Frozen percolation on the binary tree

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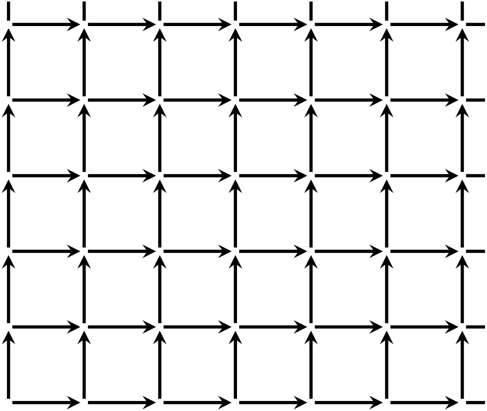
(Budapest)

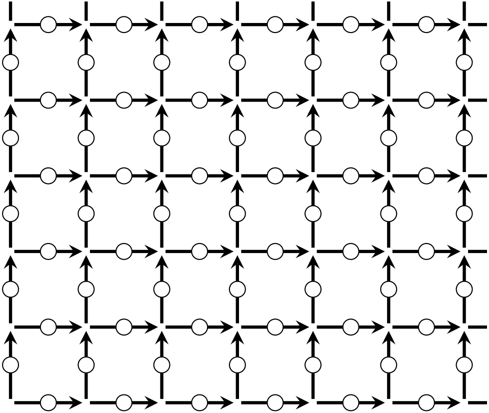
Wednesday, June 9th, 2021

Set-up



Let $G = (V, \vec{E})$ be an infinite oriented graph.

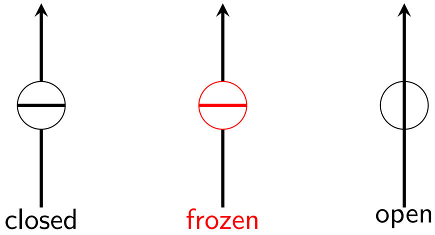
Set-up



On each edge, we place a *barrier*.

Set-up

Barriers can be in three states:



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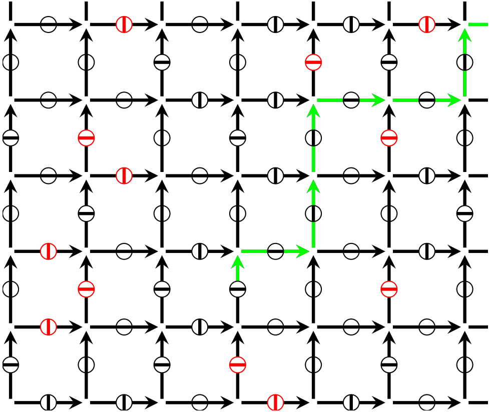

A barrier *percolates* if
there starts an infinite open path just above it.

Dynamics

We assign i.i.d. Unif$[0,1]$ *activation times* $(\tau_{\mathrm{i}})_{\mathrm{i}\in\mathbb{B}}$ to the barriers $\mathrm{i} \in \mathbb{B}$.

We fix a set $\Xi \subset (0,1]$ of *freezing times*.

- Initially, all barriers are closed.

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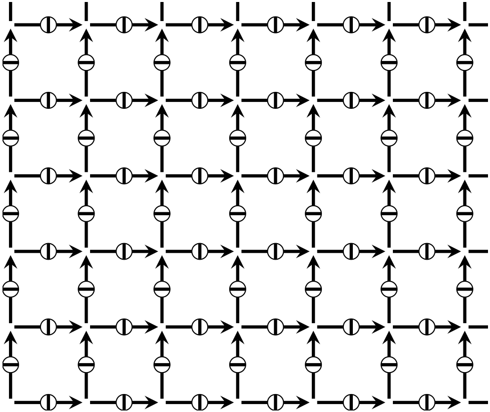
- Initially, all barriers are closed.
- At its activation time, a barrier opens, provided it is not frozen.
- At each freezing time $t \in \Xi$, all closed barriers that percolate are frozen.

Dynamics



Barriers open at their activation times.

