Peierls bounds from Toom contours

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Toom's model

Assume that $\varphi : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}$ is monotone and depends on finitely many coordinates. Example: the North East Center majority rule on $\mathbb{Z}^2$:

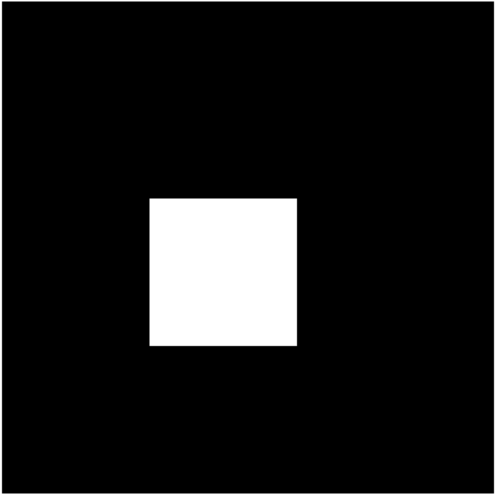
$$\varphi^{\mathrm{NEC}}(x) := \begin{cases} 1 & \text{if } x(0,0) + x(0,1) + x(1,0) \geq 2, \\ 0 & \text{if } x(0,0) + x(0,1) + x(1,0) \leq 1. \end{cases}$$

We are interested in the cellular automaton $(X_n)_{n \geq 0}$ that evolves

$$X_{n+1}(i) = \begin{cases} \varphi\big((X_n(i+j))_{j \in \mathbb{Z}^d}\big) & \text{with probability } 1-p, \\ 0 & \text{with probability } p, \end{cases}$$

independently for all $n \geq 0$ and $i \in \mathbb{Z}^d$.

Toom's model



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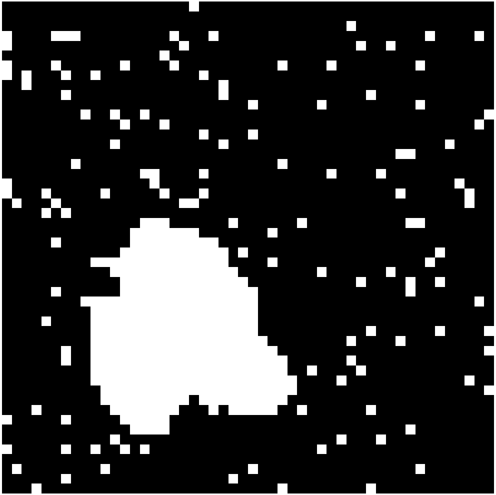

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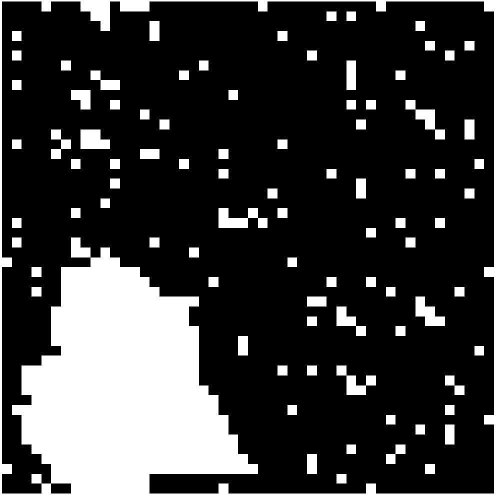

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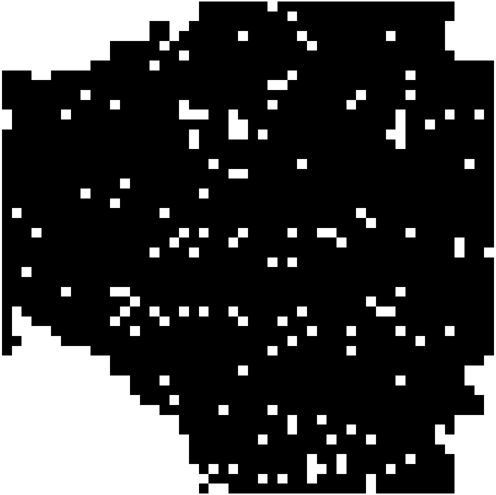

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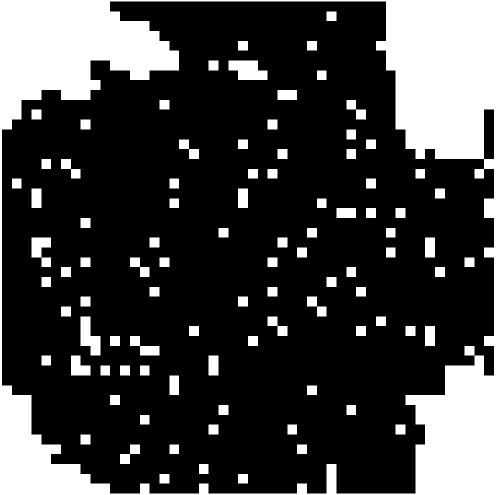

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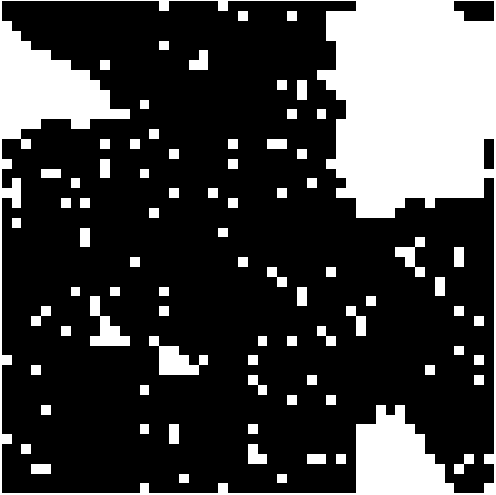

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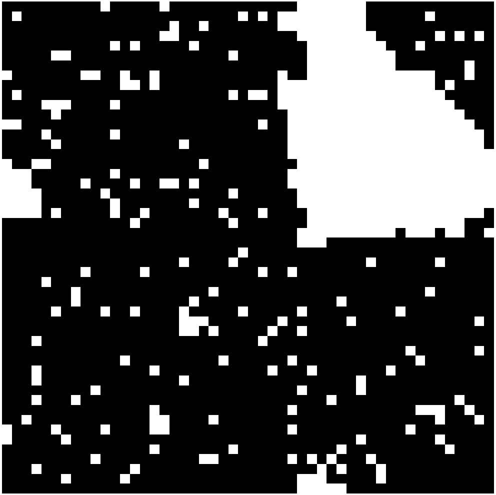

