in [V_{\rm W}, V_{\max}] : \Psi(V) \leq V_{\rm W}^{-2}\big\}.$$

Set $J = (J_-, J_+) := \big(j_-(V_{\rm L}), j_+(V_{\rm L})\big)$. If the Stigler-Luckock model has a critical window, then it is J. Conversely, if $\Psi(V_{\rm L}) = 0$ and $\overline{J} \subset I$, then J is a critical window.

For the uniform model, $V_{\rm L} = 1/z$ with z the unique solution of the equation $e^{-z} - z + 1 = 0$. Numerically, $V_{\rm L} \approx 0.78218829428020$.

Open problem:

- Show that a Stigler-Luckock model has an invariant law if and only if there exists a solution to Luckock's equation with $f_-(I_-), f_+(I_+) \geq 0$.

Results of Kelly and Yudovina

Frank Kelly and Elena Yudovina. *A Markov model of a limit order book: thresholds, recurrence, and trading strategies.* Preprint (2015 and 2016) ArXiv 1504.00579.

- **Claim** For any finite initial state, $\liminf_{t\to\infty} M_t^- = J_-$ and $\limsup_{t\to\infty} M_t^+ = J_+$ a.s.

A nice argument based on Kolmogorov's 0-1 law shows that the liminf and limsup are given by deterministic constants. The proof that they coincide with the boundary points of the critical window is rather more involved and needs additional technical assumtions. (In particular, $c\mathrm{d}x \leq \mu_\pm \leq C\mathrm{d}x$ for some $0 < c < C < \infty$.)

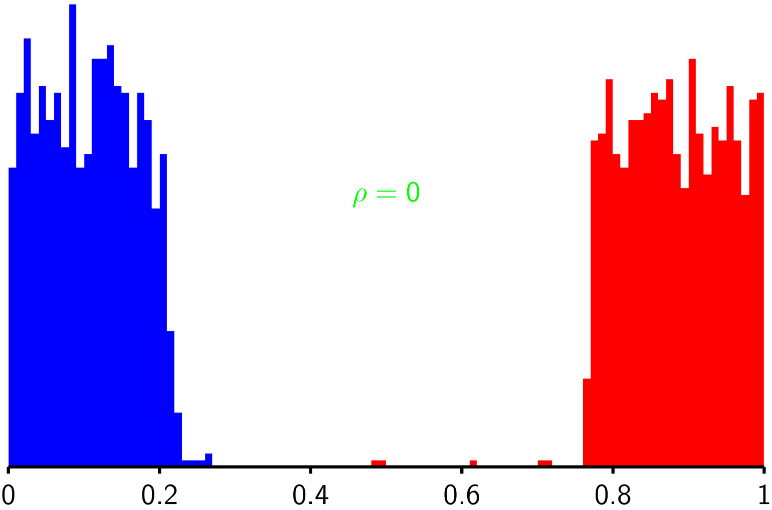
Proofs use "fluid limits".

Market makers

The Stigler-Luckock model is unrealistic because of its huge spread. In reality, this attracts *market makers* who make money from buying for a low price and selling for a higher price.

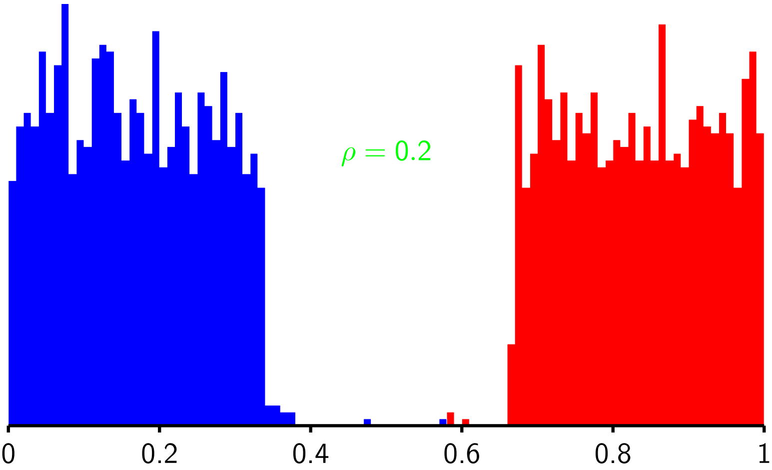
[Peržina & S., in progress] We extend the model as follows: Apart from the buyers and sellers as before, with rate $\rho \geq 0$, a market maker arrives who places both a buy and a sell order, at the prices of the current bid and ask prices, respectively.
(If there are currently no buy (sell) limit orders in the order book, then the market maker does not place a buy (sell) limit order.)

Numerical simulation



The order book after the arrival of 10,000 traders.