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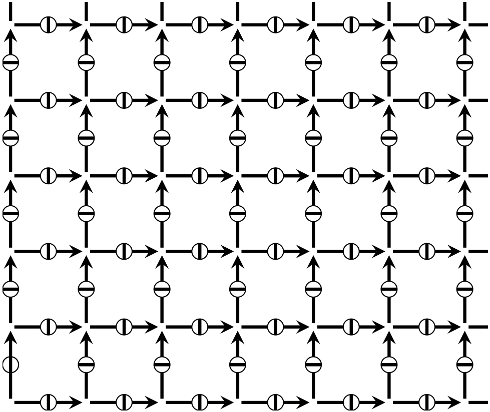

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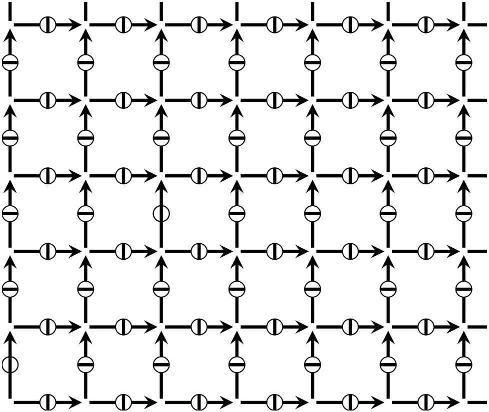

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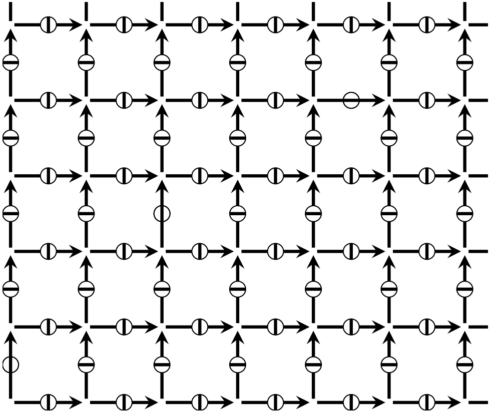

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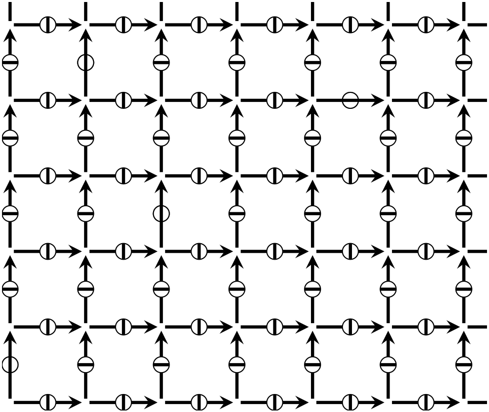

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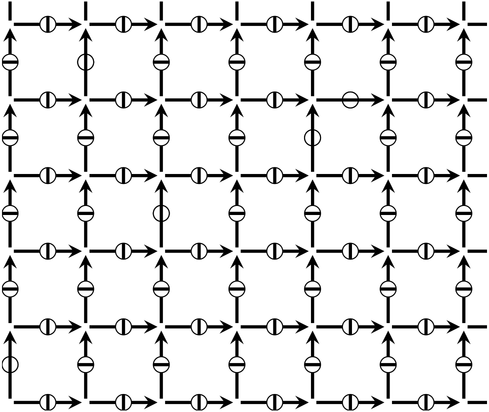

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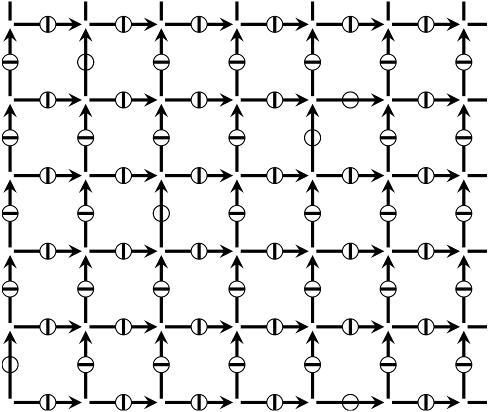

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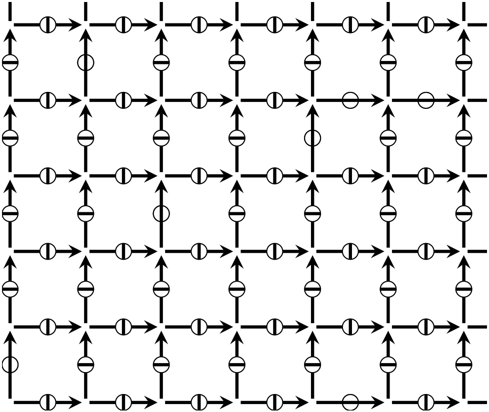

