The mean-field dual of systems with cooperative reproduction

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Continuous-time Markov chains

Let S be a finite set and let $G : S \times S \to \mathbb{R}$ satisfy

- $G(x,y) \geq 0 \; \forall x \neq y$,
- $\sum_y G(x,y) = 0$.

Then G is a *Markov generator* and

$$P_t = e^{tG} := \sum_{n=0}^{\infty} \frac{1}{n!}(tG)^n,$$

defines a semigroup of probability kernels on S, i.e., $P_t(x,y) \geq 0$ and $\sum_y P_t(x,y) = 1$. For each probability law μ on S, there exists a Markov process $(X_t)_{t\geq 0}$ with initial law $\mathbb{P}[X_0 = x] = \mu(x)$ $(x \in S)$ and transition probabilities

$$\mathbb{P}\big[X_u = y \,\big|\, (X_s)_{s\leq t}\big] = P_{u-t}(X_t, y) \quad \text{a.s.} \quad (y \in S).$$

Stochastic flows

Let $\mathcal{M}$ be a set whose elements are maps $m : S \to S$ and let $(r_m)_{m\in\mathcal{M}}$ be nonnegative constants. Then

$$Gf(x) := \sum_{m\in\mathcal{M}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}$$

defines a Markov generator. Let ω be a Poisson point set on $\mathcal{M} \times \mathbb{R}$ with intensity $r_m \mathrm{d}t$. We may order the elements of

$$\omega_{s,u} := \{(m,t) \in \omega : t \in (s,u]\} = \big\{(m_1,t_1),\ldots,(m_n,t_n)\big\}$$

in such a way that $t_1 < \cdots < t_n$. Then

$$\mathbf{X}_{s,u} := m_n \circ \cdots \circ m_1.$$

defines a stochastic flow: $\mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$ a.s.

Stochastic flows

(Poisson construction of Markov processes) *Define maps* $(\mathbf{X}_{s,t})_{s\leq t}$ *as above in terms of a Poisson point set* ω*. Let* X_0 *be an* S*-valued random variable, independent of* ω*. Then*

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

is a Markov process with generator G*.*

Remark The sample paths of X are right-continuous because we defined $\omega_{s,u} := \{(m,t) \in \omega : t \in (s,u]\}$.

The contact process

Let (Λ, E) be a finite graph with vertex set Λ and edge set E.
Let $S := \{0,1\}^\Lambda$.
For each $i \in \Lambda$, define a *death map* $\mathtt{dth}_i : S \to S$ by

$$\mathtt{dth}_i(x)(k) := \begin{cases} 0 & \text{if } j = k, \\ x(k) & \text{otherwise.} \end{cases}$$

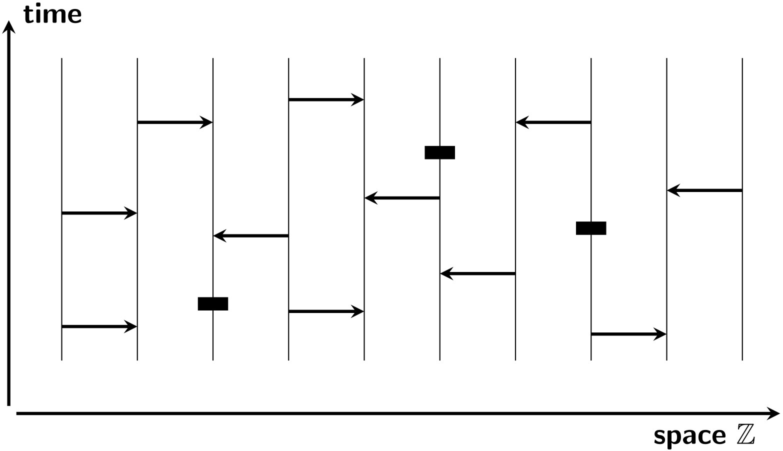
For each (i,j) with $\{i,j\} \in E$, define a *reproduction map* $\mathtt{rep}_{ij} : S \to S$ by

$$\mathtt{rep}_{ij}(x)(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

Fix $\lambda \geq 0$ and give death and reproduction maps the rates

$$r_{\mathtt{dth}_i} := 1 \quad \text{and} \quad r_{\mathtt{rep}_{ij}} := \lambda.$$

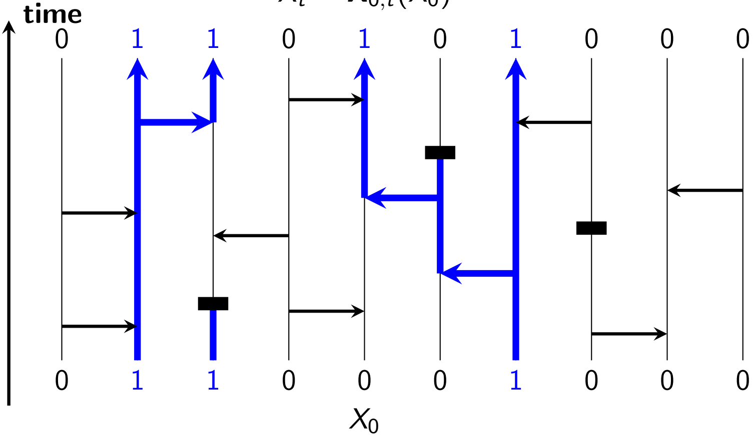
The graphical representation



We denote $\mathtt{rep}_{ij}$ by an arrow from i to j
and $\mathtt{dth}_i$ by a rectangle at i.

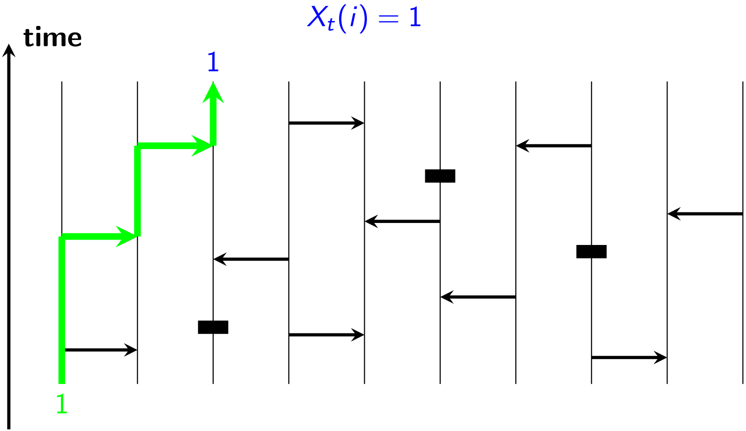
The graphical representation

$$X_t = \mathbf{X}_{0,t}(X_0)$$



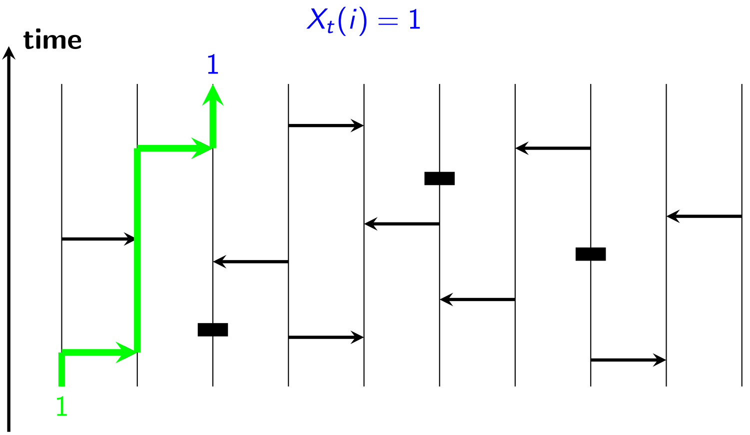
The map $x \mapsto \mathbf{X}_{0,t}(x)$.

Open paths



$X_t(i) = 1 \Leftrightarrow \exists$ open path from $(j, 0)$ with $X_0(j) = 1$ to (i, t).