Numerical simulation



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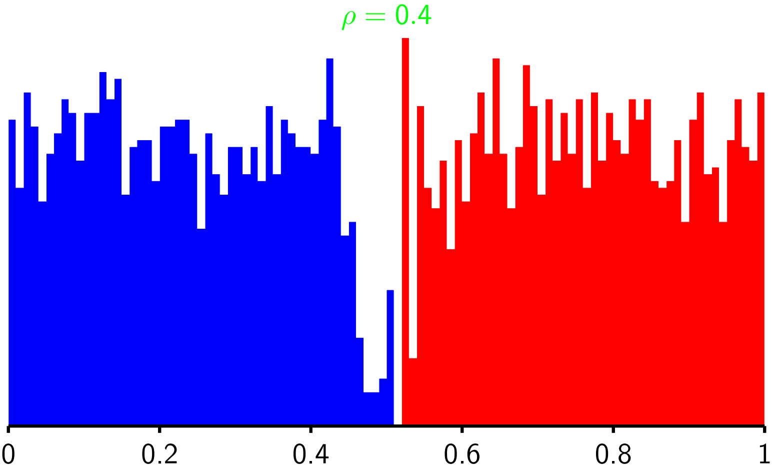

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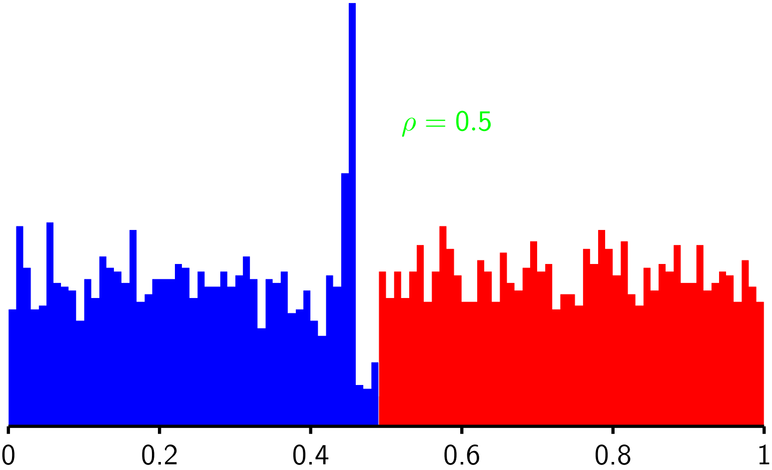

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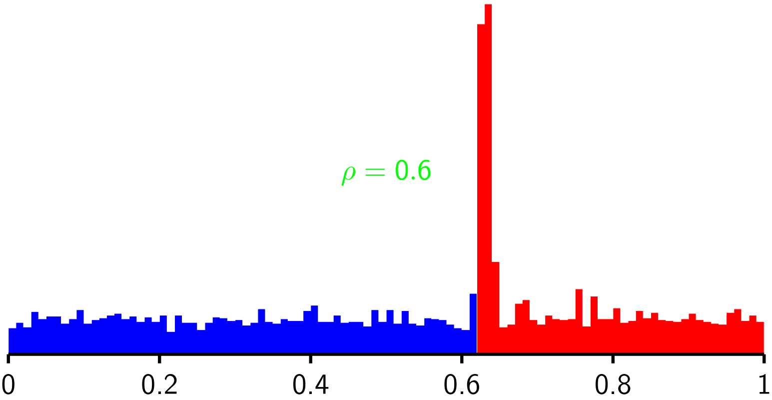

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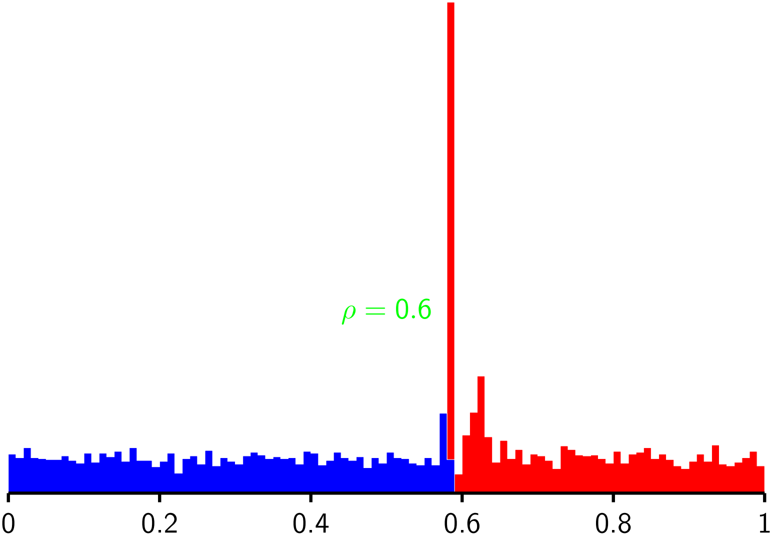

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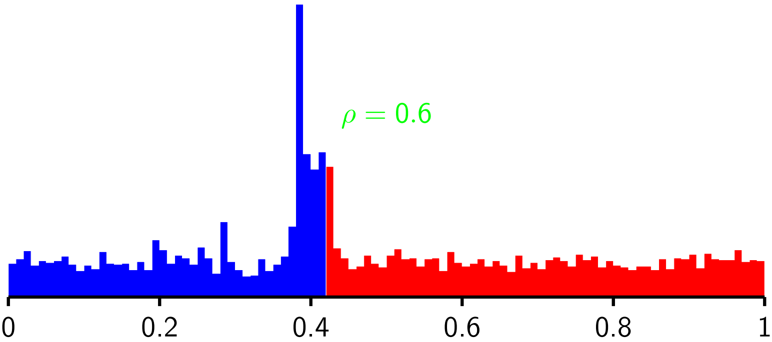