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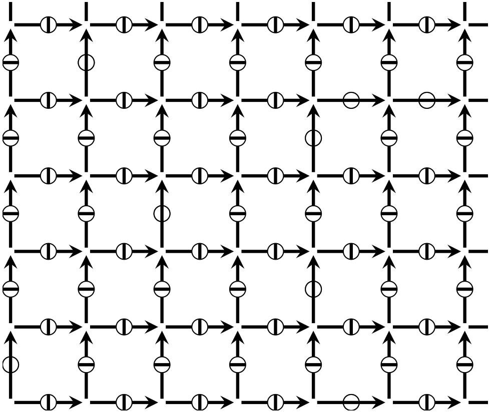

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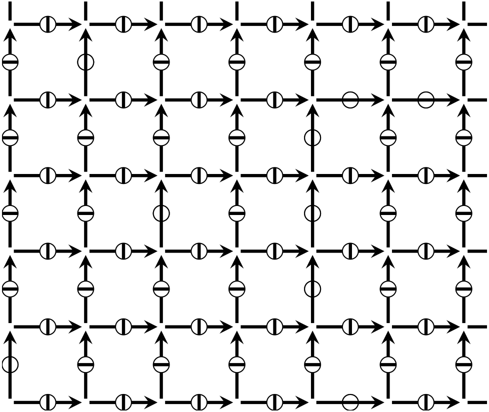

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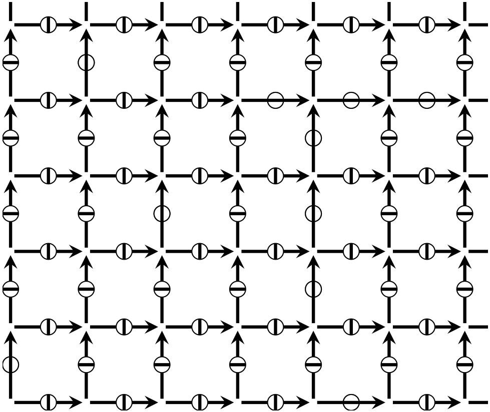

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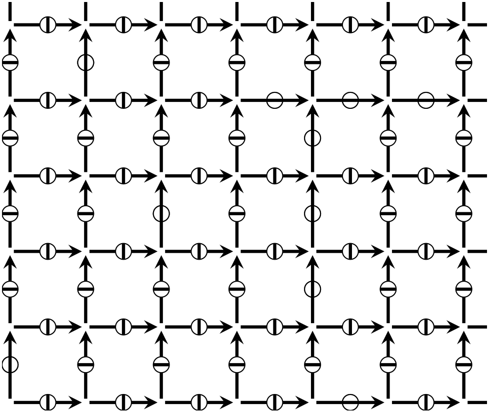

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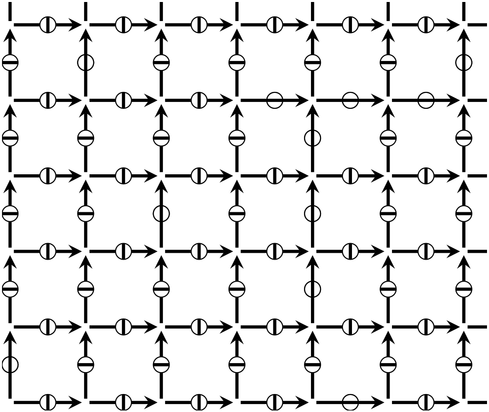

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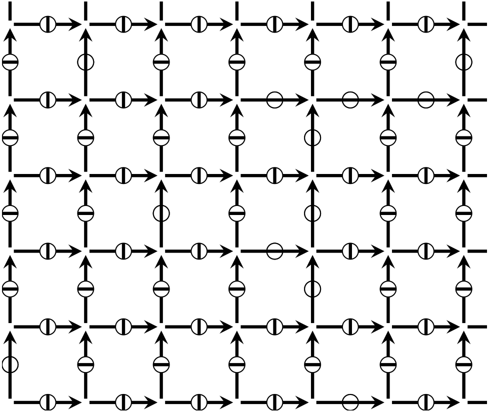

