Frozen percolation on the binary tree

Jan M. Swart (Czech Academy of Sciences)

joint with Balázs Ráth, and Tamás Terpai (Budapest)

Friday, January 3rd, 2020

Percolation on 3-regular graphs

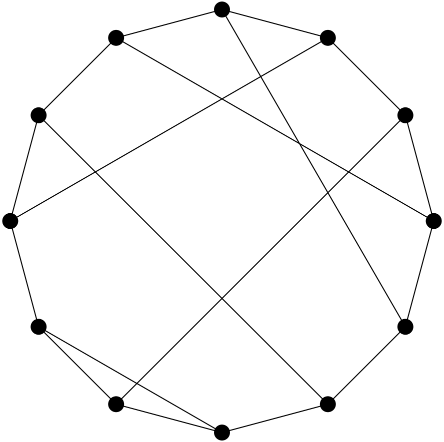
Let $G_n = (V_n, E_n)$ be random, uniformly chosen, 3-regular graphs with n vertices (n is even).

Let $(\tau_e)_{e \in E_n}$ be i.i.d. uniformly distributed $[0,1]$-valued random variables attached to the edges.

Initially, all edges are closed. At time τ_e, the edge e opens.

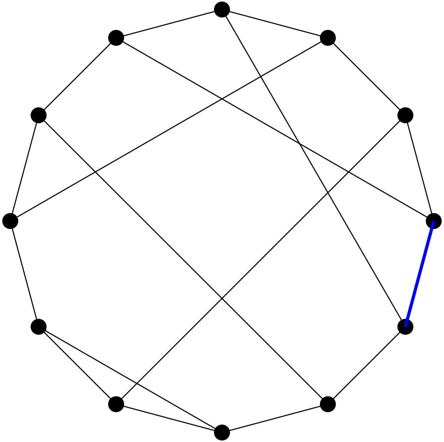
Known fact For large n, a giant component forms at time $t = \frac{1}{2}$.