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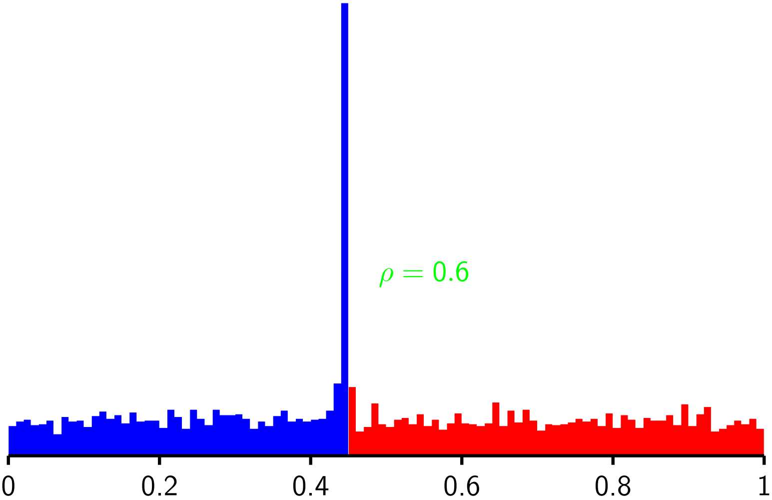

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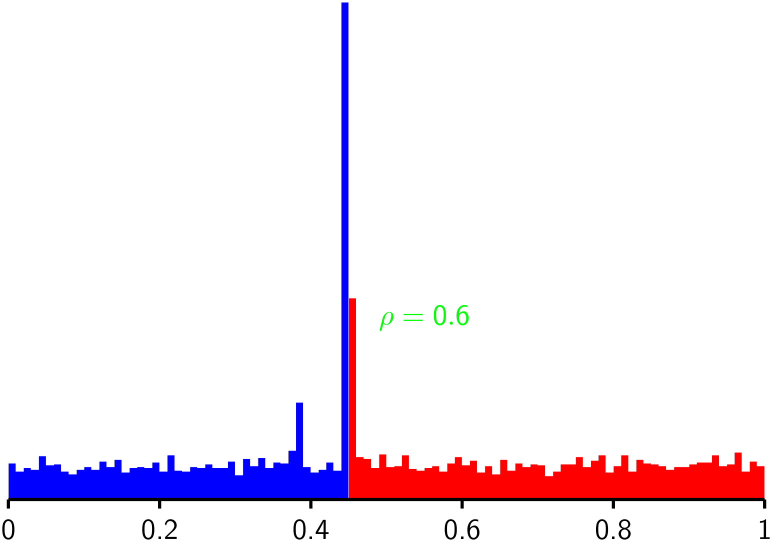

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Market makers

As long as $\rho < x_{\mathrm{W}}$ (with ρ the rate of market makers and x_{W} the Walrasian volume of trade), the model still has a critical window of positive length.

As soon as $\rho = x_{\mathrm{W}}$, the critical window closes.

When $\rho > x_{\mathrm{W}}$, we (numerically) observe "freezing", i.e., the price settles at a random level that is determined by the history of the process.

In reality, the market is only attractive to market makers as long as the spread is nonzero. A realistic model should tune itself to the critical value $\rho = x_{\mathrm{W}}$.

Another example of self-organised criticality?

Future work

Open mathematical problems for the original Stigler-Luckock model:

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- Show convergence to this invariant law started from any finite initial state.
- Study the equilibrium distribution of the time before a limit order is matched: same power law as for email model?
- Study the shape of the equilibrium process near the boundary: same scaling limit as for email model?

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- allowing orders to be cancelled
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- better strategies for buyers, sellers, and market makers
- making the supply and demand functions depend on time

Future work

A little warning: less is often more.

Before you write down the most general possible model, keep in mind that for understanding the basic principles that are at work, it is often much more useful to have a minimal working example than a model with lots of parameters.

Today we have learned that even if buyers and sellers are in one place, there is still room for market makers who make money by transporting goods not in space but in time, buying when the price is low and selling when the price is high.

And in fact, you need these people
to attain Walras' equilibrium price.

The Bak Sneppen model

Introduced by Bak & Sneppen (1993).

Consider an ecosystem with N species. Each species has a fitness in $[0, 1]$.

In each step, the species $i \in \{1, \ldots, N\}$ with the lowest fitness dies out, together with its neighbours $i - 1$ and $i + 1$ (with periodic b.c.), and all three are replaced by species with new, i.i.d. uniformly distributed fitnesses.

There is a critical fitness $f_{\rm c} \approx 0.6672(2)$ such that when N is large, after sufficiently many steps, the fitnesses are approximately uniformly distributed on $(f_{\rm c}, 1]$ with only a few smaller fitnesses. Moreover, for each $\varepsilon > 0$, the lowest fitness spends a positive fraction of time above $f_{\rm c} - \varepsilon$, uniformly as $N \to \infty$.