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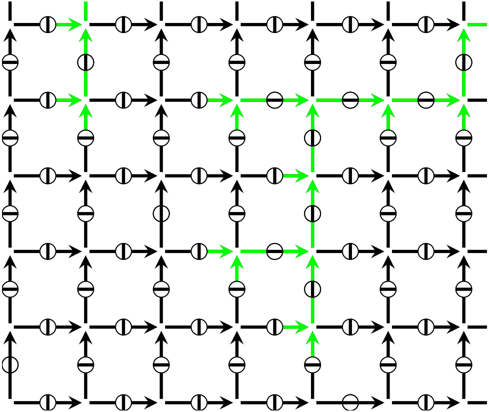

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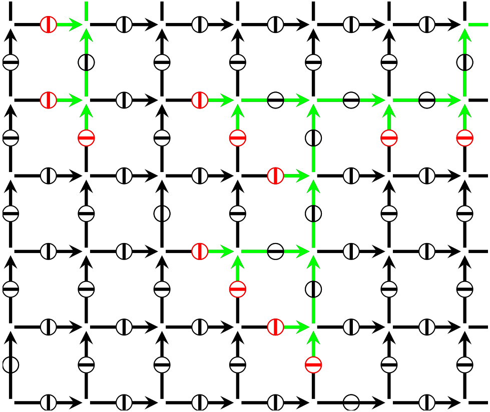

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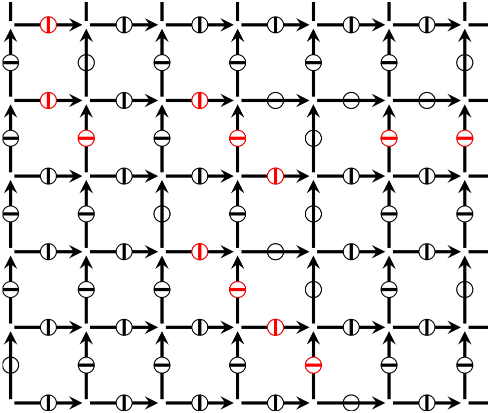

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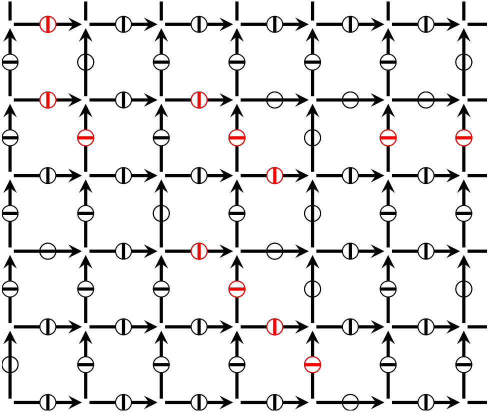

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Notation

$$\mathbb{A}_t := \{ \mathrm{i} \in \mathbb{B} : \tau_\mathrm{i} \leq t \},$$
$$\mathbb{F} := \{ \mathrm{i} \in \mathbb{B} : \mathrm{i} \text{ frozen at the final time } 1 \}.$$

Then $\mathbb{A}_t \backslash \mathbb{F}$ are the open barriers at time t.

We write $\mathrm{i} \stackrel{\mathbb{A}_t \backslash \mathbb{F}}{\longrightarrow} \infty$ if i percolates at time t.

The Frozen Percolation Equation

The *Frozen Percolation Equation* (FPE) reads:

$$\mathbb{F} = \big\{\mathrm{i} \in \mathbb{B} : \mathrm{i} \stackrel{\mathbb{A}_t\setminus\mathbb{F}}{\longrightarrow} \infty \text{ for some } t \in (0, \tau_{\mathrm{i}}] \cap \Xi\big\}.$$

If Ξ is finite, then (FPE) has a solution, which is a.s. unique.

Questions for infinite Ξ:

- Existence of solutions?
- Uniqueness of solutions?
- Uniqueness in which sense?

Existence

If $\Xi = (0, 1]$, then clusters freeze as soon as they reach infinite size. This leads to *self-organised criticality.*

Frozen percolation can also be defined on unoriented graphs. Just replace each unoriented edge by two oriented edges whose barriers are activated at the same time.

For $\Xi = (0, 1]$, David Aldous (2000) proved that (FPE) has a solution on the unoriented 3-regular tree, or equivalently on the oriented binary tree.

For $\Xi = (0, 1]$, Itai Benjamini and Oded Schramm (2001) proved that (FPE) has no solution on the unoriented square lattice $\mathbb{Z}^2$.