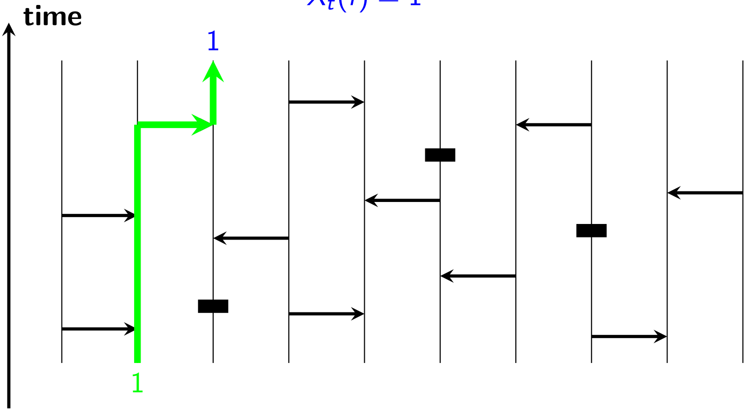
Open paths



$X_t(i) = 1 \Leftrightarrow \exists$ open path from $(j, 0)$ with $X_0(j) = 1$ to (i, t).

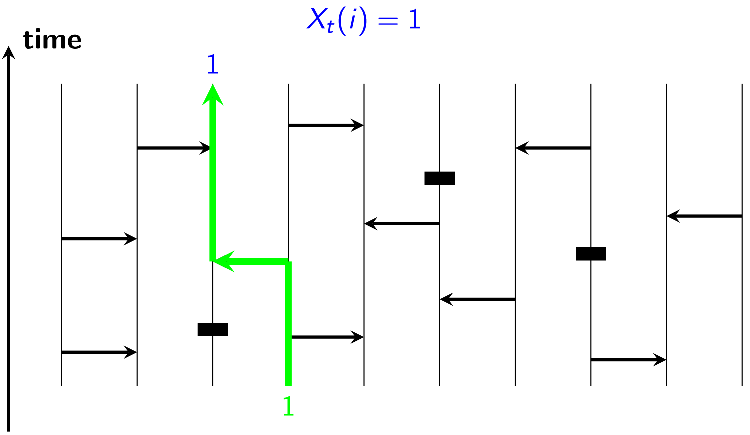
Open paths

$X_t(i) = 1$



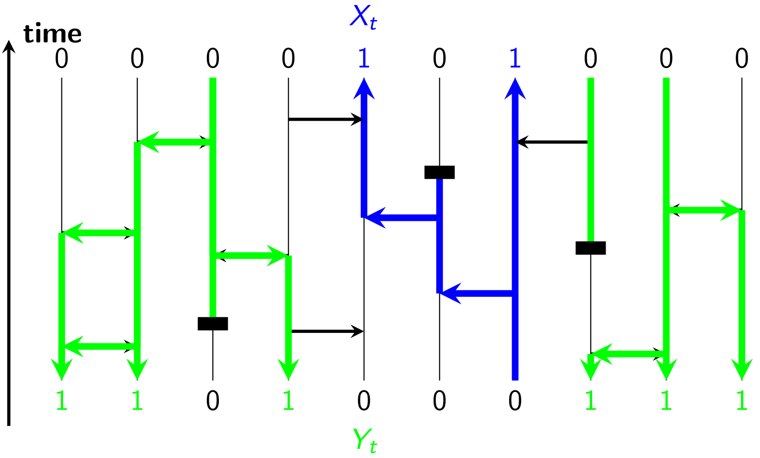
$X_t(i) = 1 \Leftrightarrow \exists$ open path from $(j, 0)$ with $X_0(j) = 1$ to (i, t).

Open paths



$X_t(i) = 1 \Leftrightarrow \exists$ open path from $(j, 0)$ with $X_0(j) = 1$ to (i, t).

Dual process



All open paths with given endpoints form a *dual process*.

$$\mathbb{P}\big[X_t \cap Y_0 \neq \emptyset\big] = \mathbb{P}\big[X_0 \cap Y_t \neq \emptyset\big] \qquad (t \geq 0).$$

Cooperative reproduction

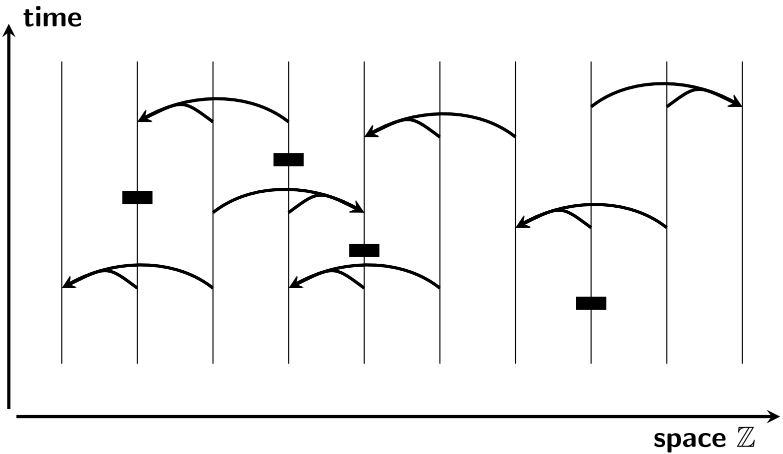
Let (Λ, E) be a finite graph with vertex set Λ and edge set E.
Let $S := \{0,1\}^\Lambda$.
For each (i,j,k) with $\{i,j\} \in E$ and $\{j,k\} \in E$, define a *cooperative reproduction map* $\mathtt{coop}_{ijk} : S \to S$ by

$$\mathtt{coop}_{ijk}(x)(l) := \begin{cases} \big(x(i) \wedge x(j)\big) \vee x(k) & \text{if } l = k, \\ x(l) & \text{otherwise.} \end{cases}$$

Give death and cooperative reproduction maps the rates

$$r_{\mathtt{dth}_i} := 1 \quad \text{and} \quad r_{\mathtt{coop}_{ijk}} := \alpha.$$

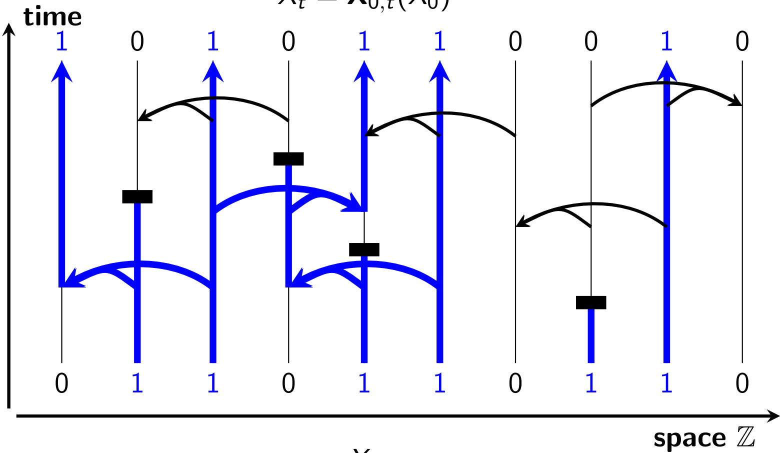
The graphical representation



We denote $\mathtt{coop}_{ijk}$ by a suitable symbol
and denote $\mathtt{dth}_i$ as before.