Percolation on 3-regular graphs



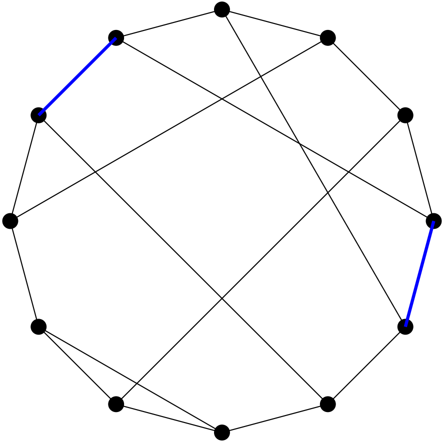
$t = 0$

Percolation on 3-regular graphs



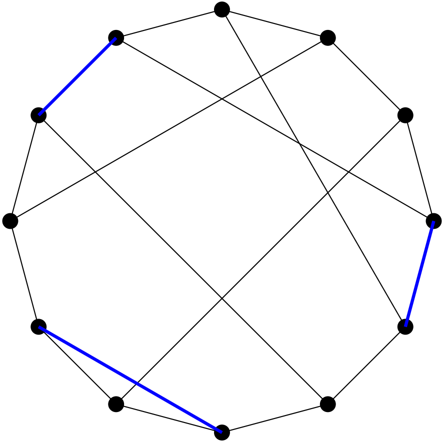
$t = 0.06$

Percolation on 3-regular graphs



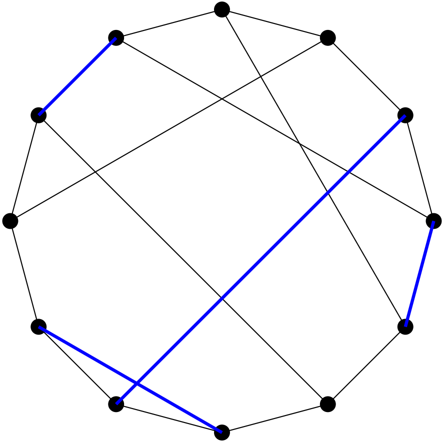
$t = 0.11$

Percolation on 3-regular graphs



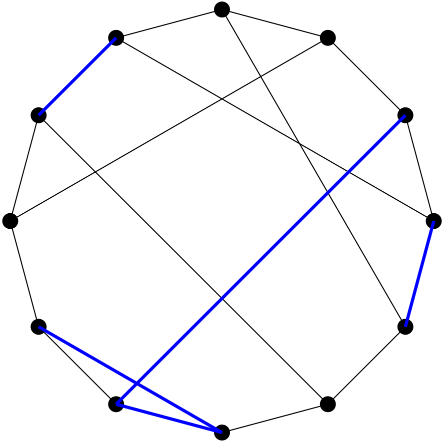
$t = 0.17$

Percolation on 3-regular graphs



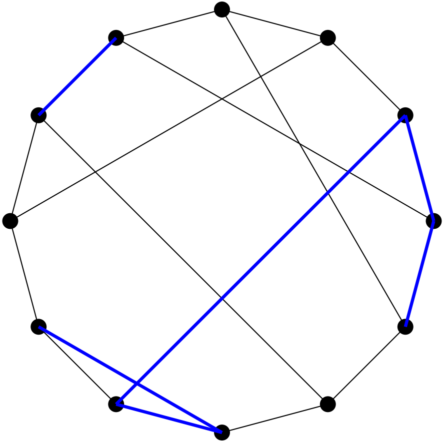
$t = 0.22$

Percolation on 3-regular graphs



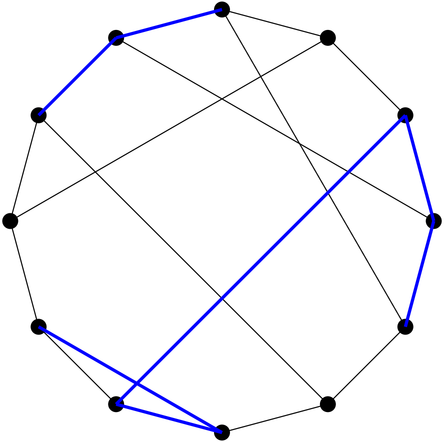
$t = 0.28$

Percolation on 3-regular graphs



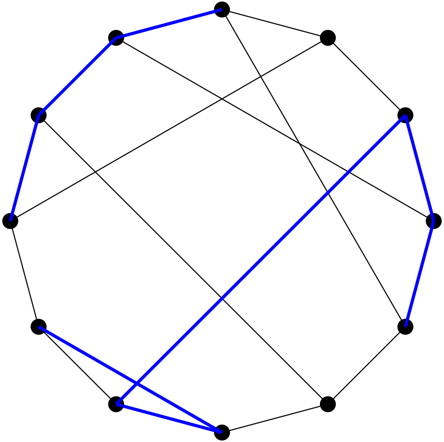
$t = 0.33$

Percolation on 3-regular graphs



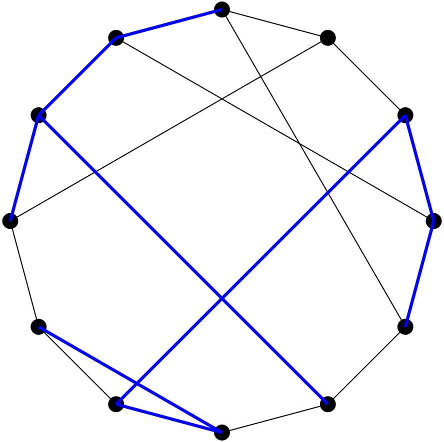
$t = 0.39$

Percolation on 3-regular graphs



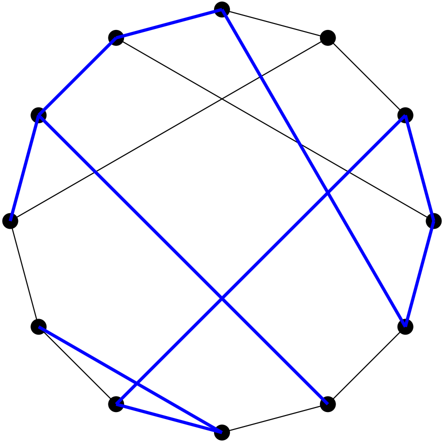
$t = 0.45$

Percolation on 3-regular graphs



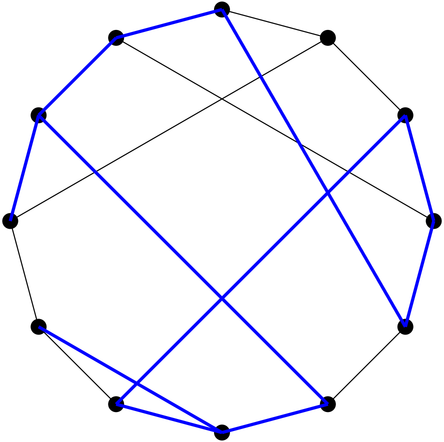
$t = 0.50$

Percolation on 3-regular graphs



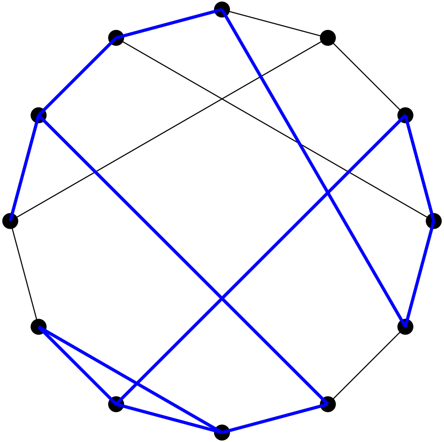
$t = 0.56$

Percolation on 3-regular graphs



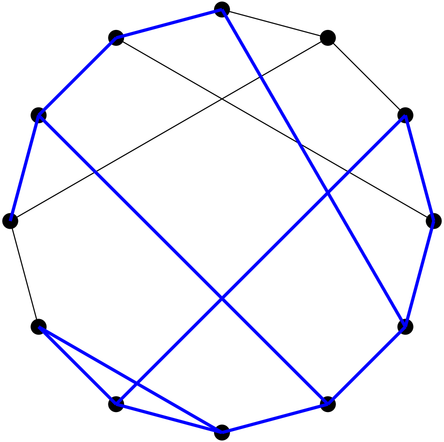
$t = 0.61$

Percolation on 3-regular graphs



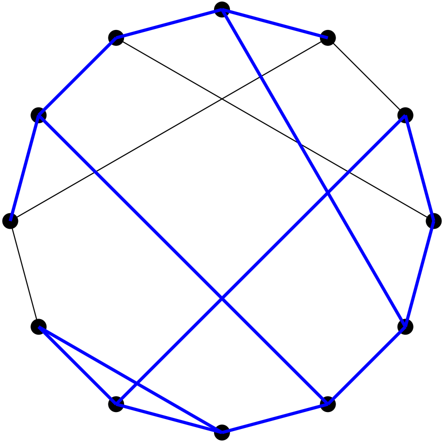
$t = 0.67$

Percolation on 3-regular graphs



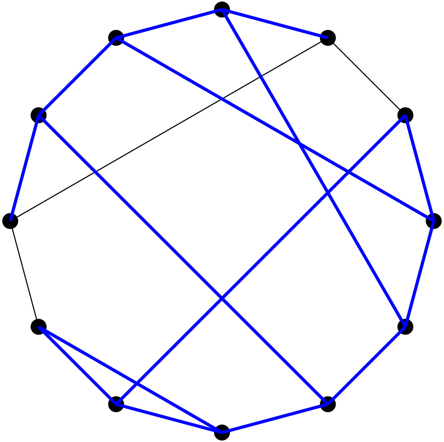
$t = 0.72$

Percolation on 3-regular graphs



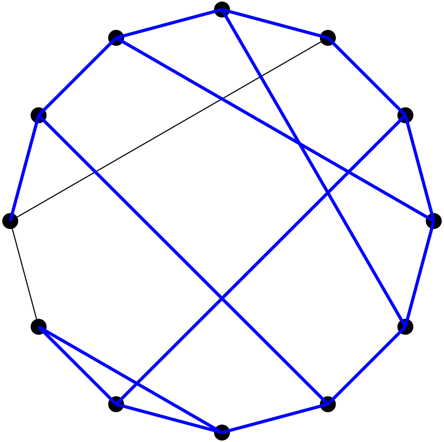
$t = 0.78$

Percolation on 3-regular graphs



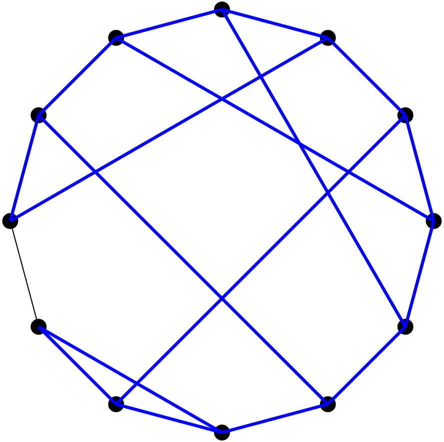
$t = 0.83$

Percolation on 3-regular graphs



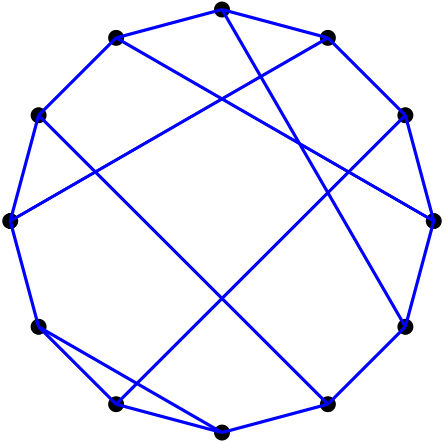
$t = 0.89$

Percolation on 3-regular graphs



$t = 0.93$

Percolation on 3-regular graphs

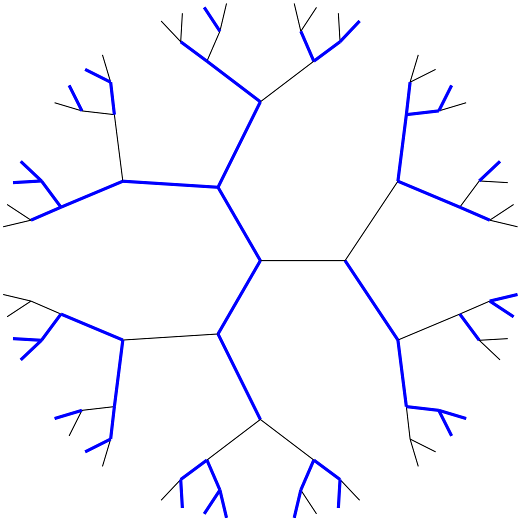


$t = 1$

The local limit

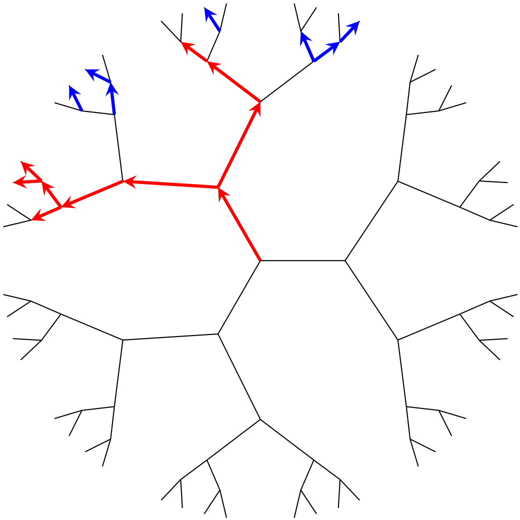
To understand why $p_{\mathrm{c}} = 1/2$ is the critical point, we look at the *weak local limit* of uniformly chosen, 3-regular graphs, as the number of vertices $n \to \infty$.