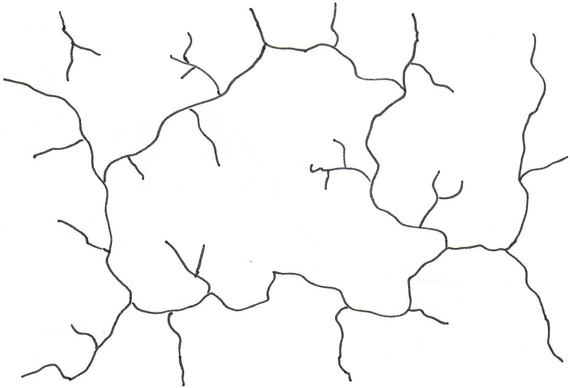
Existence

A natural approach is to choose finite Ξ_n that converge to $(0,1]$ in the sense that for each open $O \subset (0,1]$,
$\Xi_n \cap O \neq \emptyset$ for all n large enough.

On the 3-regular tree, the solutions to $(\mathrm{FPE})_{\Xi_n}$ converge to a solution of $(\mathrm{FPE})_{(0,1]}$.

But on $\mathbb{Z}^2$ this does not work: the limit exists, but does not solve $(\mathrm{FPE})_{(0,1]}$ since infinite clusters form but do not freeze.

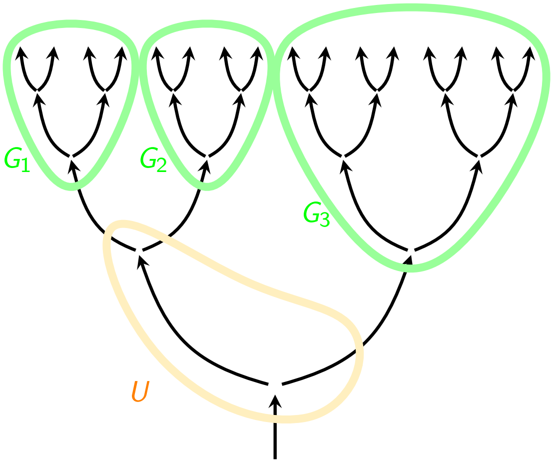
Existence



The reason is that at the first time $t \in \Xi_n$ with $t > p_c$, a very sparse cluster freezes that blocks all further paths to infinity.

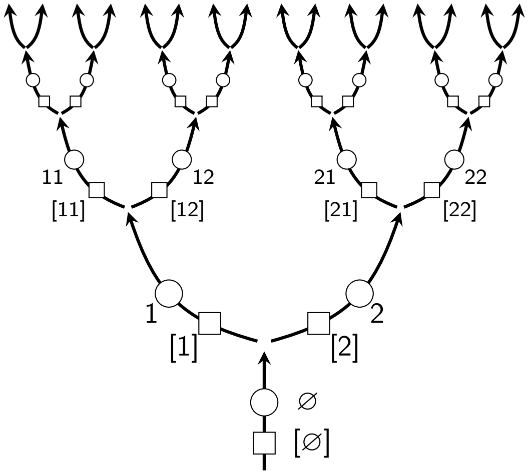
Existence on $\mathbb{Z}^3$ is an open problem.

Distributional uniqueness



On the oriented binary tree, we impose *natural conditions*:
The subtrees G_1, G_2, G_3 should be i.i.d., equally distributed with the original tree G, and independent of U.

Freezing times



We label barriers in the obvious way and let [i] denote the point just below i.

Freezing times

The freezing time of the root

$$Y_{[\varnothing]} := \inf\left\{t \in \Xi : [\varnothing] \stackrel{\mathbb{A}_t \backslash \mathbb{F}}{\longrightarrow} \infty\right\}$$

solves the *Recursive Distributional Equation* (RDE)

$$Y_{[\varnothing]} \stackrel{\mathrm{d}}{=} \gamma(\tau_\varnothing, Y_{[1]}, Y_{[2]}) := \begin{cases} Y_{[1]} \wedge Y_{[2]} & \text{if } \tau_\varnothing < Y_{[1]} \wedge Y_{[2]}, \\ \infty & \text{otherwise.} \end{cases}$$

Distributional uniqueness

[Ráth, S., Szőke '21] For each closed $\Xi \subset (0,1]$, there exists a unique solution ρ_Ξ to (RDE) that yields a solution $\mathbb{F}$ of (FPE).