The graphical representation

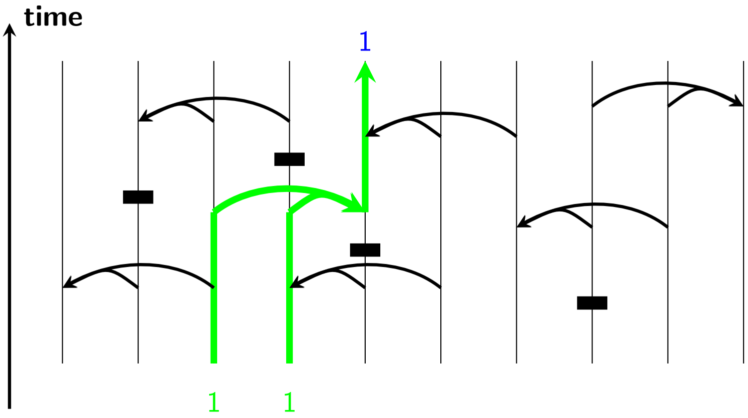
$$X_t = \mathbf{X}_{0,t}(X_0)$$



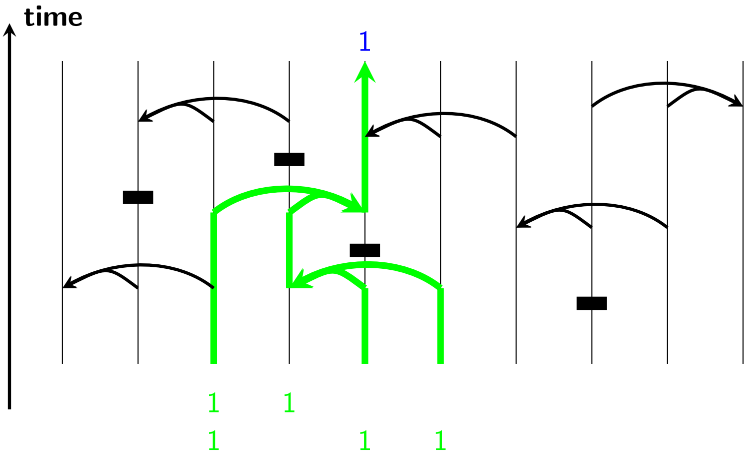
X_0

The map $x \mapsto \mathbf{X}_{0,t}(x)$.

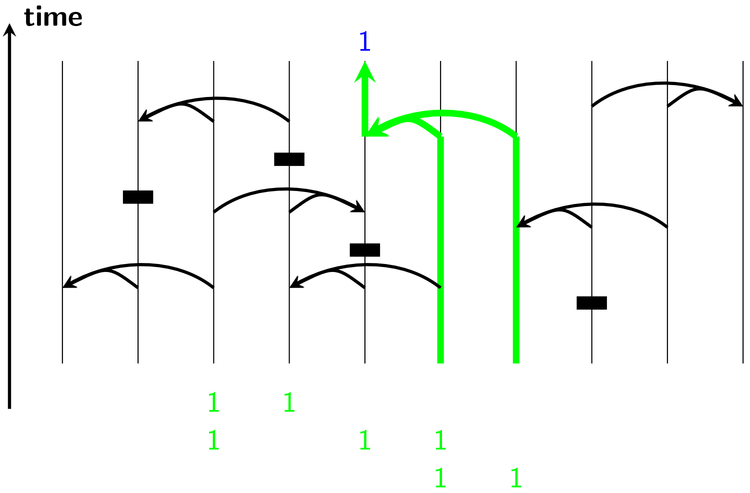
Open paths



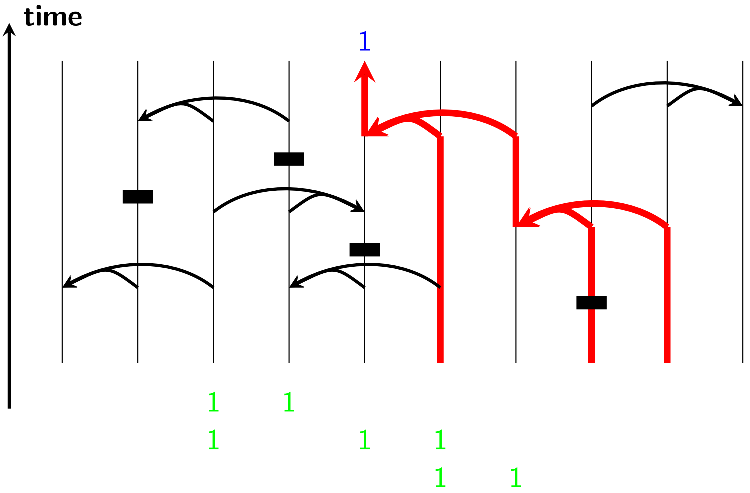
Open paths



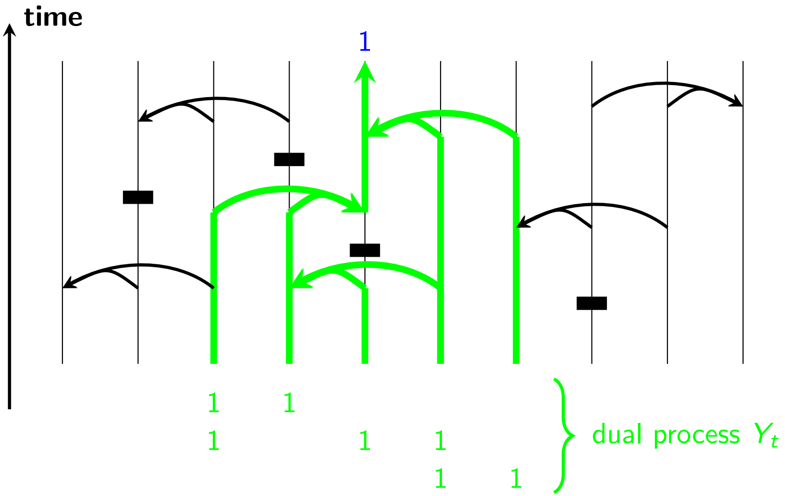
Open paths



Open paths



Open paths



$$\mathbb{P}\big[X_t \geq y \text{ for some } y \in Y_0\big] = \mathbb{P}\big[X_0 \geq y \text{ for some } y \in Y_t\big].$$

The dual process

The dual process Y_t takes value in $\mathcal{H}_0(\Lambda)$, where:

$$S_{\rm fin}(\Lambda) := \big\{y : \Lambda \to \{0,1\} : \sum_i y(i) < \infty\big\}.$$

$$\mathcal{H}_0(\Lambda) := \Big\{Y \subset S_{\rm fin}(\Lambda) : \begin{array}{l} Y \text{ is finite and each } y \in Y \\ \text{is a minimal element of } Y \end{array}\Big\}.$$

Pathwise duality:

$$1_{\{X_t \geq y \text{ for some } y \in Y_0\}} = 1_{\{X_0 \geq y \text{ for some } y \in Y_t\}} \quad \text{a.s.}$$

We can view $Y \in \mathcal{H}_0(\Lambda)$ as a *hypergraph* with vertex set Λ and set of *hyperedges* Y.

The mean-field limit of the contact process

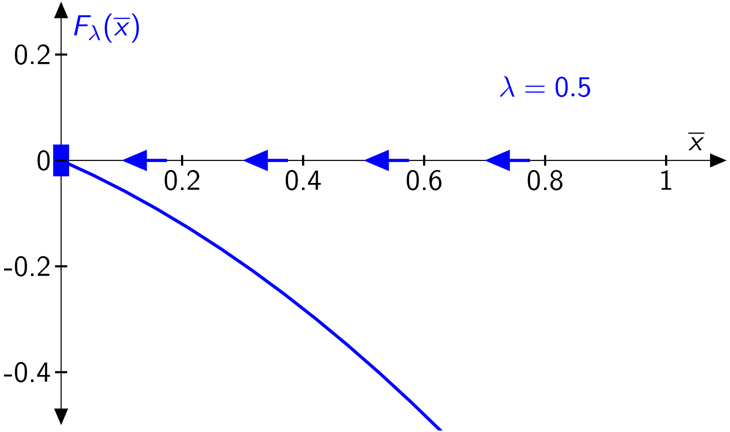
Consider the contact process on the complete graph K_N, where the following maps are applied with the following rates:

$$\begin{array}{llll} \texttt{rep}_{ij} & \text{with rate} & \lambda N/N^2 & \forall 1 \leq i, j \leq N, \\ \texttt{dth}_i & \text{with rate} & 1N/N & \forall 1 \leq i \leq N. \end{array}$$

Then the fraction of occupied sites $\overline{X}_t := N^{-1} \sum_{i=1}^{N} X_t(i)$ converges to the solution of the *mean-field ODE*