Frozen percolation on the 3-regular tree



The weak local limit of uniform 3-regular graphs is the 3-regular tree.

Frozen percolation on the 3-regular tree



Locally, we see an infinite component iff a certain branching process survives.

This branching process is supercritical for $p > 1/2$.

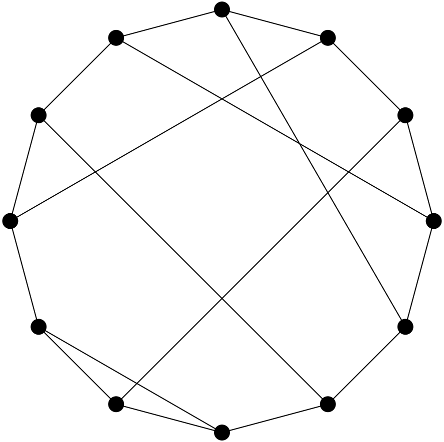
Frozen percolation on 3-regular graphs

Let $(\sigma_v)_{v \in V_n}$ be i.i.d. exponentially distributed times with mean $1/\lambda_n$, attached to the vertices.

- At time σ_v, the vertex v is struck by *lightning*.
- At time τ_e, the edge e opens *only* if the open components at either endvertex have not been struck by lightning. In the opposite case, it freezes.

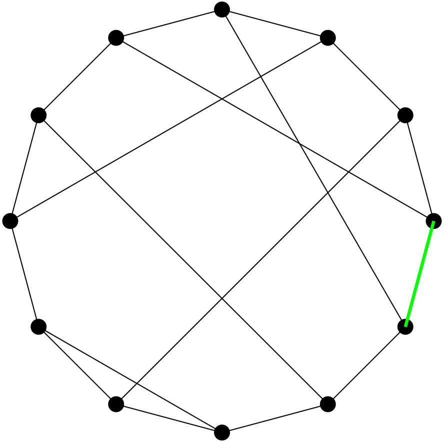
We are interested in $n^{-1} \ll \lambda_n \ll 1$, which means that w.h.p., small components are not struck by lightning, but giant components are struck immediately.