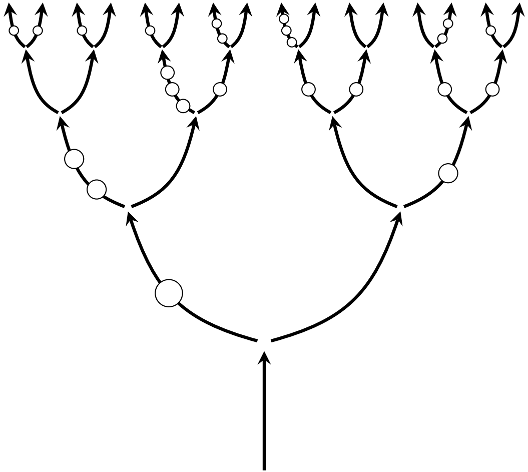
Consequence For each closed $\Xi \subset (0,1]$, (FPE) has a solution that satisfies the natural conditions, and the joint law of $\big((\tau_{\rm i})_{{\rm i}\in\mathbb{B}}, \mathbb{F}\big)$ is uniquely determined.

Aldous (2000) proved

$$\rho_{(0,1]}(\mathrm{d}y) = \frac{\mathrm{d}y}{2y^2} 1_{[\frac{1}{2},1]}(y) \qquad \rho\big(\{\infty\}\big) = \tfrac{1}{2}.$$

We discovered the problem becomes easier if we place a geometrically distributed number of barriers with mean one on each edge.

The MBBT



We call this the *Marked Binary Branching Tree*.

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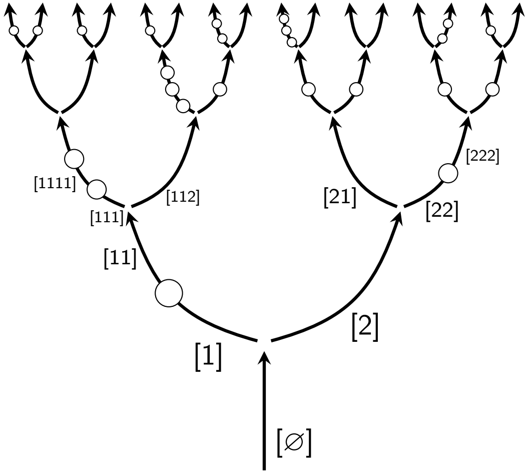

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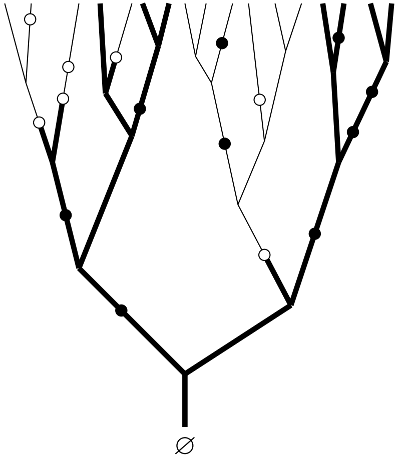
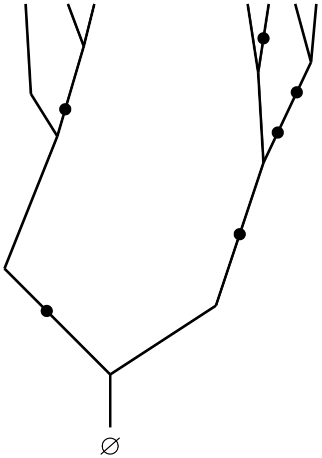
Scaling of the MBBT



If we are only interested in the time interval $[0, r]$, then we can restrict ourselves to the *pruned tree* $\{[i] : [i] \xrightarrow{\mathbb{A}_r} \infty\}$.

Scaling of the MBBT



For the MBBT, $\mathbb{P}\big[[\varnothing] \xrightarrow{\mathbb{A}_r} \infty\big] = r$ and conditional on this event, the pruned tree is equally distributed with a scaled version of the original tree.