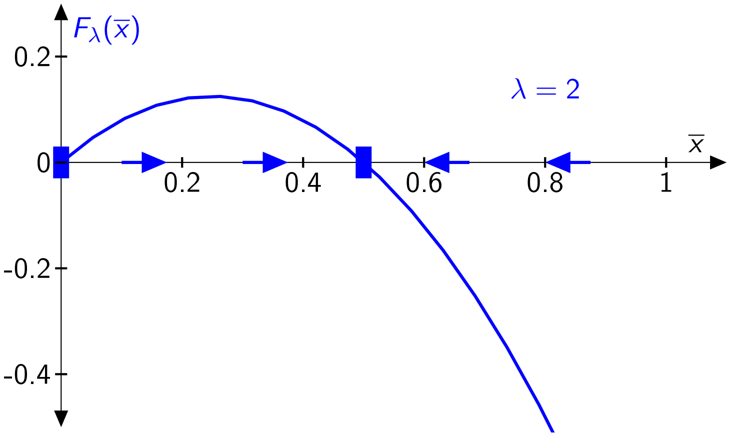
$$\frac{\partial}{\partial t}\overline{X}_t = \lambda \overline{X}_t(1 - \overline{X}_t) - \overline{X}_t =: F_\lambda(\overline{X}_t).$$

The mean-field limit of the contact process



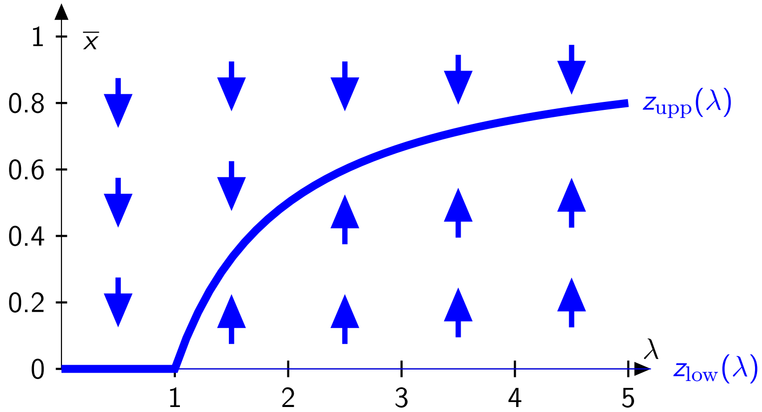
For $\lambda < 1$, the equation $\frac{\partial}{\partial t}\overline{X}_t = F_\lambda(\overline{X}_t)$ has a single, stable fixed point $\overline{x} = 0$.

The mean-field limit of the contact process



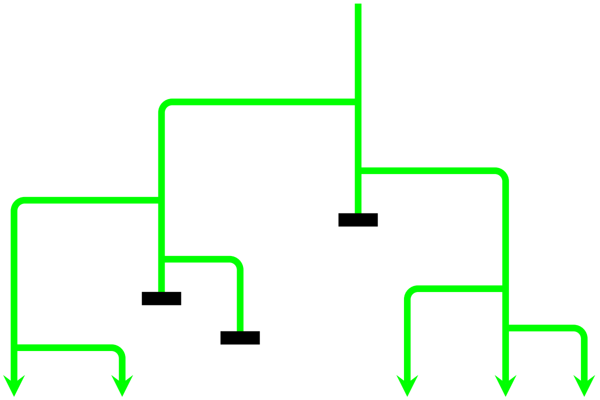
For $\lambda > 1$, the fixed point at 0 becomes unstable and a new, stable fixed point appears.

The mean-field limit of the contact process



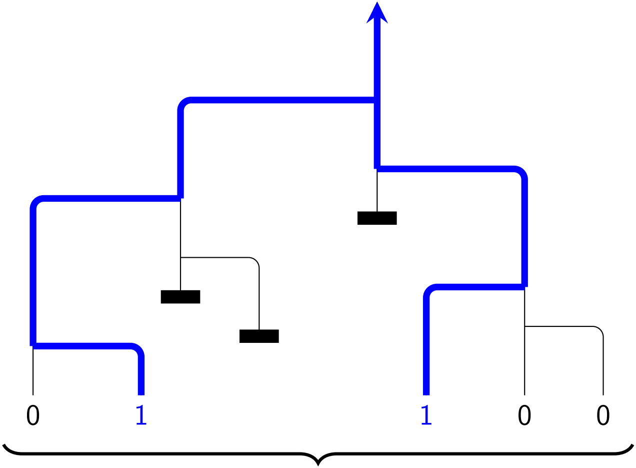
Fixed points of $\frac{\partial}{\partial t}\overline{X}_t = F_\lambda(\overline{X}_t)$ for different values of λ.

Mean-field limit of the dual process



In the mean-field limit, the dual process is a *branching process*.

Mean-field limit of the dual process



$\{B_i\}$ i.i.d. with intensity $\overline{X}_0$

Mean-field duality

Let Y be an $\mathbb{N}$-valued random variable. Let $(B_i)_{i\in\mathbb{N}_+}$ be i.i.d. Bernoulli random variables with $\mathbb{P}[B_i = 1] = \overline{x}$, independent of Y. Define

$$\mathrm{Thin}_{\overline{x}}(Y) := \sum_{i=1}^{Y} B_i.$$

Let $(\overline{Y}_t)_{t\geq 0}$ be a Markov process in $\mathbb{N}$ that jumps

$$y \mapsto y+1 \text{ with rate } \lambda y \quad \text{and} \quad y \mapsto y-1 \text{ with rate } y.$$

Then

$$\mathbb{P}\big[\mathrm{Thin}_{\overline{X}_0}(\overline{Y}_t) \neq 0\big] = \mathbb{P}\big[\mathrm{Thin}_{\overline{X}_t}(\overline{Y}_0) \neq 0\big],$$

where $(\overline{X}_t)_{t\geq 0}$ solves the mean-field ODE

$$\tfrac{\partial}{\partial t}\overline{X}_t = \lambda \overline{X}_t(1-\overline{X}_t) - \overline{X}_t.$$

The survival of the $\mathbb{N}$-valued branching process $(\overline{Y}_t)_{t\geq 0}$ started in $\overline{Y}_0 = 1$ is given by

$$\mathbb{P}^1\big[\overline{Y}_t \neq 0 \ \forall t \geq 0\big] = z_{\rm upp}(\lambda).$$

Proof

$$\begin{aligned}&\mathbb{P}^1\big[\mathrm{Thin}_1(\overline{Y}_t) \neq \emptyset\big]\\ &\quad = \mathbb{P}^1\big[\mathrm{Thin}_{\overline{X}_t}(1) \neq \emptyset\big] = \overline{X}_t \underset{t\to\infty}{\longrightarrow} z_{\rm upp}(\lambda).\end{aligned}$$

∎

The mean-field limit of cooperative reproduction

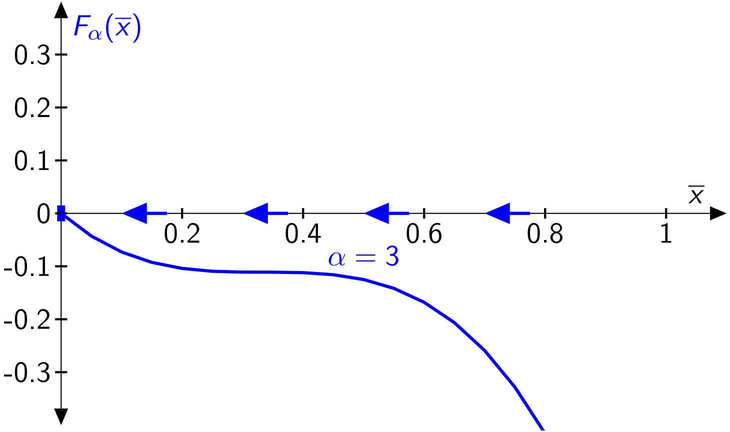
Consider a cooperative reproduction process on the complete graph K_N, where the following maps are applied with the following rates:

$$\begin{array}{llll} \mathtt{coop}_{ijk} & \text{with rate} & \alpha N/N^3 & \forall 1 \leq i,j,k \leq N, \\ \mathtt{dth}_i & \text{with rate} & 1N/N & \forall 1 \leq i \leq N. \end{array}$$