Frozen percolation on 3-regular graphs



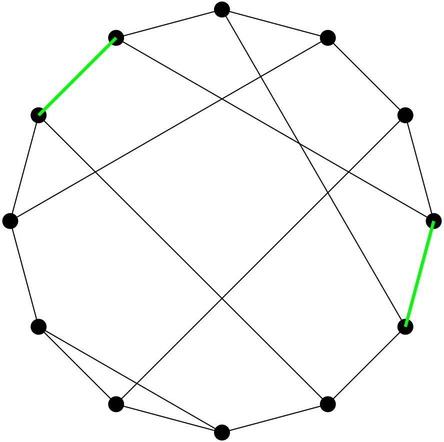
$t = 0$

Frozen percolation on 3-regular graphs



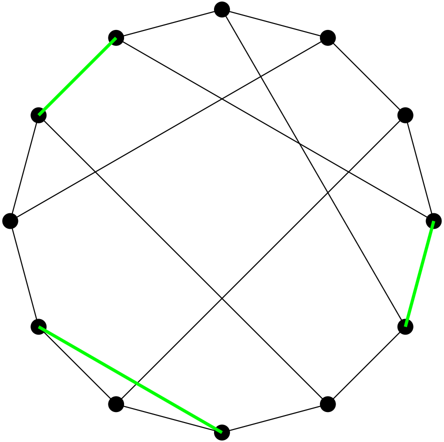
$t = 0.05$

Frozen percolation on 3-regular graphs



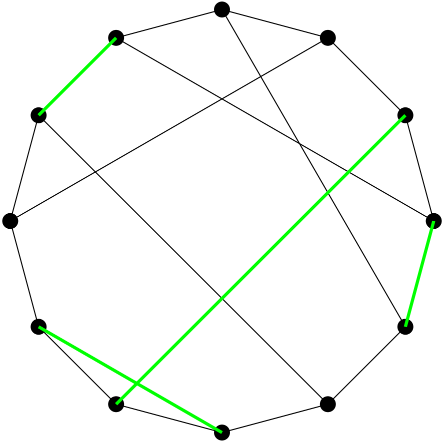
$t = 0.1$

Frozen percolation on 3-regular graphs



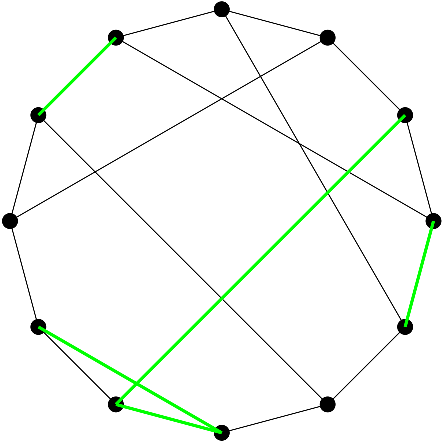
$t = 0.15$

Frozen percolation on 3-regular graphs



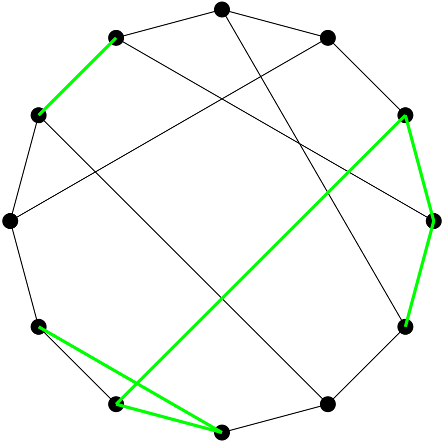
$t = 0.2$

Frozen percolation on 3-regular graphs



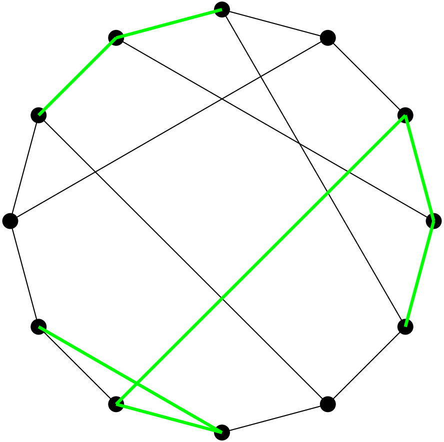
$t = 0.25$

Frozen percolation on 3-regular graphs



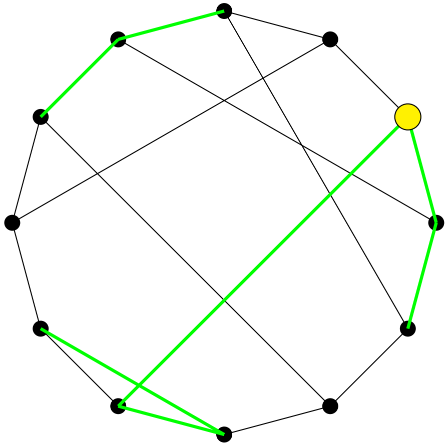
$t = 0.3$

Frozen percolation on 3-regular graphs



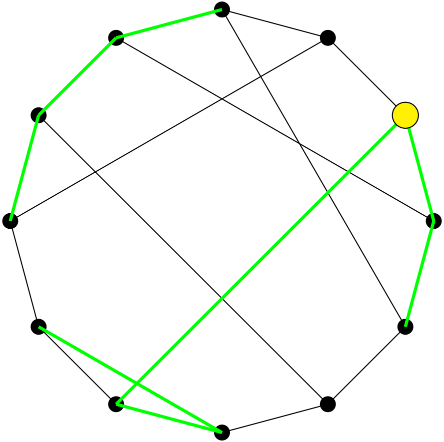
$t = 0.35$

Frozen percolation on 3-regular graphs



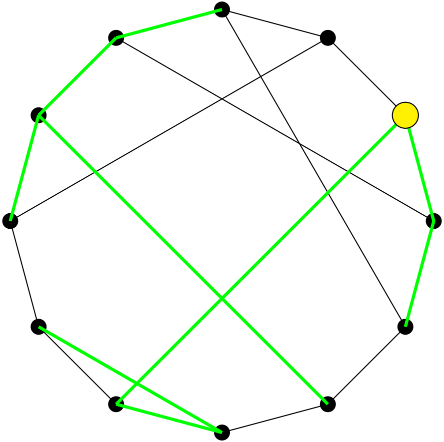
$t = 0.4$

Frozen percolation on 3-regular graphs



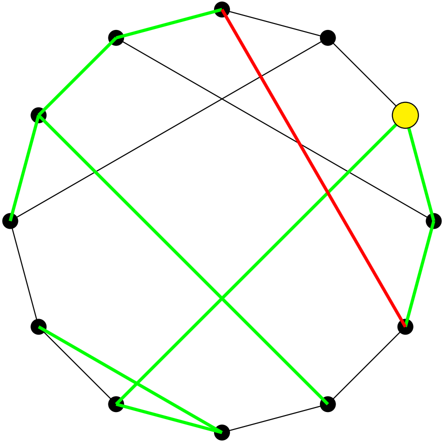
$t = 0.45$

Frozen percolation on 3-regular graphs



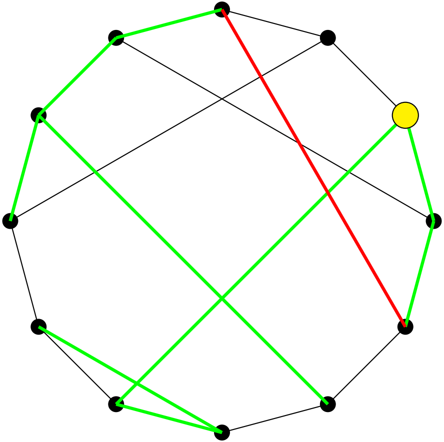
$t = 0.5$

Frozen percolation on 3-regular graphs



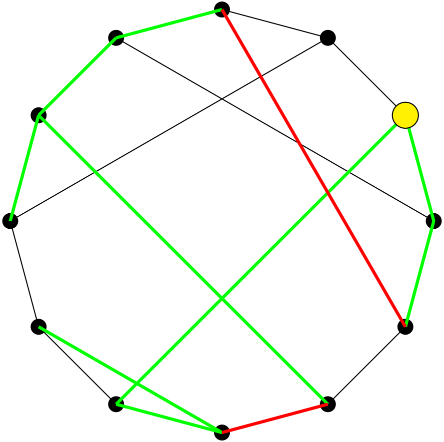
$t = 0.55$

Frozen percolation on 3-regular graphs



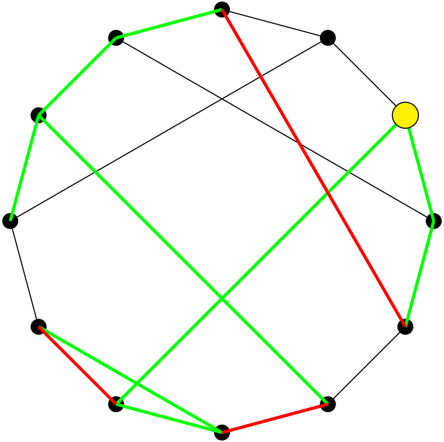
$t = 0.6$

Frozen percolation on 3-regular graphs



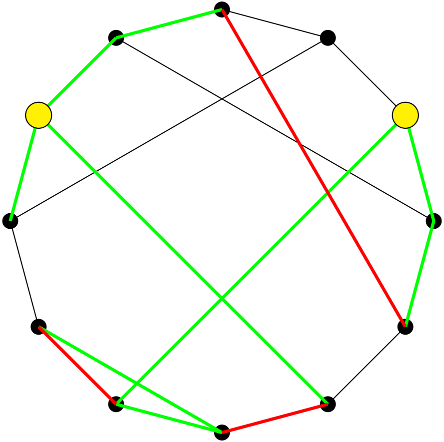
$t = 0.65$

Frozen percolation on 3-regular graphs



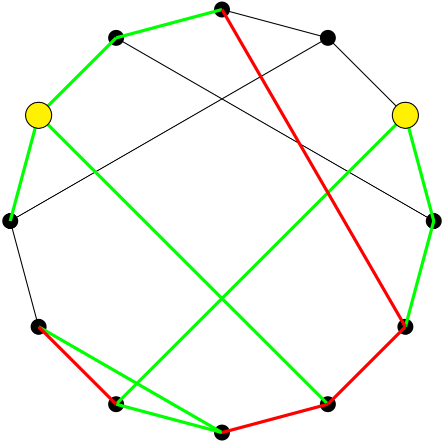
$t = 0.7$

Frozen percolation on 3-regular graphs



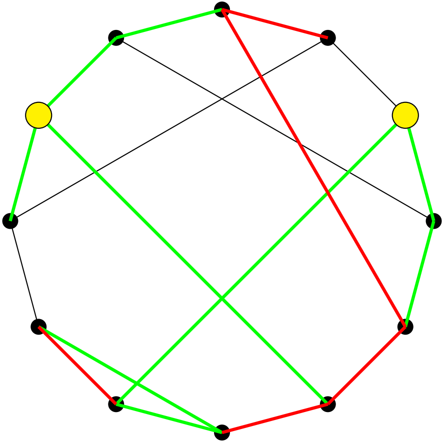