The modified Bak Sneppen model

Introduced by Meester & Sarkar (2012).

Instead of the *neighbours* of the least fit species, choose one *arbitrary other* species from the population that dies together with the least fit species.

Critical point exactly $f_{\mathrm{c}} = 1/2$.

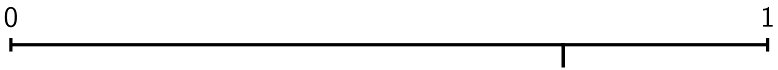
Critical behaviour at f_{c}: intervals between times when all individuals have a fitness $> f_{\mathrm{c}}$ have a power-law distribution with $\mathbb{P}[\tau \geq k] \sim k^{-1/2}$.

Proof based on coupling to a branching process.

A model for canyon formation

We start with a flat rock profile.

A model for canyon formation



The river cuts into the rock at a uniformly chosen point.

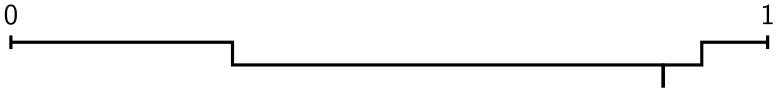
A model for canyon formation

Rock between a next point and the river is eroded one step down.

A model for canyon formation

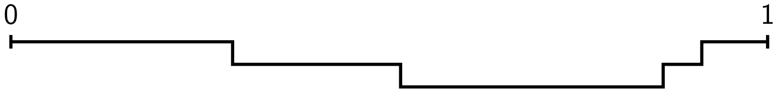
We continue in this way.

A model for canyon formation



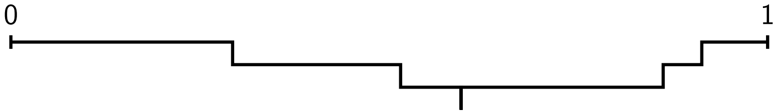
Either the river cuts deeper in the rock.

A model for canyon formation



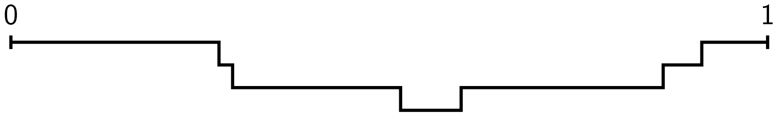
Or one side of the river is eroded down.

A model for canyon formation



We are interested in the limit profile.

A model for canyon formation



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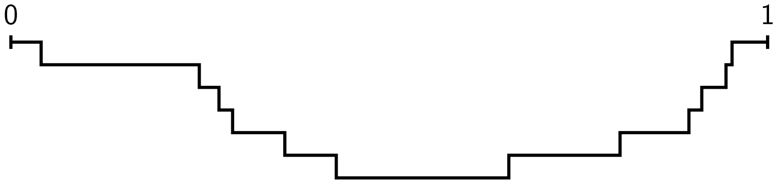

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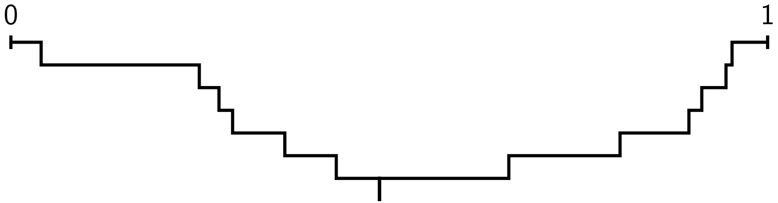

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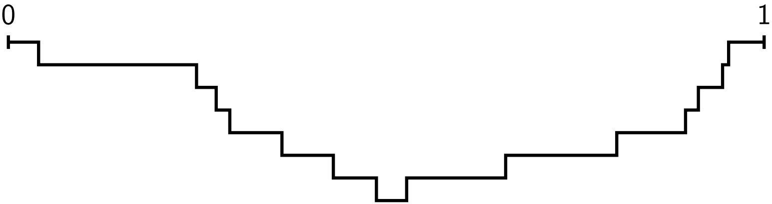

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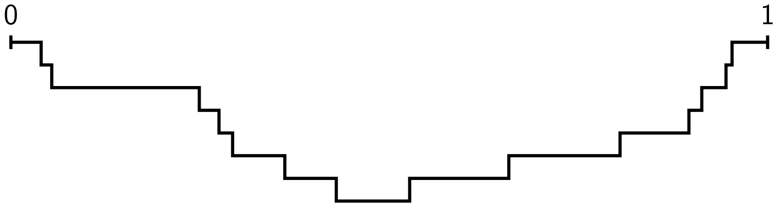

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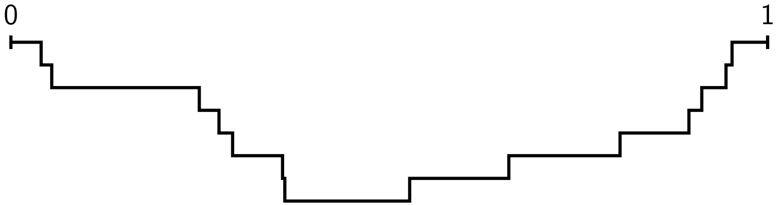