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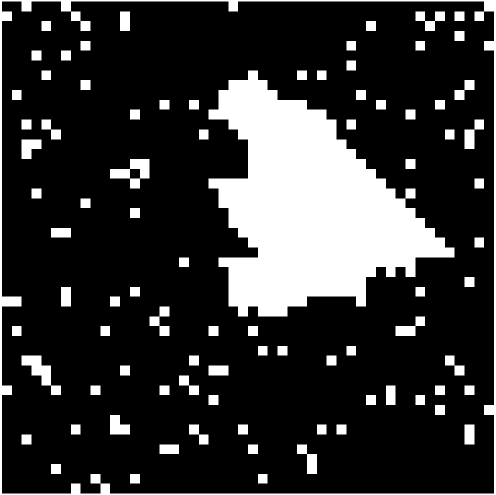

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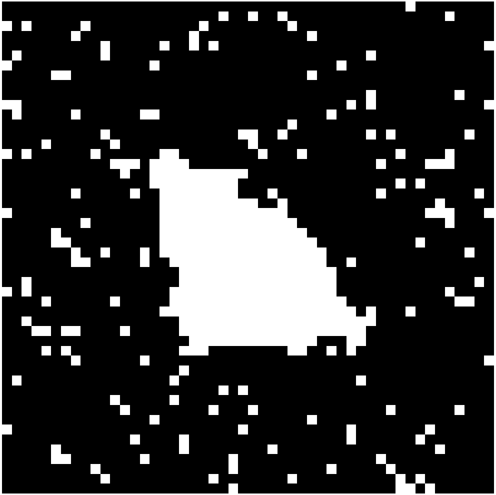

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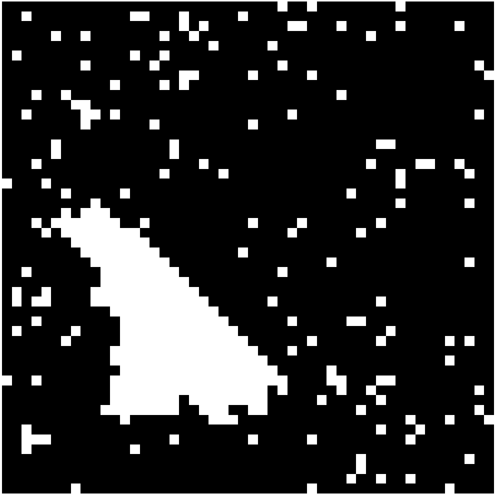

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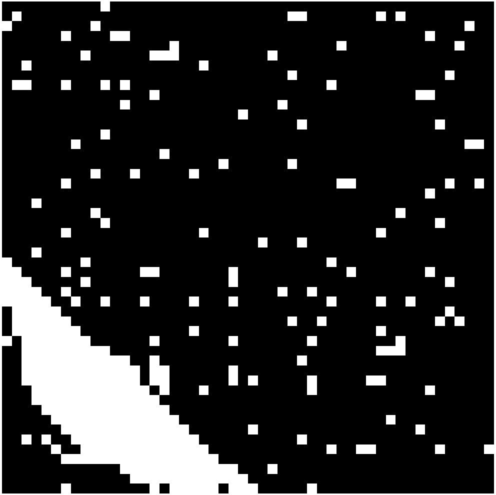

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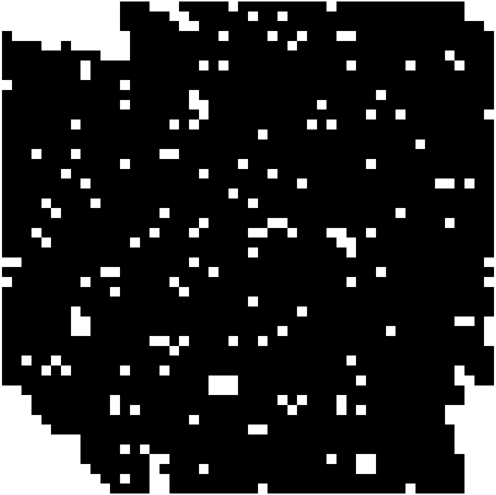

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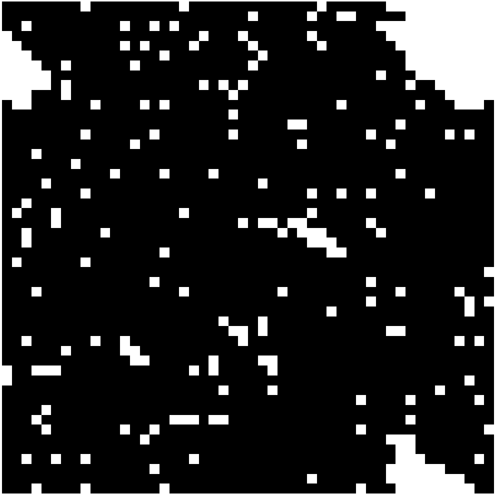

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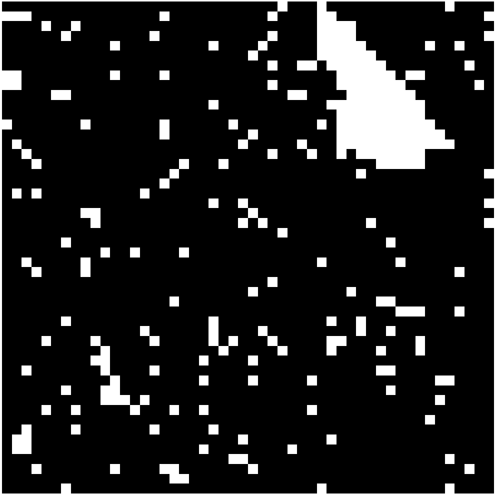

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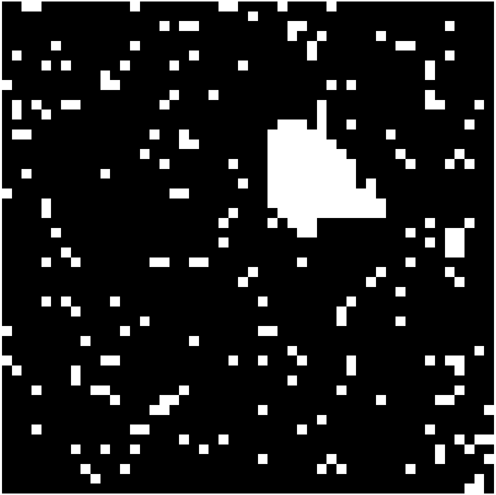

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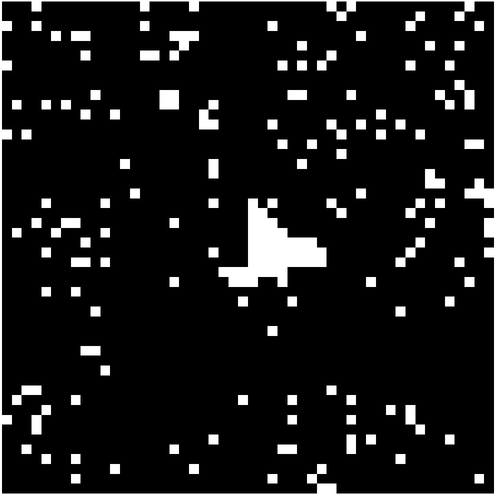