Then the fraction of occupied sites $\overline{X}_t := N^{-1} \sum_{i=1}^{N} X_t(i)$ converges to the solution of the *mean-field ODE*

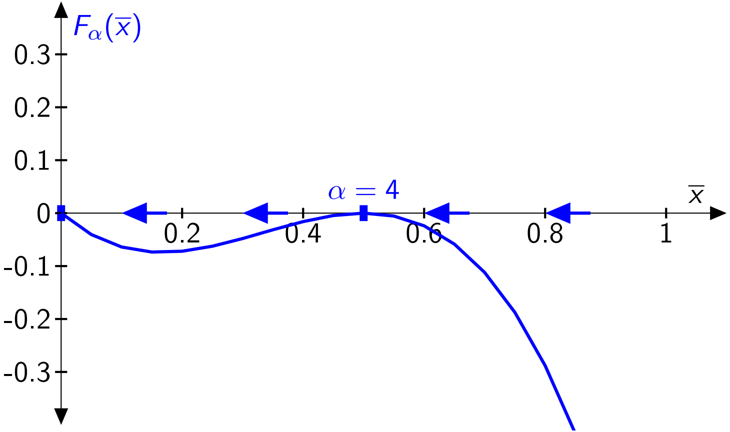
$$\frac{\partial}{\partial t} \overline{X}_t = \alpha \overline{X}_t^2 (1 - \overline{X}_t) - \overline{X}_t =: F_\alpha(\overline{X}_t).$$

The mean-field of cooperative reproduction



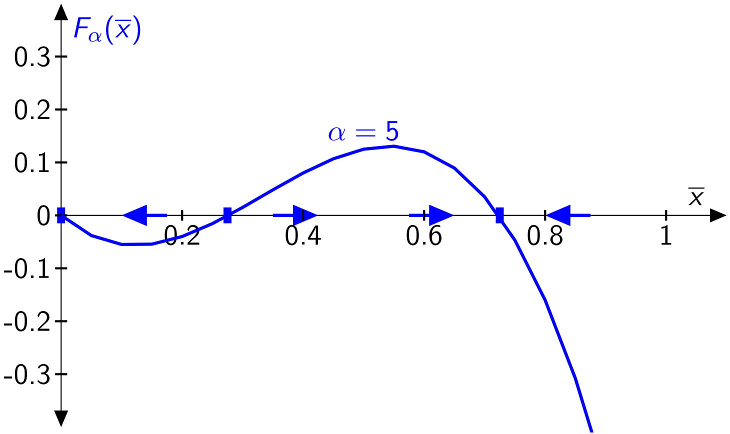
For $\alpha < 4$, the equation $\frac{\partial}{\partial t}\overline{X}_t = F_\alpha(\overline{X}_t)$ has a single, stable fixed point $\overline{x} = 0$.

The mean-field of cooperative reproduction



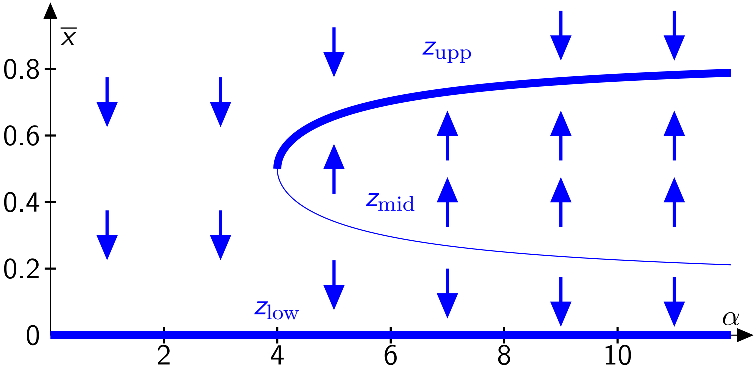
For $\alpha = 4$, a second fixed point appears at $\overline{x} = 0.5$.

The mean-field of cooperative reproduction



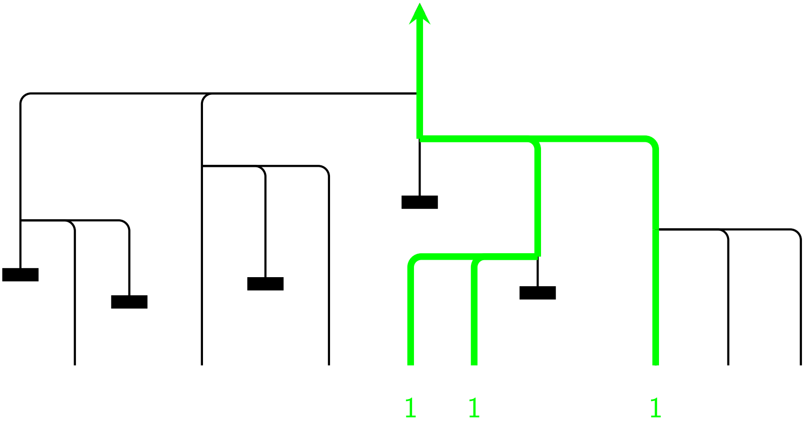
For $\alpha > 4$, there are two stable fixed points and one unstable fixed point, which separates the domains of attraction of the other two.

The mean-field of cooperative reproduction



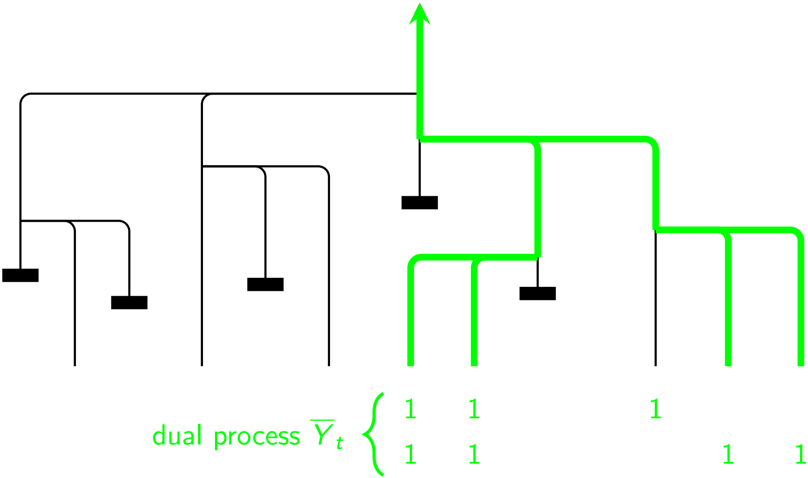
Fixed points of $\frac{\partial}{\partial t}\overline{X}_t = F_\alpha(\overline{X}_t)$ for different values of α.

Mean-field limit of the dual process



The mean-field dual can be embedded in a branching process.

Mean-field limit of the dual process



The mean-field dual

For $Y, Y' \in \mathcal{H}_0(\mathbb{N}_+)$, write $Y \sim Y'$ if they are equal up to a permutation of $\mathbb{N}_+$. Denote the corresponding equivalence class by $\overline{Y} := \{Y' \in \mathcal{H}_0(\mathbb{N}_+) : Y \sim Y'\}$ and set $\overline{\mathcal{H}}_0(\mathbb{N}_+) := \{\overline{Y} : Y \in \mathcal{H}_0(\mathbb{N}_+)\}$.