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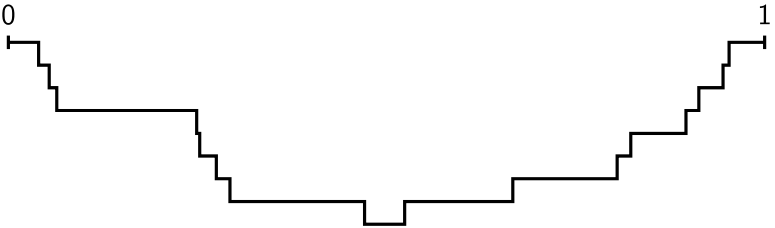

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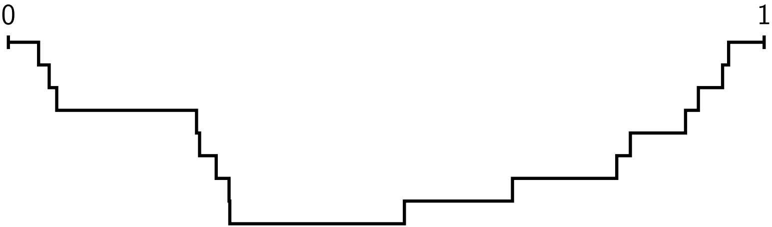

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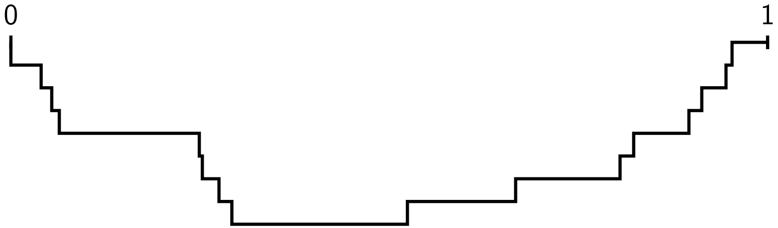

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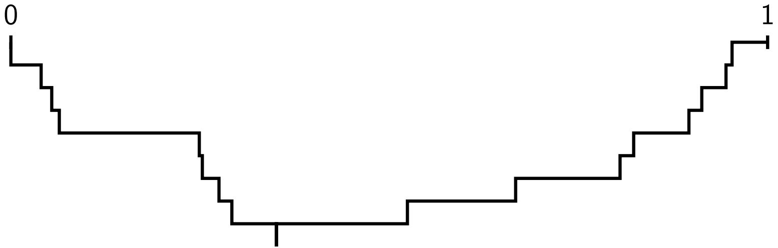

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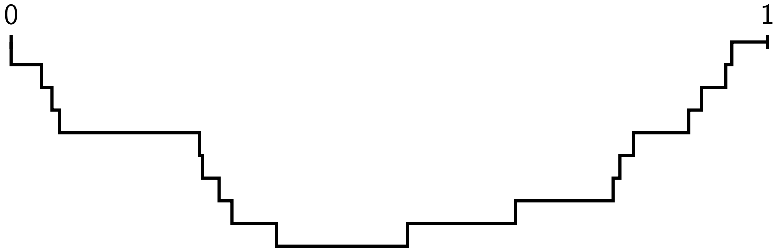

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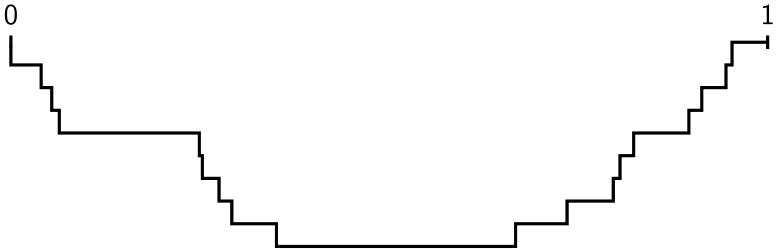

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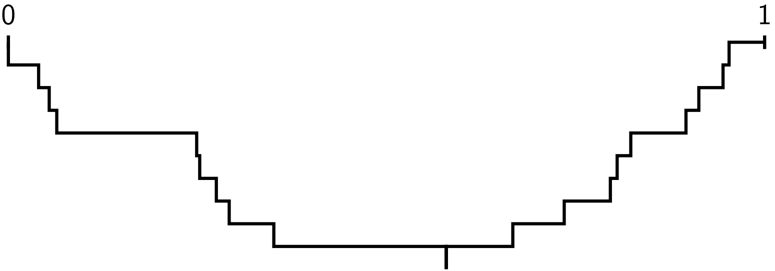

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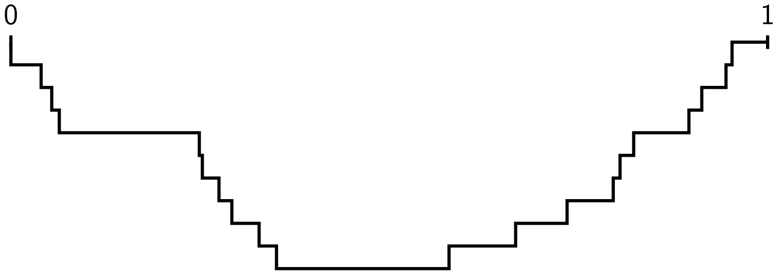