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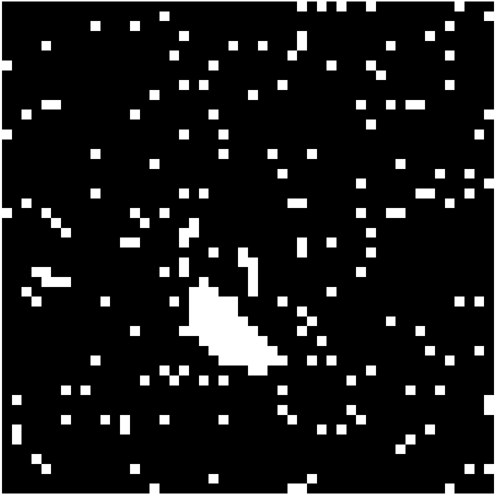

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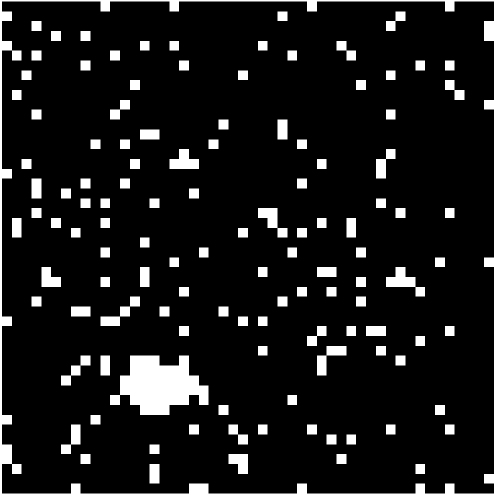

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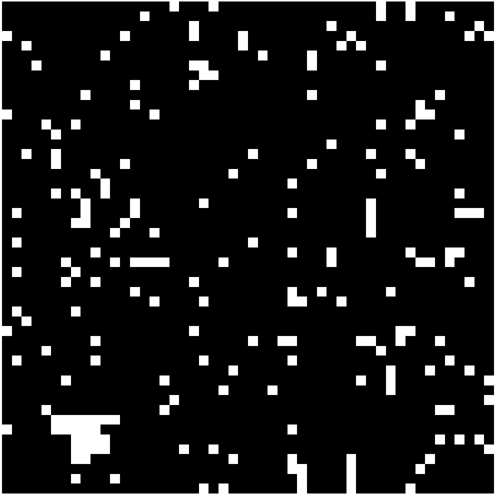

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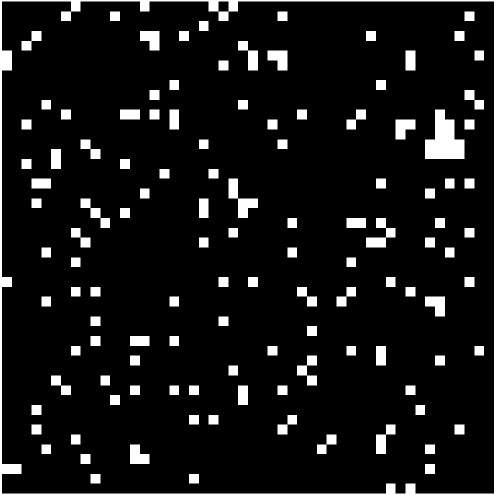

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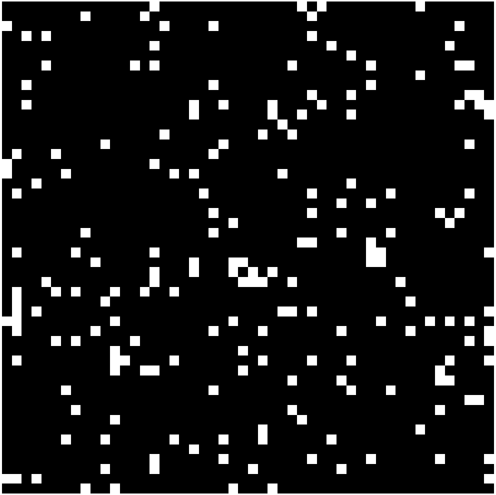


Toom's model

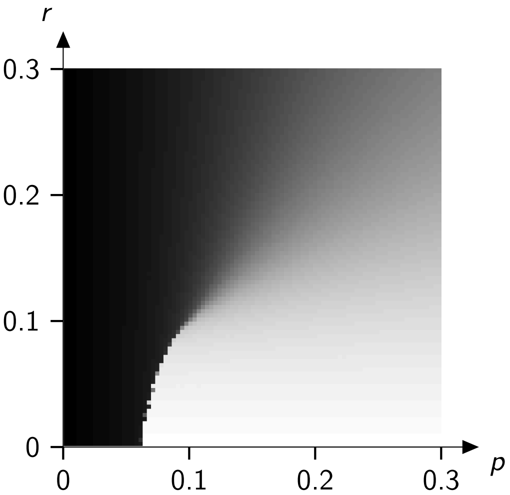
We can generalise a bit and let

$$X_{n+1}(i) = \begin{cases} \varphi\big((X_n(i+j))_{j\in\mathbb{Z}^d}\big) & \text{with probability } 1-p-r, \\ 0 & \text{with probability } p, \\ 1 & \text{with probability } r. \end{cases}$$

Let $\rho(p,r)$ denote the density of the upper invariant law.

Toom (1980) $\displaystyle\lim_{p\to 0} \rho(p,0) = 1.$

Toom's model



Density of the upper invariant law for Toom's model.

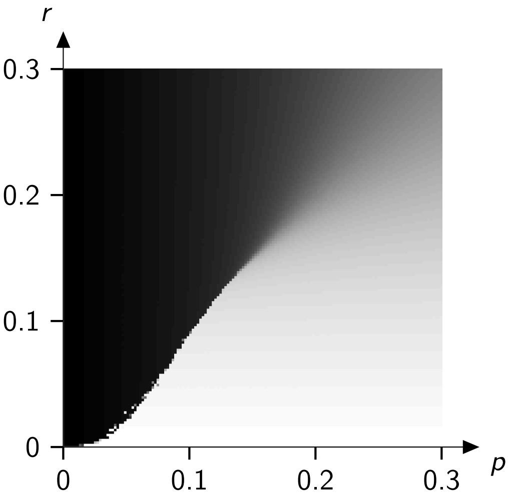
Nearest neighbour voting

The *Nearest Neighbor voting map* is defined as

$$\varphi^{\mathrm{NN}}(x) := \begin{cases} 1 & \text{if} \quad x(0,0)+x(0,1)+x(1,0) \\ & \qquad +x(0,-1)+x(-1,0) \geq 3, \\ 0 & \text{if} \quad x(0,0)+x(0,1)+x(1,0) \\ & \qquad +x(0,-1)+x(-1,0) \leq 2. \end{cases}$$

Toom (1980) $\rho(p,0)=0$ for all $p>0$.