We view $\overline{Y}_t$ as a Markov process in $\overline{\mathcal{H}}_0(\mathbb{N}_+)$.
Let $B = (B_i)_{i\in\mathbb{N}_+}$ be i.i.d. Bernoulli with $\mathbb{P}[B_i = 1] = \overline{x}$. Define

$$\mathrm{Thin}_{\overline{x}}(Y) := \{y \in Y : B \geq y\}.$$

Then

$$\mathbb{P}\big[\mathrm{Thin}_{\overline{X}_0}(\overline{Y}_t) \neq \emptyset\big] = \mathbb{P}\big[\mathrm{Thin}_{\overline{X}_t}(\overline{Y}_0) \neq \emptyset\big],$$

where $(\overline{X}_t)_{t\geq 0}$ solves the mean-field ODE

$$\frac{\partial}{\partial t}\overline{X}_t = \alpha \overline{X}_t^2(1 - \overline{X}_t) - \overline{X}_t.$$

Let $\{\{1\}\}$ denote the simplest possible initial state for $(\overline{Y}_t)_{t\geq 0}$, i.e., the hypergraph with a single vertex and a single hyperedge. Then

$$\mathbb{P}^{\{\{1\}\}}\big[\overline{Y}_t \neq \emptyset\ \forall t \geq 0\big] = z_{\rm upp}(\lambda).$$

Proof

$$\begin{aligned}&\mathbb{P}^{\{\{1\}\}}\big[{\rm Thin}_1(\overline{Y}_t) \neq \emptyset\big]\\ &\quad = \mathbb{P}^1\big[{\rm Thin}_{\overline{X}_t}(\{\{1\}\}) \neq \emptyset\big] = \overline{X}_t \underset{t\to\infty}{\longrightarrow} z_{\rm upp}(\lambda).\end{aligned}$$

∎

Distribution determining functions

The law of an $\mathbb{N}$-valued random variable Y is uniquely determined by the function $\phi : [0,1] \to [0,1]$ defined as

$$\phi(\overline{x}) := \mathbb{P}\big[\mathrm{Thin}_{\overline{x}}(Y) \neq \emptyset\big] = \mathbb{E}\big[1 - (1-\overline{x})^{Y}\big].$$

But the law of an $\overline{\mathcal{H}}_0$-valued random variable $\overline{Y}$ is *not* uniquely determined by the analogue function.

What have we missed?

Coupled processes

Recall that $(X_t)_{t\geq 0}$ is constructed from a stochastic flow $(\mathbf{X}_{s,u})_{s\leq u}$. Using *the same* stochastic flow, we can *couple* processes started in initial states $X_0^1, \ldots, X_0^n$ by setting

$$X_t^k := \mathbf{X}_{0,t}(X_0^k) \qquad (t \geq 0,\ k = 1, \ldots, n).$$

The coupled process $(X_t^1, \ldots, X_t^n)_{t\geq 0}$ is a Markov process. Pathwise duality:

$$\begin{aligned} &1_{\{X_t^1 \geq y \text{ for some } y \in Y_0\}} 1_{\{X_t^2 \geq y \text{ for some } y \in Y_0\}} \\ &\quad = 1_{\{X_0^1 \geq y \text{ for some } y \in Y_t\}} 1_{\{X_0^2 \geq y \text{ for some } y \in Y_t\}} \quad \text{a.s.} \end{aligned}$$

And similarly for three or more coupled processes.

The n-variate mean-field ODE

On the complete graph, let

$$\mu_t^{(n)}(x) := N^{-1} \sum_{i=1}^{n} 1_{\{(X_t^1,\ldots,X_t^n)=x\}} \qquad \big(x \in \{0,1\}^n\big).$$

In the mean-field limit, $(\mu_t^{(n)})_{t\geq 0}$ solves an ODE.

$$\mathrm{Test}_\mu(Y)(k) := 1_{\{B^k \geq y \text{ for some } y \in Y\}} \qquad (k = 1,\ldots,n),$$

where $(B_i)_{i\in\mathbb{N}_+} = (B_i^1,\ldots,B_i^n)_{i\in\mathbb{N}_+}$ are i.i.d. with law μ. Then

$$\mathbb{P}\big[\mathrm{Test}_{\mu_0^{(n)}}(\overline{Y}_t) = x\big] = \mathbb{P}\big[\mathrm{Test}_{\mu_t^{(n)}}(\overline{Y}_0) = x\big] \qquad \big(x \in \{0,1\}^n\big).$$

The n-variate equation tells us how $\overline{Y}_t$ reacts to thinnings with *correlated* Bernoulli random variables.

The big picture

Our aim is to understand the asymptotics as $t \to \infty$ of the random map $\mathbf{X}_{0,t}$ on the complete graph K_N when N is large. More precisely, we are interested in the limit $\lim_{t\to\infty} \lim_{N\to\infty}$ (in this order).

This leads us to study the asymptotics as $t \to \infty$ of:

- the *mean-field dual* $\overline{Y}_t$
- solutions $(\mu_t^{(n)})_{t\geq 0}$ of the *n-variate* mean-field ODE.

In particular, we want to understand $\lim_{t\to\infty} \mu_t^{(n)}$ for different initial states $\mu_0^{(n)}$ and what they tell us about $\overline{Y}_t$.

Moment measures

Let Ber_z denote the Bernoulli distribution with mean z, i.e., $\mathrm{Ber}_z(0) := 1 - z$ and $\mathrm{Ber}_z(1) := z$.
For any probability measure μ on $[0,1]$, define $\mu^{(n)}$ on $\{0,1\}^n$ by

$$\mu^{(n)}(x^1,\ldots,x^n) := \int \mu(\mathrm{d}\omega) \prod_{k=1}^{n} \mathrm{Ber}_\omega(x^k).$$

Then $\mu^{(n)}$ is the *n-th moment measure* of μ.

Examples: For $z \in [0,1]$, define

$$\underline{\mu}_z := \delta_z \quad \text{and} \quad \overline{\mu}_z := (1-z)\delta_0 + z\delta_1.$$

Then

$$\left.\begin{aligned} \underline{\mu}_z^{(n)} &= \mathbb{P}\big[(X^1,\ldots,X^n) \in \cdot\,\big], \\ \overline{\mu}_z^{(n)} &= \mathbb{P}\big[(X,\ldots,X) \in \cdot\,\big], \end{aligned}\right\} \quad X, X^1,\ldots,X^n \text{ i.i.d. } \mathrm{Ber}_z.$$

The higher-level ODE

Define $\psi : \mathcal{P}([0,1]) \to \mathcal{P}([0,1])$ by

$$\psi(\mu) := \mathbb{P}\big[\omega_1 + (1-\omega_1)\omega_2\omega_3 \in \cdot\,\big] \quad \text{with } \omega_1, \omega_2, \omega_3 \text{ i.i.d. } \mu.$$

Proposition If $(\mu_t)_{t\geq 0}$ solves the *higher-level ODE*

$$\tfrac{\partial}{\partial t}\mu_t = \alpha\big(\psi(\mu_t) - \mu_t\big) + \big(\delta_0 - \mu_t\big),$$

then its n-th moment measures $(\mu_t^{(n)})_{t\geq 0}$ solve the n-variate ODE.

Remark Not every solution of the n-variate ODE arises in this way.

Upper and lower invariant measures

For measures μ, ν on $[0,1]$, define the *convex order*

$$\mu \leq_{\mathrm{cv}} \nu \quad \Leftrightarrow \quad \int f \mathrm{d}\mu \leq \int f \mathrm{d}\nu \quad \forall \text{convex } f.$$

Let $(\underline{\mu}_{z,t})_{t \geq 0}$ denote the solution of the higher-level ODE with initial state $\underline{\mu}_{z,0} := \underline{\mu}_z$.

Proposition If $z = z_{\mathrm{low}}, z_{\mathrm{mid}}, z_{\mathrm{upp}}$ is a fixed point of the mean-field ODE, then

(a) $\overline{\mu}_z$ is a fixed point of the higher-level ODE.

(b) There exists a fixed point $\underline{\nu}_z$ of the higher-level ODE such that $\underline{\mu}_{z,t} \underset{t \to \infty}{\Longrightarrow} \underline{\nu}_z$.

(c) Any fixed point ν of the higher-level ODE with mean z satisfies $\underline{\nu}_z \leq_{\mathrm{cv}} \nu \leq_{\mathrm{cv}} \overline{\mu}_z$.

Fixed points and domains of attraction

Write $\overline{\mu}_{\rm low} := \overline{\mu}_{z_{\rm low}}$ etc.

Proposition $\underline{\nu}_{\rm low} = \overline{\mu}_{\rm low}$ and $\underline{\nu}_{\rm upp} = \overline{\mu}_{\rm upp}$, but $\underline{\nu}_{\rm mid} \neq \overline{\mu}_{\rm mid}$.

Theorem Let $\alpha > 4$ and let $(\mu_t)_{t\geq 0}$ be a solution of the higher-level ODE with $\int x\,\mu_0(\mathrm{d}x) = z$.