$t = 0.74$

Frozen percolation on 3-regular graphs



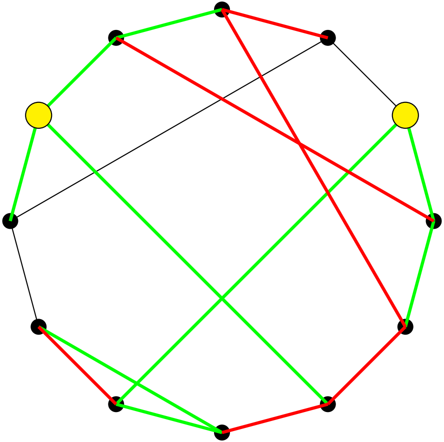
$t = 0.78$

Frozen percolation on 3-regular graphs



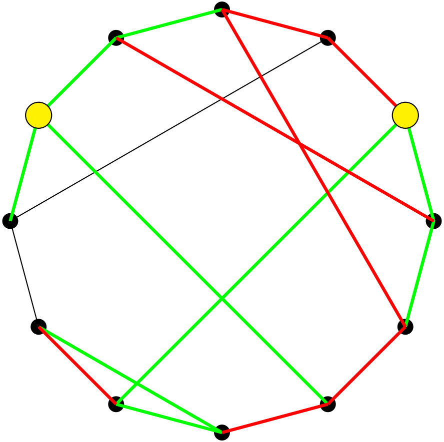
$t = 0.82$

Frozen percolation on 3-regular graphs



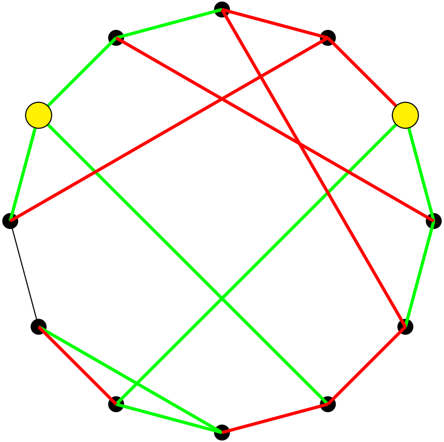
$t = 86$

Frozen percolation on 3-regular graphs



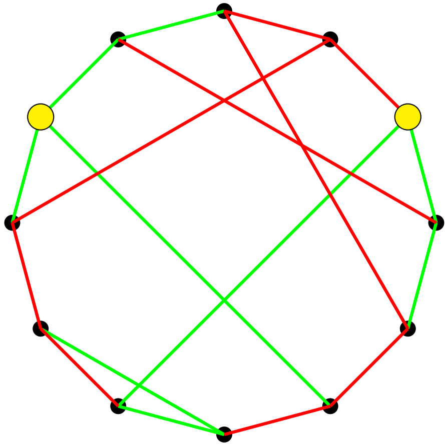
$t = 0.9$

Frozen percolation on 3-regular graphs



$t = 0.95$

Frozen percolation on 3-regular graphs



$t = 1$

Frozen percolation on the 3-regular tree

The *local weak limit* of G_n is the infinite 3-regular tree $G = (V, E)$.

Let $\tau = (\tau_e)_{e \in E}$ be a collection of i.i.d. uniformly distributed $[0, 1]$-valued *activation times* attached to the edges.

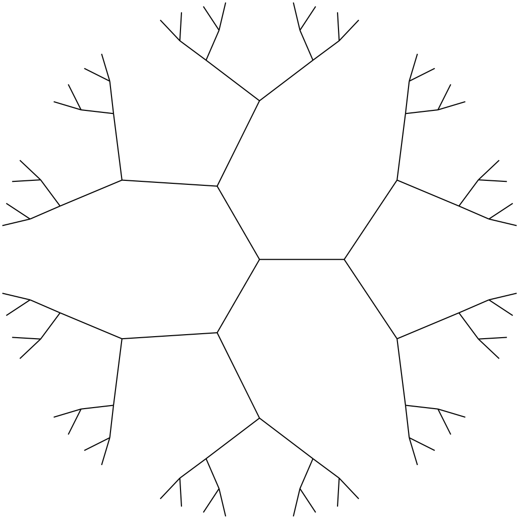
$$E_t := \{e \in E : \tau_e \leq t\}.$$

Aldous (2000) has constructed a random subset $F \subset E$ such that:

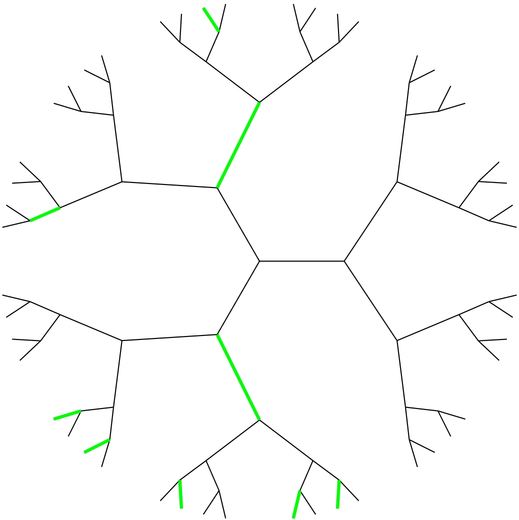
- $e \in F$ if and only if at least one endvertex of e is part of an infinite cluster of $E_{\tau_e} \backslash (F \cup \{e\})$.
- The law of (τ, F) is invariant under automorphisms of the tree.

$E \backslash E_t =$ *closed edges*, $E_t \backslash F =$ *open edges*, $E_t \cap F =$ *frozen edges*.

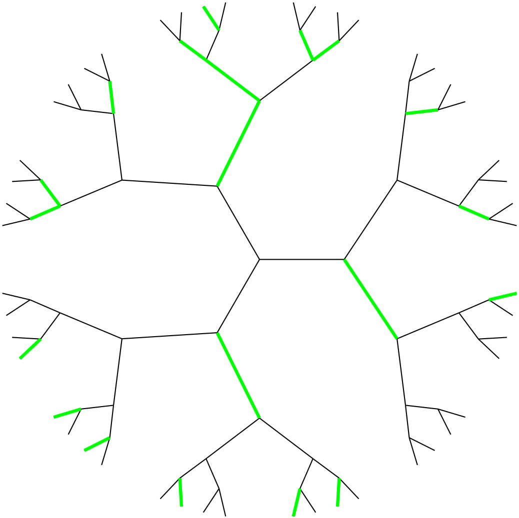
Frozen percolation on the 3-regular tree



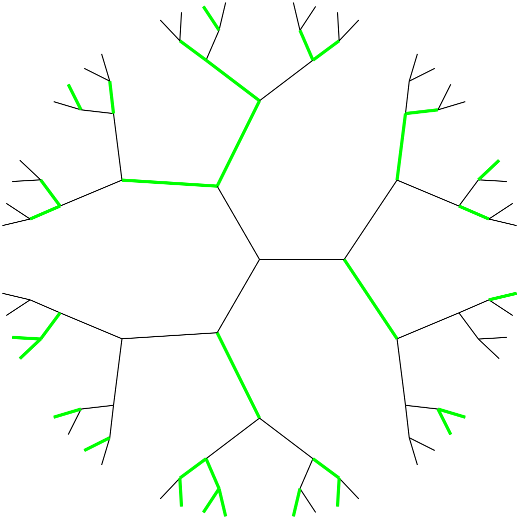
Frozen percolation on the 3-regular tree



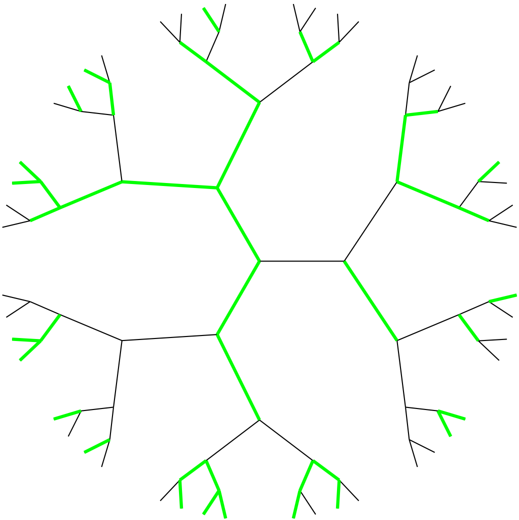
Frozen percolation on the 3-regular tree