Nearest neighbour voting



Density of the upper invariant law for nearest neighbour voting.

Toom's stability theorem

Def φ is an *eroder* if for the unperturbed cellular automaton, any finite collection of zeros disappears in finite time.

Toom's stability theorem (1980) If φ is an eroder, then $\rho(p) \to 1$ as $p \to 0$. If φ is not an eroder, then $\rho(p) = 0$ for all $p > 0$.

The eroder property

Each monotone map $\varphi : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}$ can uniquely be written as

$$\varphi(x) = \bigvee_{A \in \mathcal{A}(\varphi)} \bigwedge_{i \in A} x(i),$$

$A \in \mathcal{A}(\varphi)$ is a *minimal collection of ones* needed for $\varphi(x) = 1$.

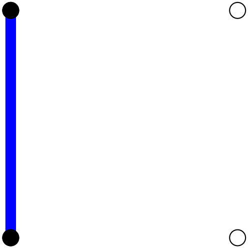
Theorem (Toom 1980, Ponselet 2013) φ is an eroder if and only if

$$\bigcap_{A \in \mathcal{A}(\varphi)} \mathrm{Conv}(A) = \emptyset,$$

where $\mathrm{Conv}(A)$ is the convex hull of A.

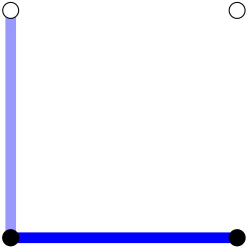
By *Helly's theorem* w.l.o.g. $|\mathcal{A}(\varphi)| \leq d + 1$.

The eroder property



Toom's model φ^{NEC}

The eroder property



Toom's model φ^{NEC}

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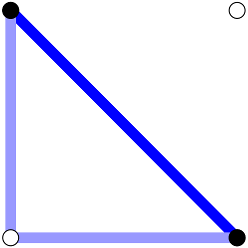

Toom's model φ^{NEC}

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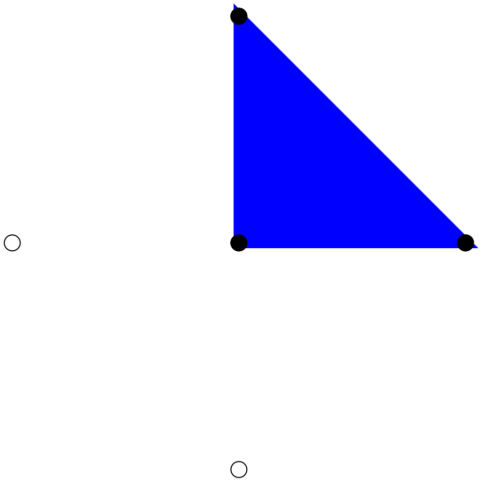
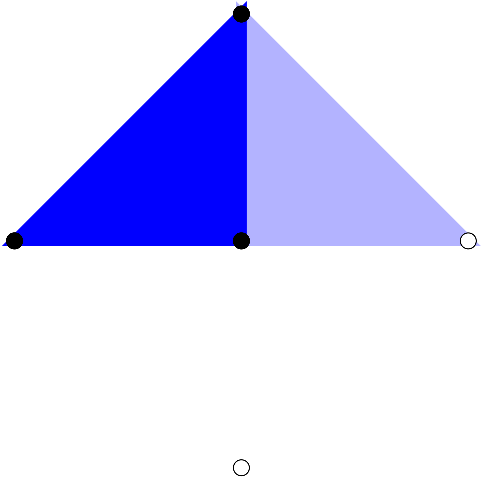


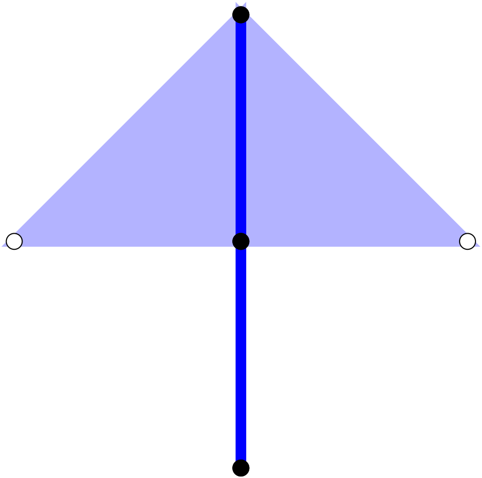
Nearest neighbour voting φ^{NN}.

The eroder property



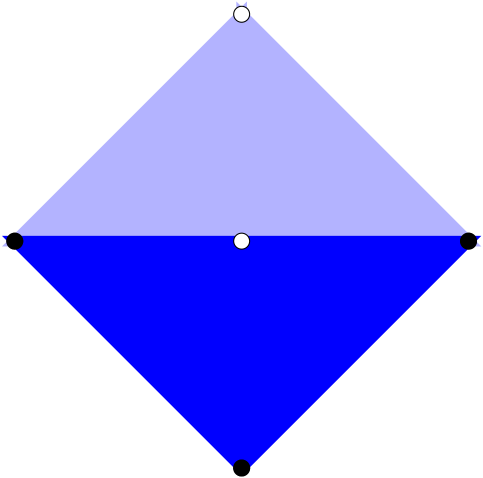
Nearest neighbour voting φ^{NN}.

The eroder property



Nearest neighbour voting φ^{NN}.

The eroder property



Nearest neighbour voting φ^{NN}.

Def A *linear polar function* is a linear function

$$\mathbb{R}^d \ni z \mapsto \big(L_1(z), \ldots, L_\sigma(z)\big) \in \mathbb{R}^\sigma$$

such that $\displaystyle\sum_{s=1}^{\sigma} L_s(z) = 0 \ (z \in \mathbb{R}^d)$.

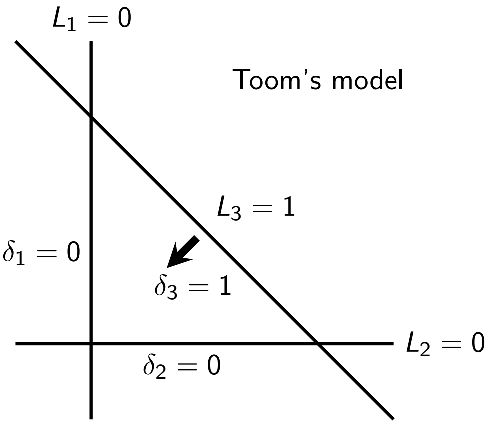
For $x \in \{0,1\}^{\mathbb{Z}^d}$, let $\displaystyle\ell_s(x) := \sup_{i \in \mathbb{Z}^d :\, x(i)=0} L_s(i)$.