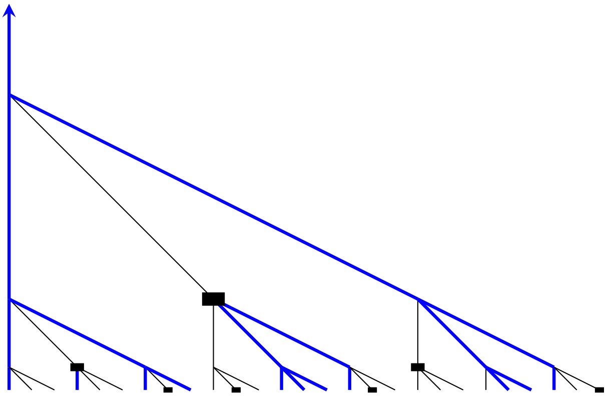
(a) If $z > z_{\rm mid}$, then $\mu_t \underset{t\to\infty}{\Longrightarrow} \overline{\mu}_{\rm upp}$.

(b) If $z < z_{\rm mid}$, then $\mu_t \underset{t\to\infty}{\Longrightarrow} \overline{\mu}_{\rm low}$.

(c) If $z = z_{\rm mid}$ and $\mu_0 \neq \overline{\mu}_{\rm mid}$, then $\mu_t \underset{t\to\infty}{\Longrightarrow} \underline{\nu}_{\rm mid}$.

(d) If $\mu_0 = \overline{\mu}_{\rm mid}$, then $\mu_t = \overline{\mu}_{\rm mid}$ $\forall t \geq 0$.

A random tree process



In a 3-regular tree, place death symbols with probability $1/(1+\alpha)$ and color the leaves blue with probability $z_{\rm mid}$. In the limit of an infinite tree, this yields a stationary picture. Such a process is called a *Random Tree Process*. A Markov chain with *tree-like time*.

Endogeny

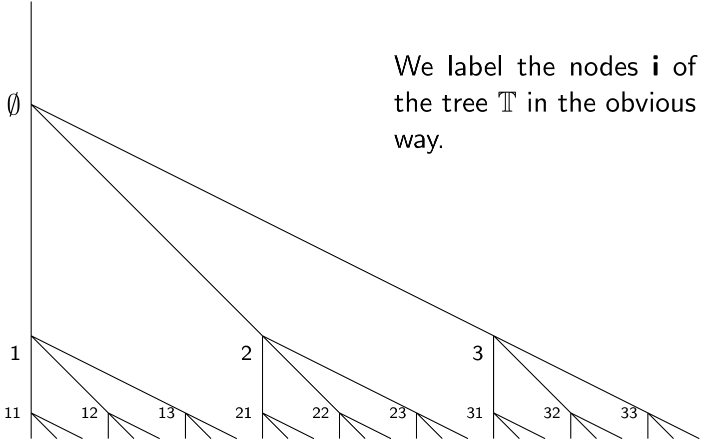
Each fixed point $z = z_{\rm low}, z_{\rm mid}, z_{\rm upp}$ of the mean-field ODE defines a Random Tree Process (RTP).

Following Aldous and Bandyopahyay [AB '04], we call a RTP *endogenous* if the state at the root (blue or black) is a function of the random variables at the nodes (death or coop maps).

Proposition The RTPs corresponding to $z_{\rm low}$ and $z_{\rm upp}$ are endogenous, but the RTP corresponding to $z_{\rm mid}$ is not.

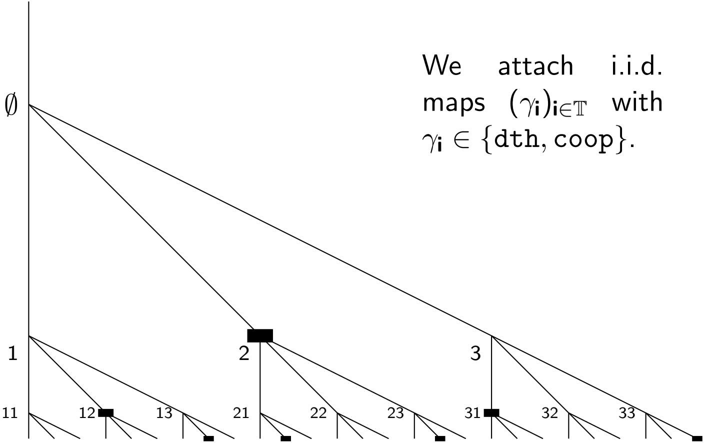
Proof Following [AB '04], this follows from an analysis of the bivariate ODE. Alternatively, for $z_{\rm low}$ and $z_{\rm upp}$, in [AB '04] it is proved that for monotone systems, the RTP corresponding to a lower or upper fixed point is always endogenous. ∎

A random tree process



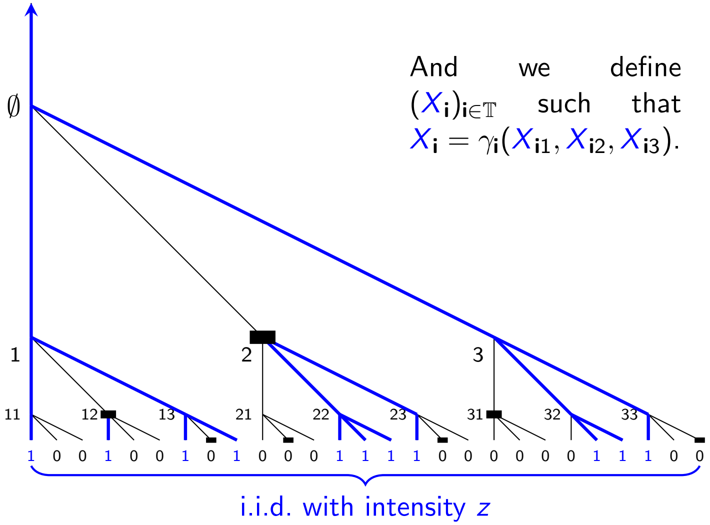
We label the nodes $\mathbf{i}$ of the tree $\mathbb{T}$ in the obvious way.

A random tree process



We attach i.i.d. maps $(\gamma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ with $\gamma_{\mathbf{i}} \in \{\mathtt{dth}, \mathtt{coop}\}$.

A random tree process



And we define $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ such that $X_{\mathbf{i}} = \gamma_{\mathbf{i}}(X_{\mathbf{i}1}, X_{\mathbf{i}2}, X_{\mathbf{i}3})$.

A higher level RTP

The Random Tree Process $(\gamma_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is endogenous iff

$$X_\emptyset = \mathbb{P}\big[X_\emptyset = 1 \,\big|\, (\gamma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\big] \quad \text{a.s.}$$

Observation: Setting

$$\omega_{\mathbf{i}} := \mathbb{P}\big[X_{\mathbf{i}} = 1 \,\big|\, (\gamma_{\mathbf{ij}})_{\mathbf{j}\in\mathbb{T}}\big]$$

defines a *higher-level RTP* $(\check{\gamma}_{\mathbf{i}}, \omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the *higher-level maps*

$$\check{\mathrm{coop}}(\omega_1, \omega_2, \omega_3) = \omega_1 + (1-\omega_1)\omega_2\omega_3 \quad \text{and} \quad \check{\mathtt{dth}}(\omega_1, \omega_2, \omega_3) := 0.$$