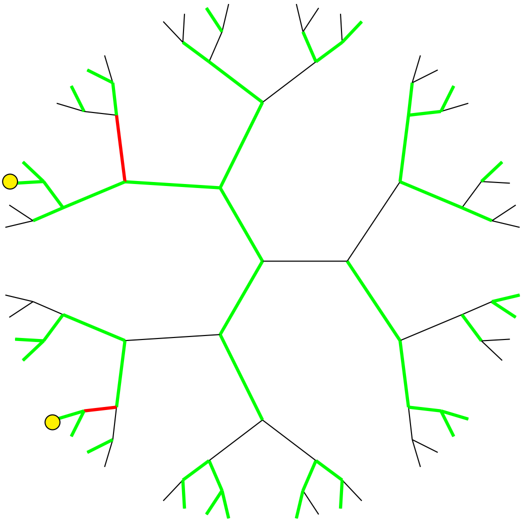
Frozen percolation on the 3-regular tree



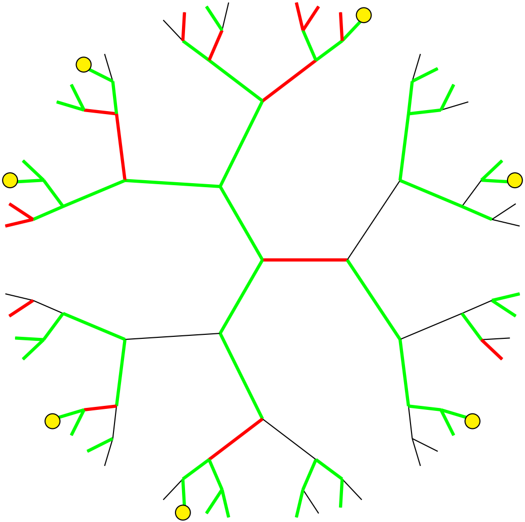
Frozen percolation on the 3-regular tree



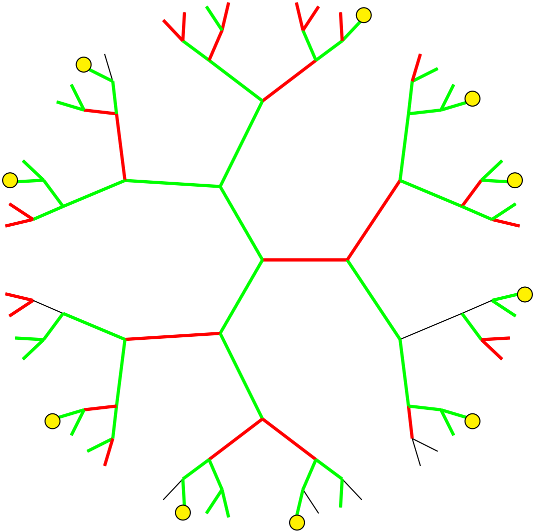
Frozen percolation on the 3-regular tree



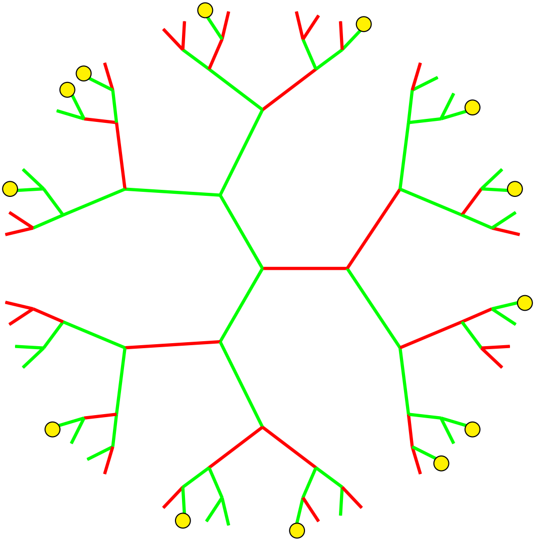
Frozen percolation on the 3-regular tree



Frozen percolation on the 3-regular tree



Frozen percolation on the 3-regular tree



Frozen percolation on the complete graph

Remark Instead of a random 3-regular graph, we could have started with the complete graph.
In this case, it is more natural to take $(\tau_e)_{e \in E}$ uniformly $[0, n]$-valued.

Frozen percolation on the complete graph has been studied by Balázs Ráth (2009). Merle and Normand (2015) studied a configuration model with freezing.

The local limit of the complete graph equipped with i.i.d. times $(\tau_e)_{e \in E}$ is called the PWIT (Aldous & Steele, 2004).

Frozen percolation on the complete graph models the growth of polymers. Giant polymers are part of the gel and cannot grow further.
Related to the discrete Smoluchowski coagulation equation with a multiplicative kernel.

Exhibits *self-organized criticality*.

Frozen percolation on the oriented 3-regular tree

Each *undirected edge* $\{v,w\}\in E$ corresponds to two *directed edges* $(v,w),(w,v)\in\vec{E}$.

For $A\subset\vec{E}$, write $v\stackrel{A}{\longrightarrow}w$ if there exist $v=v_0,\ldots,v_n=w$ $(n\geq 0)$, such that $(v_{k-1},v_k)\in A$ $(k=1,\ldots,n)$.

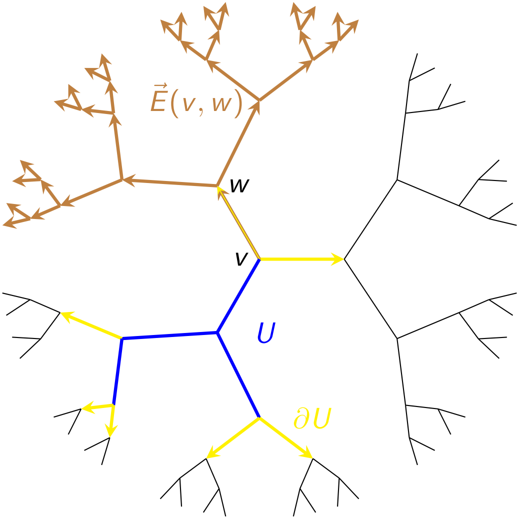
For each finite subtree $U\subset T$, let

$$\partial U:=\big\{(v,w):v\in U,\ w\in T\backslash U\big\}.$$

For each $(v,w)\in\vec{E}$, let

$$\begin{aligned}\vec{E}(v,w):=\big\{(v',w')\in\vec{E}:\exists v_0,\ldots,v_n\\ \text{s.t. } (v,w)=(v_0,v_1),\ (v_{n-1},v_n)=(v',w'),\\ (v_{k-1},v_k)\in\vec{E}\ (k=1,\ldots,n),\ \text{and } v_k\neq v_{k-2}\ (k=2,\ldots,n)\big\}.\end{aligned}$$

Frozen percolation on the 3-regular tree



Frozen percolation on the 3-regular graph

Let $\tau=(\tau_e)_{e\in E}$ and $\tau(v,w):=\big(\tau_{\{v',w'\}}\big)_{(v',w')\in\vec E_{(v,w)}}$.
$\vec E_t:=\big\{(v,w)\in\vec E:\tau_{\{v,w\}}\leq t\big\}$ and $\vec E_t(v,w):=\vec E_t\cap\vec E(v,w)$.