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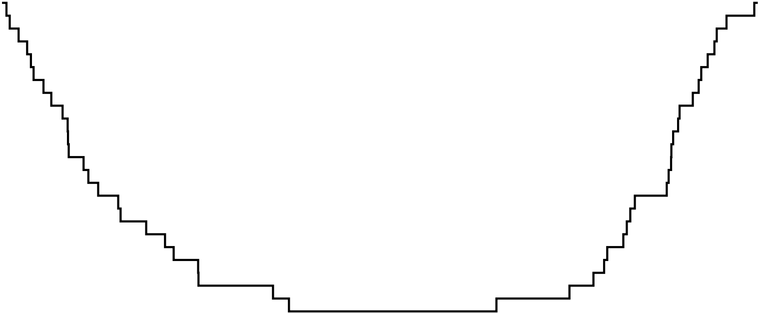
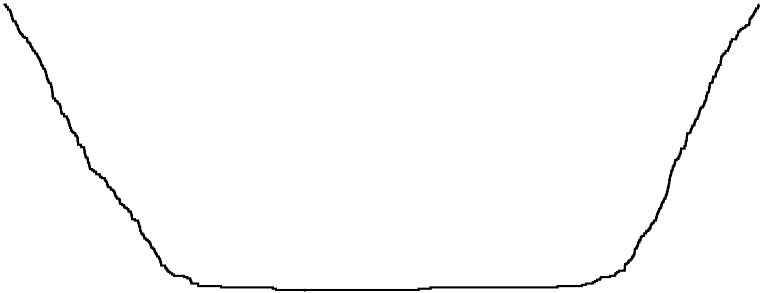
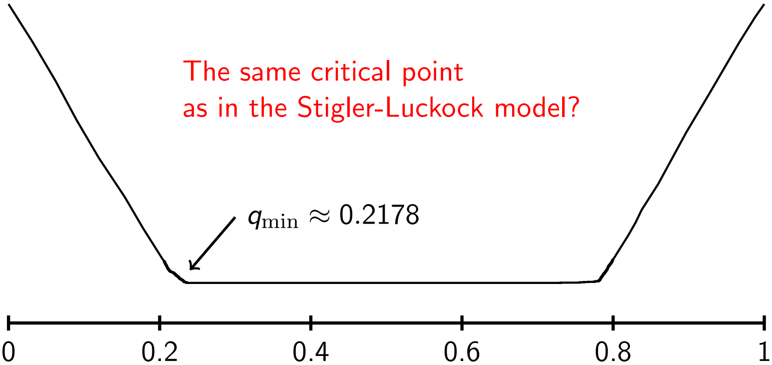


The profile after 100 steps.



The profile after 1000 steps.

A model for canyon formation

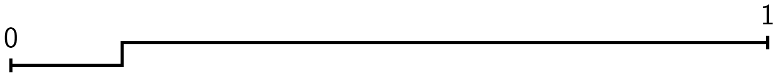


The profile after 10,000 steps.

A one-sided canyon model

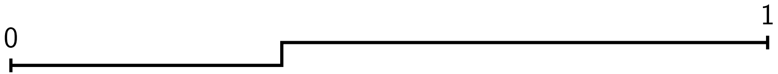
A river flows on the left.

A one-sided canyon model



The river either cuts deeped into the rock.

A one-sided canyon model



Or the shore is eroded down, starting from a random point.

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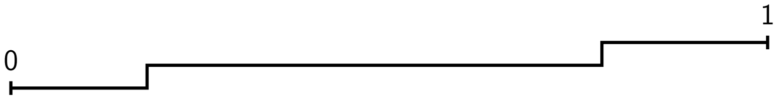
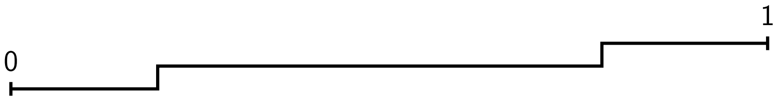


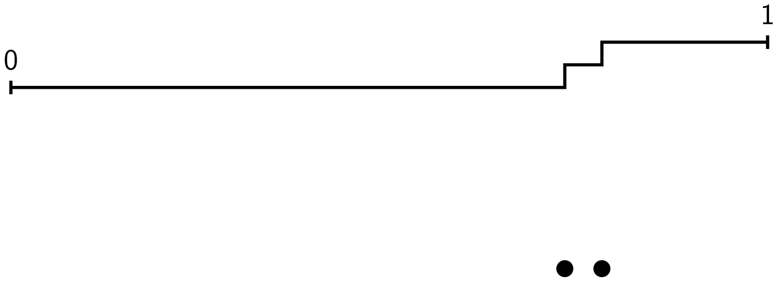
We either make the river deeper. . .