Then for the unperturbed cellular automaton:

$$\ell_s(X_n) \leq \ell_s(X_0) - \delta_s n \quad \text{with} \quad \delta_s := \sup_{A \in \mathcal{A}(\varphi)} \inf_{i \in A} L_s(i).$$

The constants δ_s $(1 \leq s \leq \sigma)$ are *edge speeds*.

The eroder property



$$L_1(z) := -z_1, \quad L_2(z) := -z_2, \quad L_3(z) := z_1 + z_2.$$

The eroder property

Lemma (Toom 1980, Ponselet 2013) φ is an eroder if and only if there exists a linear polar function L such that

$$\delta := \sum_{s=1}^{\sigma} \delta_s > 0 \quad \text{with} \quad \delta_s := \sup_{A \in \mathcal{A}(\varphi)} \inf_{i \in A} L_s(i).$$

Proof of sufficiency Define the *extent* of x by

$$\mathrm{ext}(x) := \sum_{s=1}^{\sigma} \ell_s(x) \quad \text{with} \quad \ell_s(x) := \sup_{i \in \mathbb{Z}^d :\, x(i)=0} L_s(i).$$

Then $\mathrm{ext}(x) \geq 0$ if there is at least one zero since $\displaystyle\sum_{s=1}^{\sigma} L_s(z) = 0$.

Moreover $\mathrm{ext}(X_n) \leq \mathrm{ext}(X_0) - \delta n$. ∎

The maximal trajectory

Let $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t)\in\mathbb{Z}^d\times\mathbb{Z}}$ be an i.i.d. collection of maps with

$$\mathbb{P}\big[\Phi_{(i,t)} = \varphi^0\big] = p \quad \text{and} \quad \mathbb{P}\big[\Phi_{(i,t)} = \varphi\big] = 1 - p,$$

where $\varphi^0(x) := 0$ denotes the *trivial zero map*. A *trajectory* of Φ is a function $(i,t) \mapsto x_t(i)$ such that

$$x_t(i) = \Phi_{(i,t)}\big((x_{t-1}(i+j))_{j\in\mathbb{Z}^d}\big) \qquad \forall (i,t).$$

Lemma There a.s. exists a *maximal* trajectory $\overline{X}$.

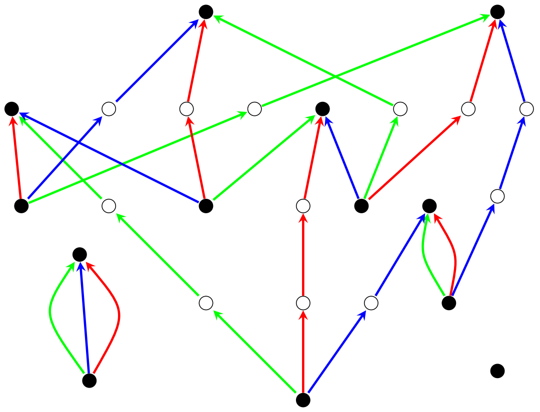
Aim For small p, derive a lower bound on $\rho(p) := \mathbb{P}[\overline{X}_0(0) = 1]$.

Toom contours

Def A *Toom graph* is a directed graph with edges of σ different *charges* and three types of vertices:

- At a *source*, σ directed edges emerge, one of each charge.
- At a *sink*, σ directed edges converge, one of each charge.
- At an *internal vertex*, there is one incoming edge and one outgoing edge, and they are of the same charge.

In addition, there can be *isolated vertices* which we can think of as a source and sink at the same time.



A Toom graph with three charges.

Peierls argument

Main idea A *Toom contour* is a connected Toom graph embedded in the plain, with one special source called the *root*.

Theorem (incomplete statement) If $\overline{X}_0(0) = 0$, then there exists a Toom contour T rooted at $(0,0)$ such that the sinks of T correspond to *defective* space-time points, where the trivial map φ^0 is applied. Consequently:

$$\mathbb{P}\big[\overline{X}_0(0) = 0\big] \leq \sum_T \mathbb{P}\big[T \text{ is present in } \Phi\big] \leq \sum_T p^{n_{\rm sink}(T)}.$$

This tends to zero as $p \to 0$ provided

$$N_n^{\rm sink} := \#\{T : n_{\rm sink}(T) = n\}$$

grows at most exponentially in n.

It is not hard to show that there exists a $R < \infty$ such that

$$N_n^{\mathrm{edge}} \leq R^n \quad \text{with} \quad N_n^{\mathrm{edge}} := \#\{T : n_{\mathrm{edge}}(T) = n\}.$$

Need to show that $n_{\mathrm{sink}}(T) \geq c n_{\mathrm{edge}}(T)$ for some $c > 0$.

Idea: edges with charge s move in the direction where L_s increases, *except* for edges coming out of sources. As a result:

$$n_{\mathrm{sink}}(T) = n_{\mathrm{source}}(T) \geq c n_{\mathrm{edge}}(T)$$

for some $c > 0$.

Toom contours

Def An *embedding* of a Toom graph with vertex set V is a map

$$V \ni v \mapsto \big(\psi(v), -h(v)\big) \in \mathbb{Z}^d \times \mathbb{Z}$$

- The *height* (=negative time) h increases by 1 along each directed edge.
- Sinks do not overlap with any other vertices.
- Internal vertices of the same charge do not overlap.

A *Toom contour* is an embedded connected Toom graph with one special source, the *root*, whose height is minimal among all vertices.

Toom contours

Let φ be an eroder. For each $1 \leq s \leq \sigma$, choose $A_s(\varphi) \in \mathcal{A}(\varphi)$ such that

$$\delta_s := \sup_{A \in \mathcal{A}(\varphi)} \inf_{i \in A} L_s(i) = \inf_{i \in A_s(\varphi)} L_s(i).$$

Def A Toom contour is *present* in $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t) \in \mathbb{Z}^d \times \mathbb{Z}}$ if:

- Sinks correspond to vertices where the trivial map φ^0 is applied.
- If (v, w) is a directed edge of charge s coming out of an internal vertex or the root, then $\psi(w) - \psi(v) \in A_s(\varphi)$.
- For directed edges emerging at other sources $\psi(w) - \psi(v) \in \bigcup_{s=1}^{\sigma} A_s(\varphi)$.

Theorem (complete statement) If $\overline{X}_0(0) = 0$, then there is a Toom contour rooted at $(0, 0)$ present in Φ.