Theorem There exists a random subset $\vec F\subset\vec E$ such that:

(i) $(v,w)\in\vec F$ if and only if $w\overset{\vec E_{\tau_{\{v,w\}}}(v,w)\backslash\vec F}{\longrightarrow}\infty$.

(ii) The joint law of $(\tau,\vec F)$ is invariant under automorphisms of the tree.

(iii) Let $\vec F(v,w):=\vec F\cap E(v,w)$. Then for each finite subtree (U,E_U), the r.v.'s $\big(\tau(v,w),\vec F(v,w)\big)_{(v,w)\in\partial U}$ are i.i.d. and independent of $(\tau_e)_{e\in E_U}$.

Moreover, (i)–(iii) uniquely determine the joint law of $(\tau,\vec F)$.
Setting $F:=\big\{\{v,w\}\in E:(v,w)\in\vec F\text{ or }(w,v)\in\vec F\big\}$ yields the frozen percolation process of Aldous.

Frozen percolation on the 3-regular graph

Aldous (2000) proved existence and asked:

Question Is $\vec{F}$ measurable w.r.t. the σ-field generated by $\tau = (\tau_e)_{e \in E}$?

Short answer No.

Frozen percolation on the oriented binary tree

Let $\mathbb{T}$ denote the space of all finite words $\mathbf{i} = i_1 \cdots i_n$ $(n \geq 0)$ made up from the alphabet $\{1, 2\}$.

Elements $\mathbf{i} \in \mathbb{T}$ label oriented edges in an infinite binary tree. Let $\mathbb{T}_t := \{\mathbf{i} \in \mathbb{T} : \tau_\mathbf{i} \leq t\}$ and let $\vec{F} \subset \mathbb{T}$ as before.

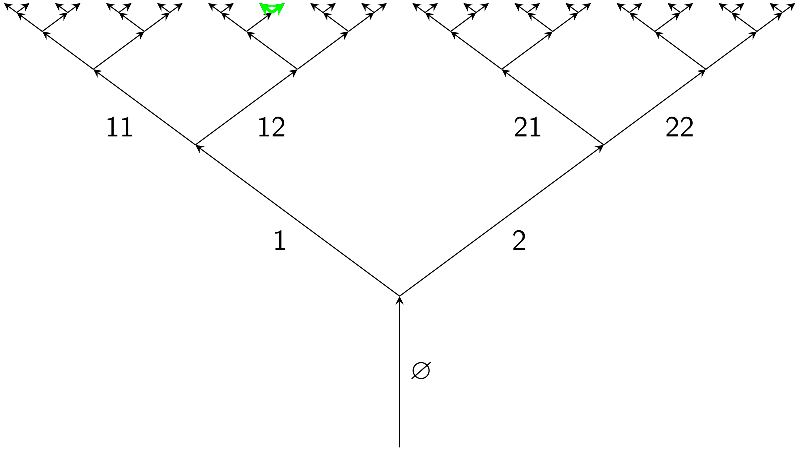
$$X_\mathbf{i} := \inf \Big\{ t \in [0, 1] : \mathbf{i} \xrightarrow{\mathbb{T}_t \backslash \vec{F}} \infty \Big\}$$

with $\inf \emptyset := \infty$. Then

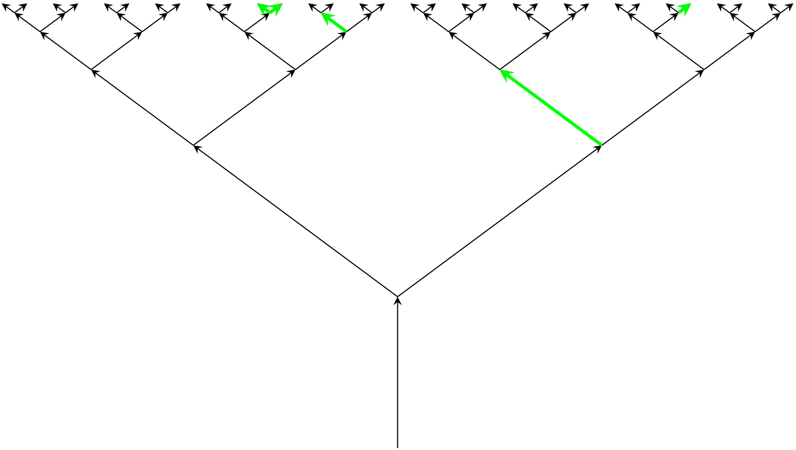
(i) For each $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(X_\mathbf{i})_{\mathbf{i} \in \partial \mathbb{U}}$ are i.i.d. and independent of $(\tau_\mathbf{i})_{\mathbf{i} \in \mathbb{U}}$.

(ii) $X_\mathbf{i} = \Phi[\tau_\mathbf{i}](X_{\mathbf{i}1} \wedge X_{\mathbf{i}2})$ with $\Phi[t](x) := \begin{cases} x & \text{if } x > t, \\ \infty & \text{if } x \leq t. \end{cases}$

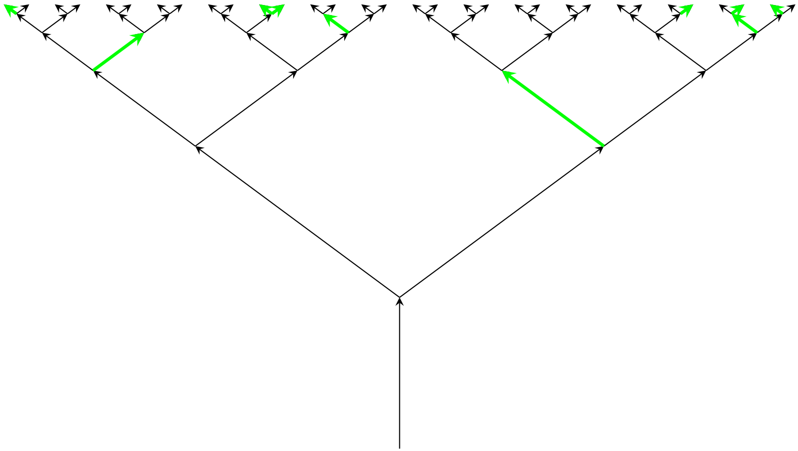
Frozen percolation on the oriented binary tree



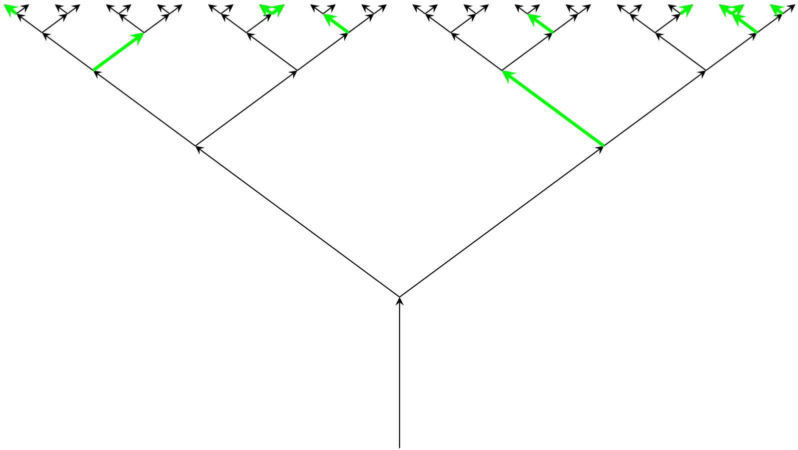
Frozen percolation on the oriented binary tree