A one-sided canyon model



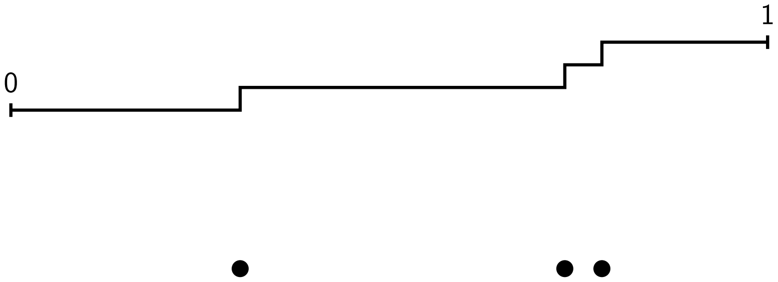
. . . or we erode the shore,

A one-sided canyon model



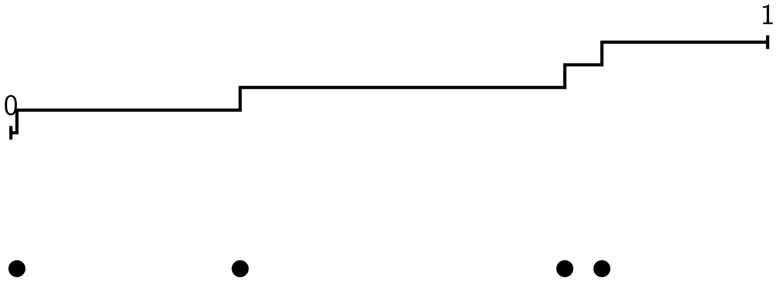
...depending on where the new point falls.

A one-sided canyon model



Points on the left of all others are simply added.

A one-sided canyon model



Points on the left of all others are simply added.

A one-sided canyon model

Otherwise, we remove the left-most point.

A one-sided canyon model

In other words, we always add the new point.

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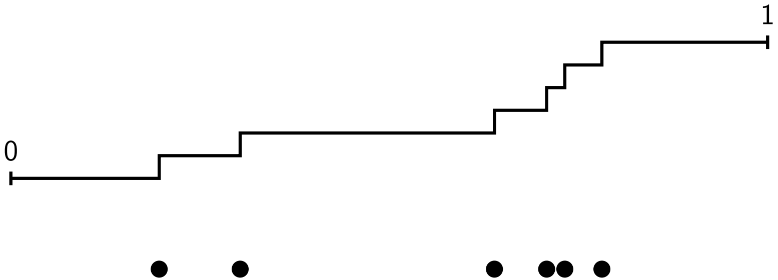
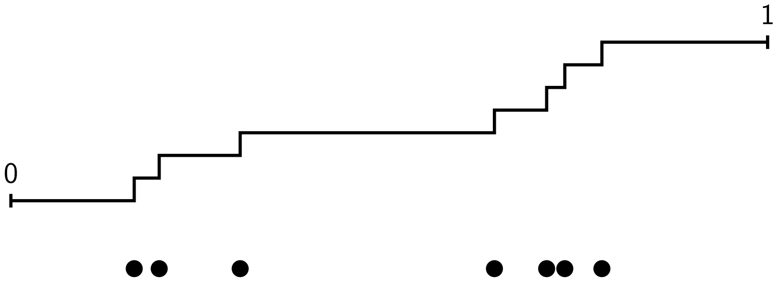


If the new point is not the left-most, then we remove the left-most.

A one-sided canyon model



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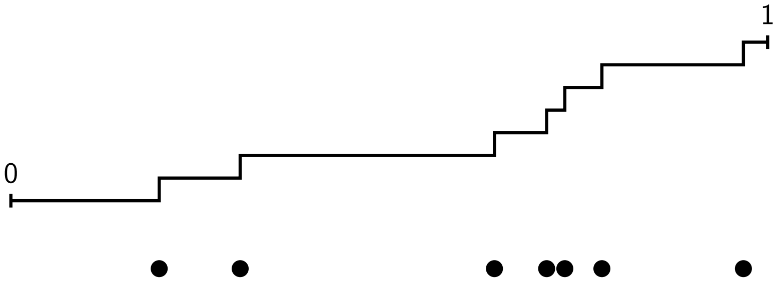

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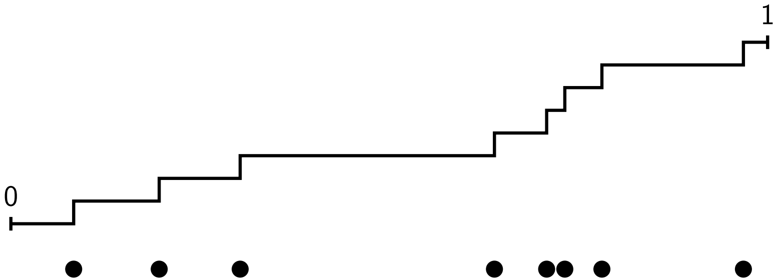