Cooperative branching

The cooperative branching map is defined as

$$\varphi^{\text{coop}}(x) := \big(x(0,1) \wedge x(1,0)\big) \vee x(0,0).$$

One has $\mathcal{A}(\varphi^{\text{coop}}) = \{A_1, A_2\}$ with

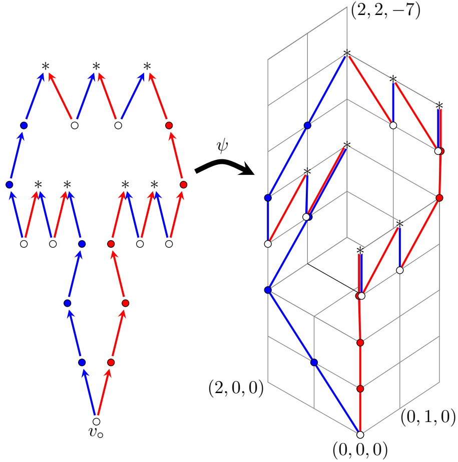
$$A_1 := \{(0,1),(1,0)\} \quad \text{and} \quad A_2 := \{(0,0)\}.$$

We choose the linear polar function

$$L_1(z) := z_1 + z_2, \quad L_2(z) := -z_1 - z_2.$$

The corresponding edge speeds are given by

$$\begin{aligned}
\delta_1 &= \sup_{A \in \mathcal{A}(\varphi)} \inf_{i \in A} L_1(i) = \inf_{i \in A_1} L_1(i) = 1, \\
\delta_2 &= \sup_{A \in \mathcal{A}(\varphi)} \inf_{i \in A} L_2(i) = \inf_{i \in A_2} L_2(i) = 0.
\end{aligned}$$



A Toom contour for the cooperative branching map.

Lemma There exists a $c > 0$ such that $n_{\mathrm{sink}} \geq c n_{\mathrm{edge}} + 1$.

Proof
$$\sum_{s=1}^{\sigma} \sum_{(v,w)\in E_s} \big(L_s(\psi(w)) - L_s(\psi(v))\big)$$
$$= \sum_{v\in V} \sum_{s=1}^{\sigma} \Big\{ \sum_{u:\,(u,v)\in E_s} L_s(\psi(v)) - \sum_{w:\,(v,w)\in E_s} L_s(\psi(v)) \Big\} = 0.$$

Let E_s° denote the edges of charge s out of a source different from the root and E_s^{*} the other edges. Then

$$0 = \sum_{s=1}^{\sigma} \sum_{(v,w)\in E_s^{*}} \underbrace{\big(L_s(\psi(w)) - L_s(\psi(v))\big)}_{\geq \delta_s}$$
$$+ \sum_{s=1}^{\sigma} \sum_{(v,w)\in E_s^{\circ}} \underbrace{\big(L_s(\psi(w)) - L_s(\psi(v))\big)}_{\geq -K}. \qquad \blacksquare$$

Lemma The number of Toom contours rooted at $(0,0)$ with N edges is bounded by R^n for some $R<\infty$.

Let $\mathcal{T}_0$ denote the set of all Toom contours rooted at $(0,0)$.
Let $n_{\rm sink}(T)$ denote the number of sinks of T.
Let $N_n^{\rm edge}$ denote the number of $T\in\mathcal{T}_0$ with n edges. Then

$$\mathbb{P}\big[\overline{X}_0(0)=0\big]\leq\sum_{T\in\mathcal{T}_0}\mathbb{P}\big[T\text{ is present in }\Phi\big]\leq\sum_{T\in\mathcal{T}_0}p^{n_{\rm sink}(T)}$$

$$\leq p\sum_{T\in\mathcal{T}_0}p^{cn_{\rm edge}(T)}=p\sum_{n=0}^\infty N_n^{\rm edge}p^{cn}\leq p\sum_{n=0}^\infty R^np^{cn}.$$

Intrinsic randomness

Let $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t)\in\mathbb{Z}^d\times\mathbb{Z}}$ be an i.i.d. collection of maps with

$$\mathbb{P}\big[\Phi_{(i,t)} = \varphi^0\big] = p \quad \text{and} \quad \mathbb{P}\big[\Phi_{(i,t)} = \varphi_k\big] = (1-p)r_k,$$

where φ^0 denotes the trivial zero map and $\varphi_1, \ldots, \varphi_m$ are nontrivial monotone local maps. A *trajectory* of Φ is a function $(i,t) \mapsto x_t(i)$ such that

$$x_t(i) = \Phi_{(i,t)}\big((x_{t-1}(i+j))_{j\in\mathbb{Z}^d}\big) \qquad \forall (i,t).$$

Lemma There a.s. exists a *maximal* trajectory $\overline{X}$.

Aim Fix a probability law $r_1, \ldots, r_m$. For small p, derive a lower bound on $\rho(p) := \mathbb{P}[\overline{X}_0(0) = 1]$.

Intrinsic randomness

Example

$\varphi_1 = \varphi^{\rm NEC},\ \varphi_2 = \varphi^{\rm NWC},\ \varphi_3 = \varphi^{\rm SWC},\ \varphi_4 = \varphi^{\rm SEC}$,
$r_1 = r_2 = r_3 = r_4 = 1/4$.
In spite of $\varphi^{\rm NEC}, \varphi^{\rm NWC}, \varphi^{\rm SWC}, \varphi^{\rm SEC}$ being eroders, this random cellular automaton is believed to be *unstable*.

Intuitively, the "edge speed" in each direction is zero.

On closer look, under the unperturbed evolution, half-space configurations no longer evolve into half-space configurations, so it is a priori not even clear how to define edge speeds in the presence of intrinsic randomness.

Nevertheless, it is believed that
edge speeds should still determine stability.

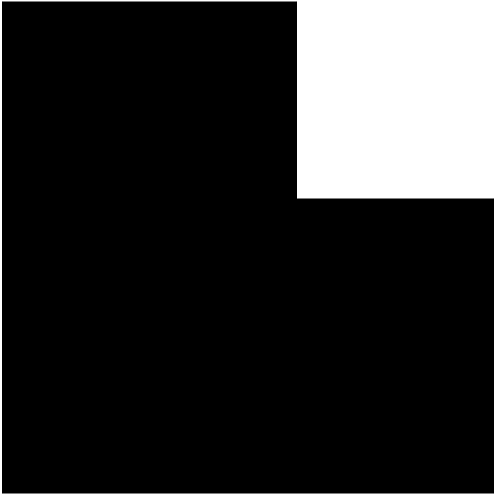
Continuous time

Consider the interacting particle system $(X_t)_{t\geq 0}$ with

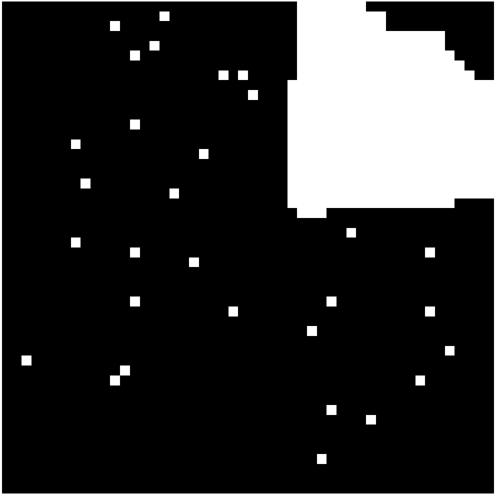
$$X_t(i) \mapsto \begin{cases} \varphi^{\mathrm{NEC}}\big((X_t(i+j))_{j\in\mathbb{Z}^d}\big) & \text{with rate } 1, \\ 0 & \text{with rate } p. \end{cases}$$