Moreover

$$\underline{\nu}_{\mathrm{mid}} = \mathbb{P}[\omega_\emptyset \in \cdot\,].$$

A discrete evolution

On finite trees, if we assign the leaves i.i.d. $\omega_{\mathbf{i}}$ with law μ_0, then t levels above this the $\omega_{\mathbf{i}}$ are i.i.d. with law μ_n, where

$$\mu_n = \frac{\alpha}{\alpha+1}\psi(\mu_{n-1}) + \frac{1}{\alpha+1}\delta_0.$$

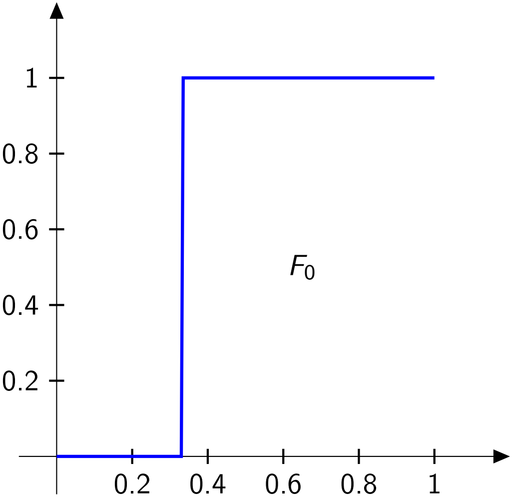
We start with $\mu_0 = \delta_{z_{\rm mid}}$ and plot the distribution function

$$F_n(s) := \mu\big([0,s]\big) \qquad \big(s \in [0,1]\big)$$

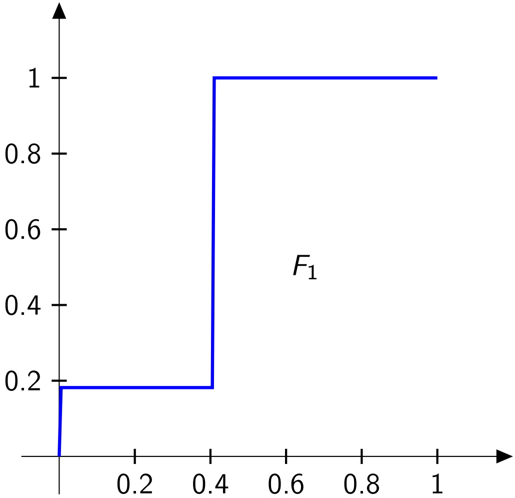
for the parameters $\alpha = 9/2$, $z_{\rm mid} = 1/3$, $z_{\rm upp} = 2/3$.

As $n \to \infty$, this converges to the distribution function of $\underline{\nu}_{\rm mid}$.

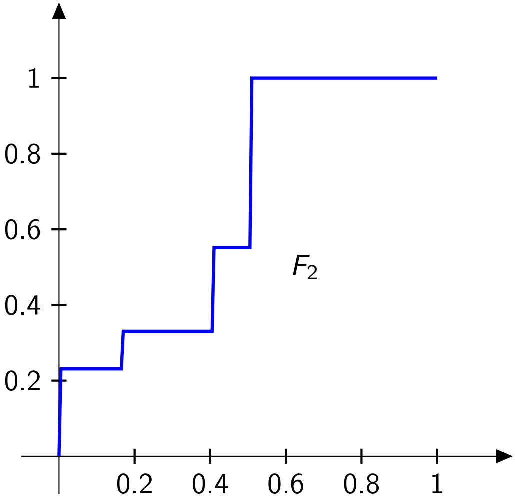
Numerical results



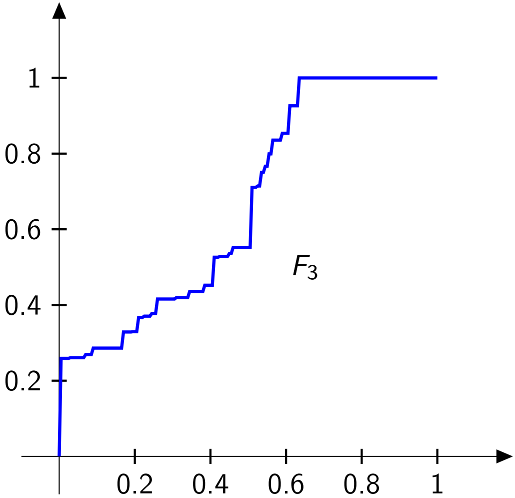
Numerical results



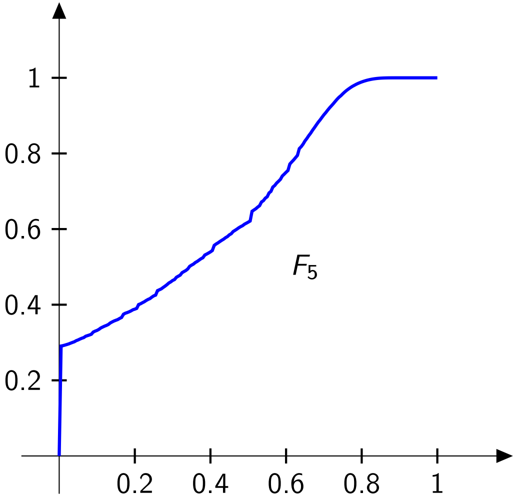
Numerical results



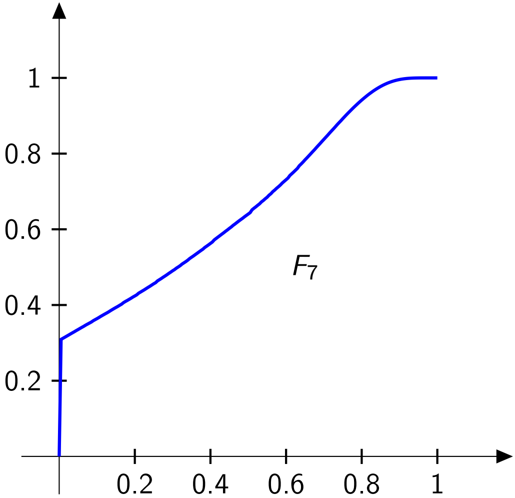
Numerical results



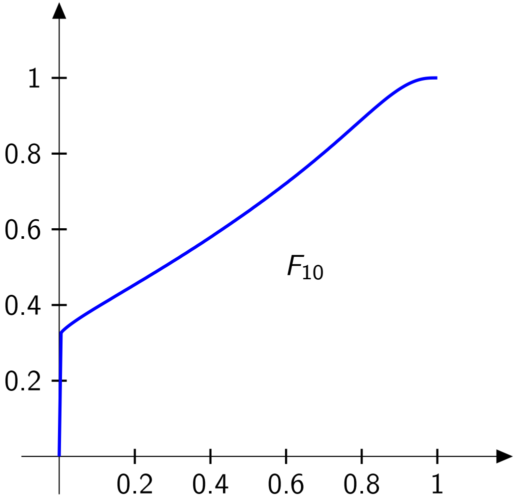
Numerical results



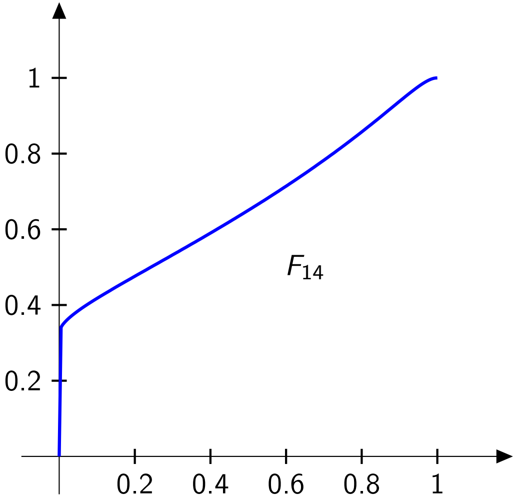
Numerical results



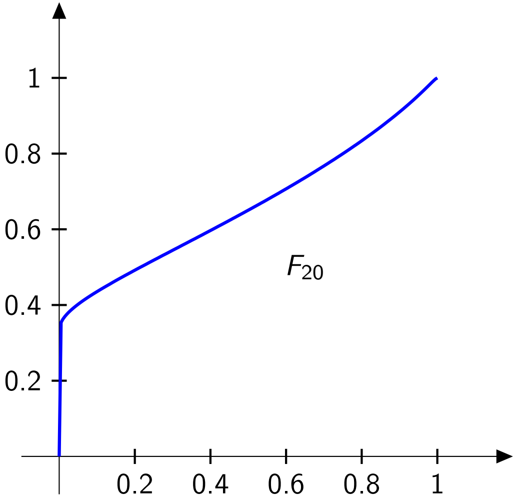
Numerical results



Numerical results



Numerical results



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