The MBBT

Letting κ_i denote the number of offspring of i, the Recursive Distributional Equation (RDE) now takes the form

$$Y_{[\varnothing]} \stackrel{\mathrm{d}}{=} \gamma(\tau_\varnothing, \kappa_\varnothing, Y_{[1]}, Y_{[2]}) := \begin{cases} Y_{[1]} \wedge Y_{[2]} & \text{if } \kappa_\varnothing = 2, \\ Y_{[1]} & \text{if } \kappa_\varnothing = 1 \text{ and } \tau_\varnothing < Y_{[1]}, \\ \infty & \text{otherwise.} \end{cases}$$

A probability measure ρ on $[0,1] \cup \{\infty\}$ solves this RDE iff

$$\int_{[0,t]} \rho(\mathrm{d}y)y = \rho\big([0,t]\big)^2 \qquad (0 \leq t \leq 1).$$

Aldous' solution takes the simple form

$$\rho_{(0,1]}(\mathrm{d}y) = \tfrac{1}{2} 1_{[0,1]}(y)\,\mathrm{d}y \qquad \rho\big(\{\infty\}\big) = \tfrac{1}{2}.$$

Almost sure uniqueness

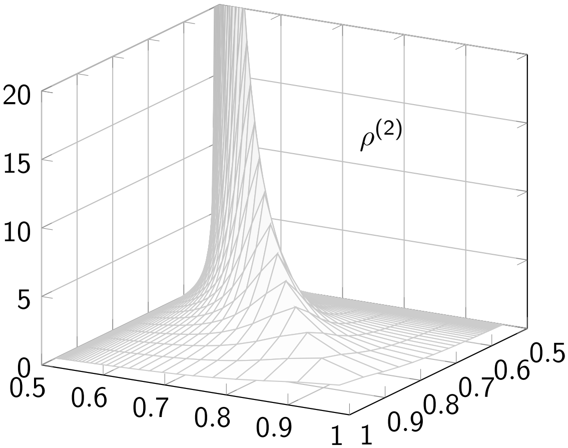
Let $\mathbb{F}, \mathbb{F}'$ be solutions to (FPE) that satisfy the natural conditions and are conditionally independent given $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{B}}$.

Then $(Y_{[\varnothing]}, Y'_{[\varnothing]})$ solves the *bivariate RDE*

$$(Y_{[\varnothing]}, Y'_{[\varnothing]}) \stackrel{\mathrm{d}}{=} \big(\gamma(\tau_\varnothing, Y_{[1]}, Y_{[2]}), \gamma(\tau_\varnothing, Y'_{[1]}, Y'_{[2]})\big).$$

David Aldous and Antar Bandyopadhyay (2005) proved that $\mathbb{F} = \mathbb{F}'$ a.s. if and only if each solution $\rho^{(2)}$ to the bivariate RDE that has marginals $\rho_{\underline{=}}$ is *concentrated on the diagonal.*

Nontrivial solution of the bivariate RDE



For $\Xi = (0, 1]$, numerical calculations by Bandyopadhyay (2004) suggested the existence of a *nontrivial* solution, that is not concentrated on the diagonal.

Almost sure uniqueness

For 15 years, nobody could prove this.
Several people tried, no one harder than Balázs Ráth.

The bivariate RDE gives an integral equation for

$$F(s,t) := \mathbb{P}\big[Y_{[\varnothing]} \leq s,\ \ Y'_{[\varnothing]} \leq t\big].$$

For the MBBT, scaling gives $F(rs, rt) = rF(s,t)$ $(r, s, t \in [0,1])$.

The bivariate RDE now reduces to an integral equation for a function of one variable, which can be solved.

[Ráth, S., Terpai accepted AoP] For $\Xi = (0,1]$, frozen percolation on the binary tree is not a.s. unique.

Other sets of freezing times

For $0 < \theta < 1$, set $\Xi_\theta := \{\theta^n : n \geq 0\}$.

[Ráth, S., Szőke '21] There exists a parameter $\theta^* = 0.636\ldots$ such that all solutions of the bivariate RDE with marginals ρ_{Ξ_θ} are concentrated on the diagonal if and only if $0 < \theta \leq \theta^*$.

In other words, if $\mathbb{F}, \mathbb{F}'$ are solutions to (FPE) for Ξ_θ, that satisfy the natural conditions and are conditionally independent given $(\tau_{\rm i})_{\rm i \in \mathbb{B}}$, then $\mathbb{F} = \mathbb{F}'$ a.s. if and only if $\theta \leq \theta^*$.

Open problems

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- For $\theta \leq \theta^*$, are solutions to (FPE) a.s. unique even if we drop the natural conditions?
- other sets of freezing times,
- n-regular oriented trees with $n \geq 3$,
- other graphs such as $\mathbb{Z}^3$.