Gray (1999) $\lim_{p\to 0} \rho(p) = 1.$

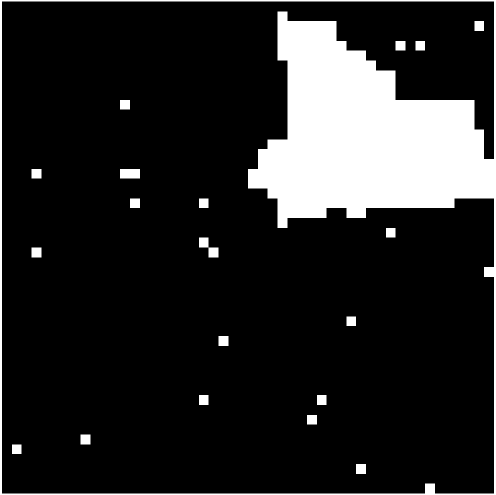
Toom's model in continuous time



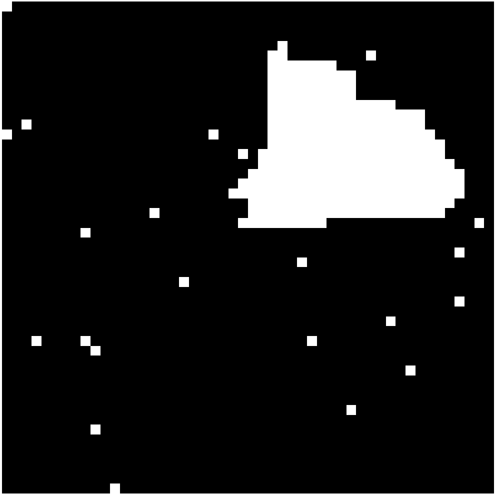
Toom's model in continuous time



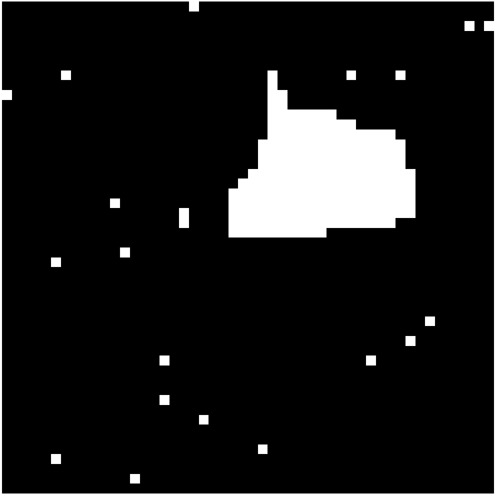
Toom's model in continuous time



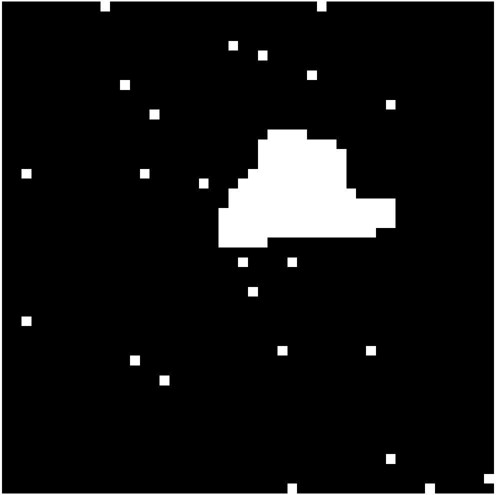
Toom's model in continuous time



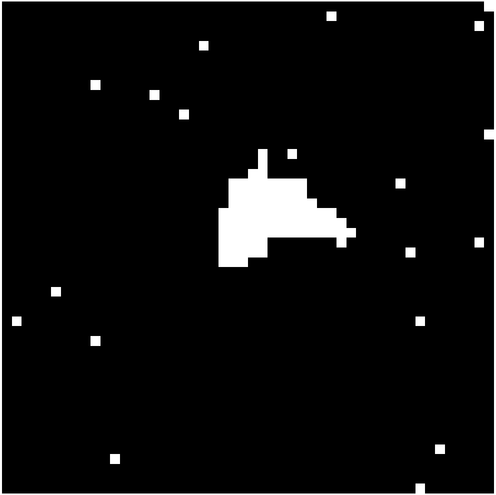
Toom's model in continuous time



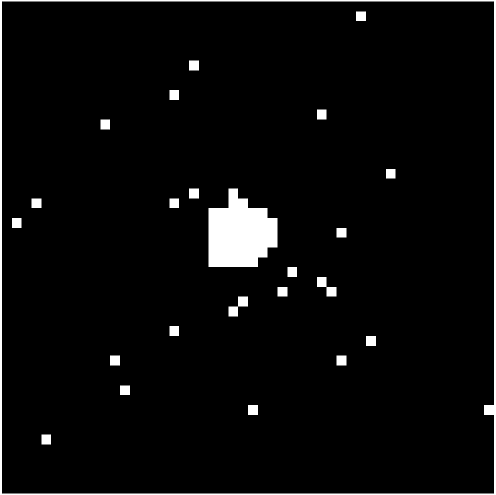
Toom's model in continuous time



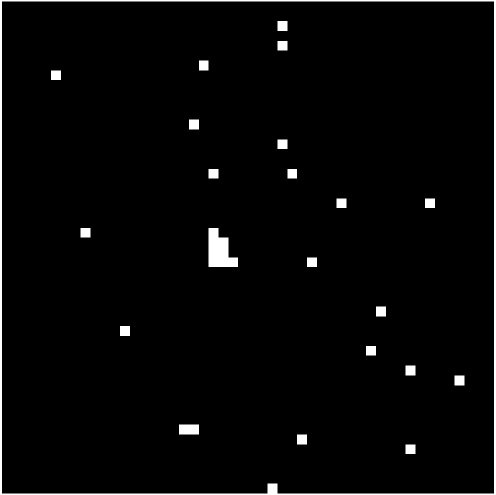
Toom's model in continuous time



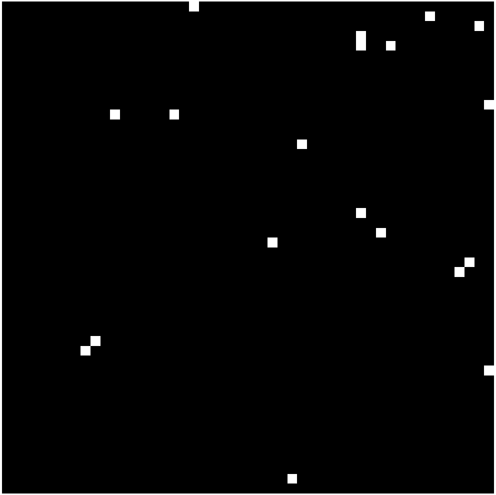
Toom's model in continuous time



Toom's model in continuous time



Toom's model in continuous time



Continuous time

We can think of the continuous-time model as the $\varepsilon \to 0$ limit of a discrete-time model that applies three maps:

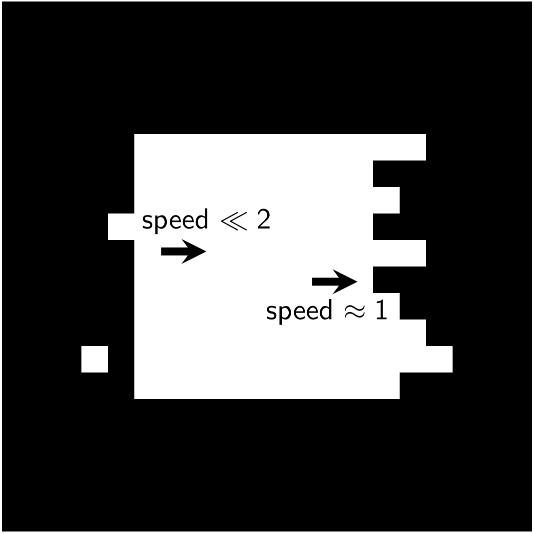
$$\begin{array}{ll} \varphi^{\mathrm{NEC}} & \text{with probability } \varepsilon, \\ \varphi^{0} & \text{with probability } \varepsilon p, \\ \varphi^{\mathrm{id}} & \text{with the remaining probability,} \end{array}$$

where $\varphi^{\mathrm{id}}(x) := x(0)$ is the *identity map*.

Gray (1999) has shown that combining the identity map with an eroder can spoil stability. Let:

$$\varphi(x) := \begin{cases} 0 & \text{if } x(-2,0) = x(-1,0) = 0, \\ 1 & \text{if } x(-3,k) = x(-2,k) = 1 \ \forall |k| \leq n, \\ x(0,0) & \text{in all other cases.} \end{cases}$$

Continuous time



Toom contours and intrinsic randomness

Let $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t)\in\mathbb{Z}^d\times\mathbb{Z}}$ be an i.i.d. collection of maps with

$$\mathbb{P}\big[\Phi_{(i,t)} = \varphi^0\big] = p \quad \text{and} \quad \mathbb{P}\big[\Phi_{(i,t)} = \varphi_k\big] = (1-p)r_k,$$

where φ^0 denotes the trivial zero map and $\varphi_1, \ldots, \varphi_m$ are nontrivial monotone local maps.

Let $L : \mathbb{R}^d \to \mathbb{R}^\sigma$ be a linear polar function.

For each $1 \leq s \leq \sigma$ and $1 \leq k \leq m$, choose $A_s(\varphi_k) \in \mathcal{A}(\varphi_k)$ such that

$$\delta_s(\varphi_k) := \sup_{A\in\mathcal{A}(\varphi_k)} \inf_{i\in A} L_s(i) = \inf_{i\in A_s(\varphi_k)} L_s(i).$$

For a vertex v of a Toom contour, let $\kappa(v)$ indicate the map that is applied at $\big(\psi(v), -h(v)\big)$, i.e.,

$$\Phi_{(\psi(v),-h(v))} = \varphi_{\kappa(v)},$$

where $\varphi_0 = \varphi^0$ is the zero map and $\varphi_1, \ldots, \varphi_m$ are non-constant.

Def A Toom contour is *present* in $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t)\in\mathbb{Z}^d\times\mathbb{Z}}$ if:

- Sinks correspond to vertices where the trivial map φ^0 is applied.
- If (v, w) is a directed edge of charge s coming out of an internal vertex or the root, then $\psi(w) - \psi(v) \in A_s(\varphi_{\kappa(v)})$.
- For directed edges emerging at other sources $\psi(w) - \psi(v) \in \bigcup_{s=1}^{\sigma} A_s(\varphi_{\kappa(v)})$.

Theorem If $\overline{X}_0(0) = 0$, then there is a Toom contour rooted at $(0, 0)$ present in Φ.

Question *Is the Peierls sum finite?*

Trivial case If

$$\delta := \sum_{s=1}^{\sigma} \inf_{k=1,\ldots,m} \delta_s(\varphi_k) > 0,$$

Then $\mathrm{ext}(X_n) \leq \mathrm{ext}(X_0) - \delta n$ *almost surely* and Toom's argument carries over without a change.

This condition exlcudes many interesting cases, including the case where $\varphi_k = \varphi^{\mathrm{id}}$ for some $1 \leq k \leq m$.

Can we go beyond the trivial case?

Intrinsic randomness: beyond the trivial case

Work in progress

Positive result Let $m = 2$, $\varphi_1 = \varphi^{\mathrm{coop}}$, $\varphi_2 = \varphi^{\mathrm{id}}$, $r_1 > 0$. Then we can prove stability using Toom's Peierls argument.

Negative result Let $m = 2$, $\varphi_1 = \varphi^{\mathrm{cc}}$, $\varphi_2 = \varphi^{\mathrm{id}}$, with

$$\varphi^{\mathrm{cc}}(x) := \big(x(0,1) \wedge x(1,0)\big) \vee \big(x(-1,1) \wedge x(0,0) \wedge x(1,-1)\big).$$

Then for r_1 small enough the Peierls sum is infinite for any $p > 0$, in spite of the heuristics and numerics suggesting stability.

In several other cases, we still don't know...

Previous work

Toom (1980) Simple necessary and sufficient conditions for a monotone cellular automaton to be stable. Peierls argument.

Durrett & Gray (1985) Announce a number of deep results for cooperative branching. Referee asks for revision that never materialises.

Berman & Simon (1988), Gács & Reif (1988), Gács (1995,2021) Alternative proofs in a more restricted setting.

Bramson & Gray (1991) Alternative proof of Toom's result using a multiscale block construction.

Chen (1992,1994) Stability w.r.t. initial state & other perturbations. Proofs partly depend on [Durrett & Gray (1985)].

Gray (1999) Sufficient conditions for a monotone interacting particle system to be stable. Combines Toom's Peierls argument with the multiscale approach.

Previous work

Preskill (2007) Note on minimal explanations.

Maere & Ponselet (2011) Exponential decay of correlations.

Ponselet (2013) PhD thesis.

Our contributions

- Clean-up of Toom's argument;
  introduction of sources and sinks.
- Toom contours in the presence of intrinsic randomness.
- A probabilistic method for estimating the Peierls sum.
- Some explicit bounds.

Explicit bounds

Cooperative branching discrete time $p_c \geq 1/64$. (Numerics suggest $p_c \approx 0.105$.)

Cooperative branching continuous time $\lambda_c \leq 162$. (Durrett & Gray (1985) announced $\lambda_c \leq 110$. Numerics suggest $\lambda_c \approx 12.4$.)

Toom's model $p_c \geq 3^{-21}$. (Numerics suggest $p_c \approx 0.053$.)