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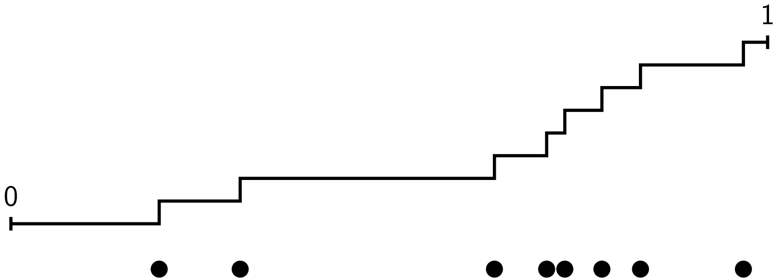

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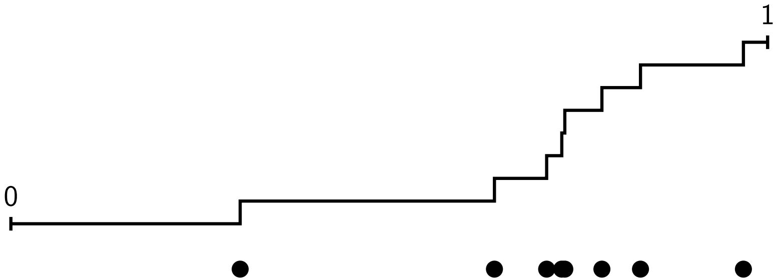

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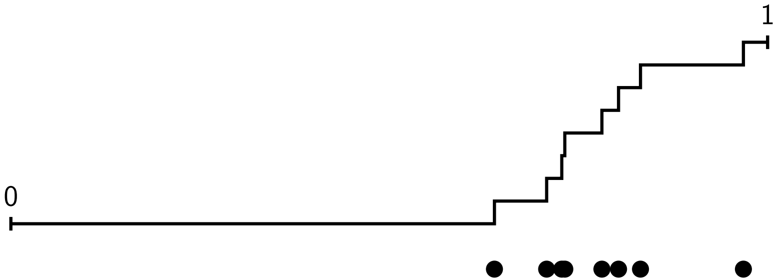

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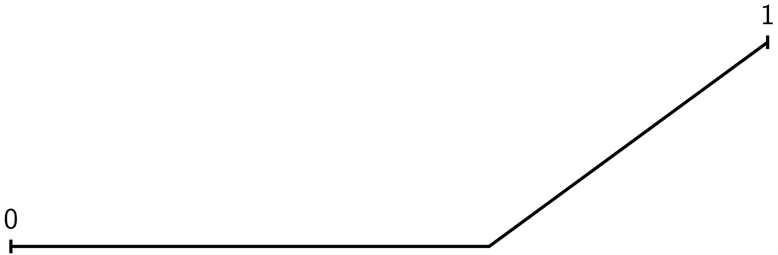
If the new point is not the left-most, then we remove the left-most.

A one-sided canyon model



In this model, the critical point is $p_{\rm c} = 1 - e^{-1} \approx 0.63212$.

The process just described defines a Markov chain $(X_k)_{k\geq 0}$ where $X_k \subset [0,1]$ is a finite set.

Consistency: For each $0 < q < 1$, we observe that the *restricted process*

$$\big(X_k \cap [0,q]\big)_{k\geq 0}$$

is a Markov chain.

Theorem 1 The restricted process is positively recurrent for $q < 1 - e^{-1}$ and transient for $q > 1 - e^{-1}$.

Theorem 2 The restricted process is null recurrent at $q = 1 - e^{-1}$.

The critical point

Proof of Theorems 1 and 2 Since only the relative order of the points matters, transforming space we may assume that the $(U_k)_{k\geq 1}$ are i.i.d. *exponentially* distributed with mean one and $X_k \subset [0, \infty]$.

For the modified model, we must prove that $p_{\rm c} = 1$.

Start with $X_0 = \emptyset$ and define

$$F_t(k) := \big|X_k \cap [0, t]\big| \qquad (k \geq 0,\ \ t \geq 0).$$

Interesting observation $(F_t)_{t\geq 0}$ is a continuous-time Markov process taking values in the functions $F : \mathbb{N} \to \mathbb{N}$.

A weight function

Proof of Theorems 1 and 2

For $t > 0$, consider the weighted sum over points in X_k

$$W_k^{(t)} := \sum_{x \in X_k} e^x 1_{[0,t]}(x).$$

Then

$$\mathbb{E}\big[W_{k+1}^{(t)} - W_k^{(t)} \,\big|\, \min(X_k) = m\big] = t - 1_{[0,t]}(m).$$

In particular, the process $W^{(t)}$ stopped at the first time that $\min(X_k) > t$ is

- A supermartingale for $t < 1$,
- A martingale for $t = 1$,
- A submartingale for $t > 1$.

Rank-based models

The email model, Stigler-Luckock model, (modified) Bak Sneppen model, and (one- or two-sided) canyon models share some common features:

- Only the relative order of the priorities/fitnesses matter. As a result, replacing the uniform distribution with any other atomless law basically yields the same model (up to a transformation of space).
- All models use some version of the rule "kill the minimal element".
- All models exhibit *self-organised criticality*.

Somewhat similar are also branching Brownian motions with killing of the lowest particle to keep the population size constant, as studied by Berestycki, Berestycki, and Schweinsberg (2010-13) and Maillard (2013).