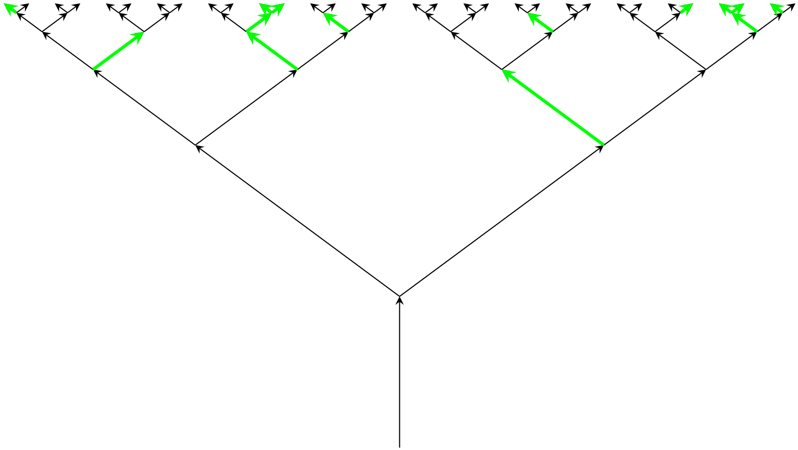
Frozen percolation on the oriented binary tree



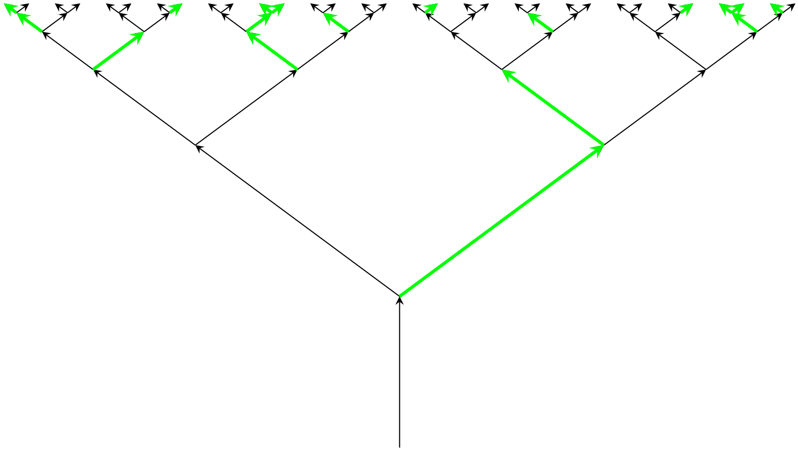
Frozen percolation on the oriented binary tree



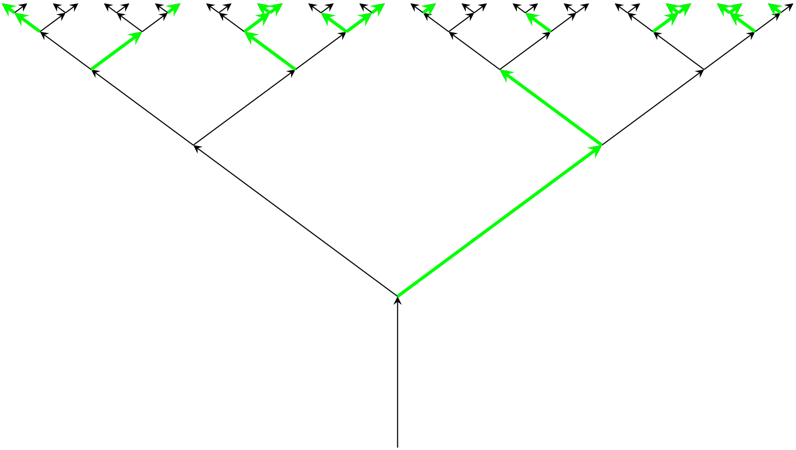
Frozen percolation on the oriented binary tree



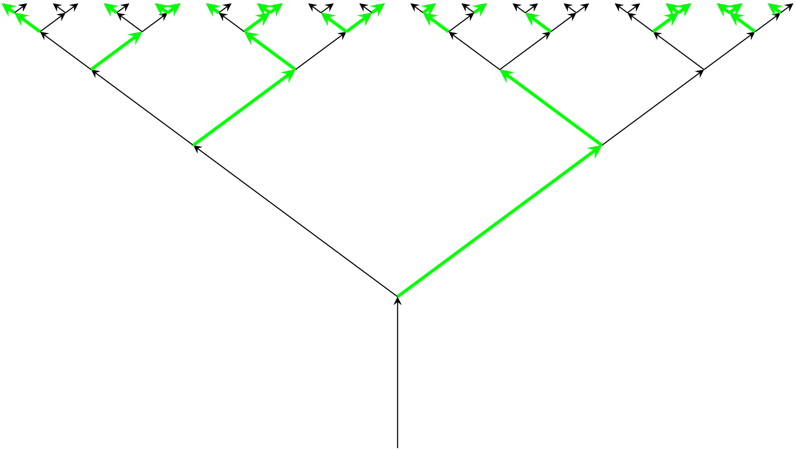
Frozen percolation on the oriented binary tree



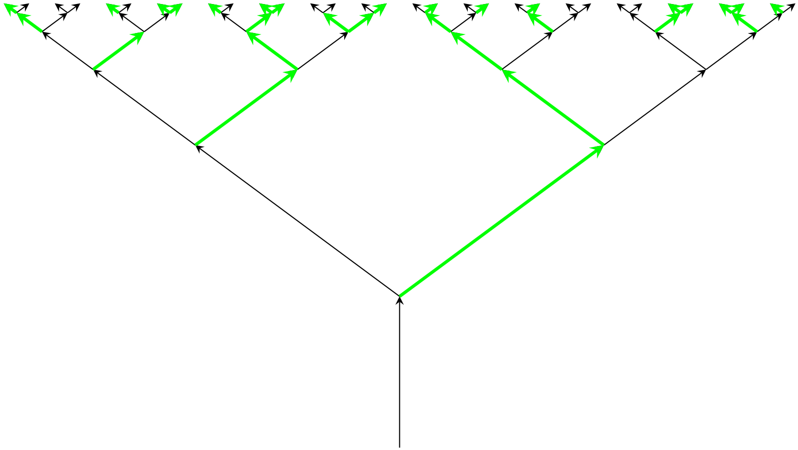
Frozen percolation on the oriented binary tree



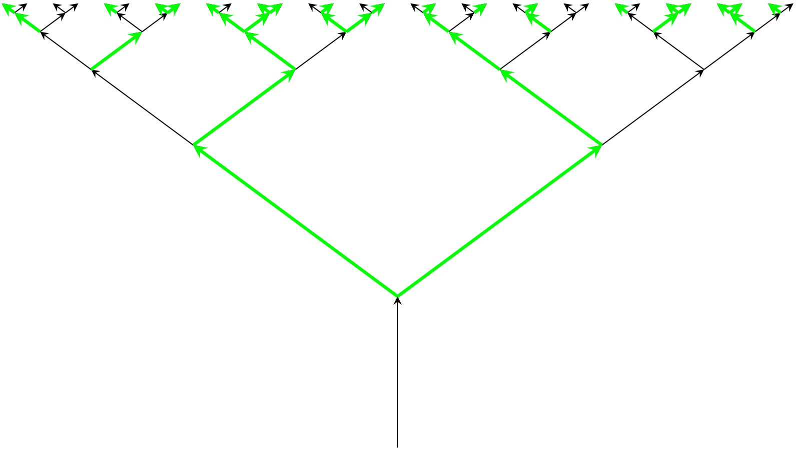
Frozen percolation on the oriented binary tree



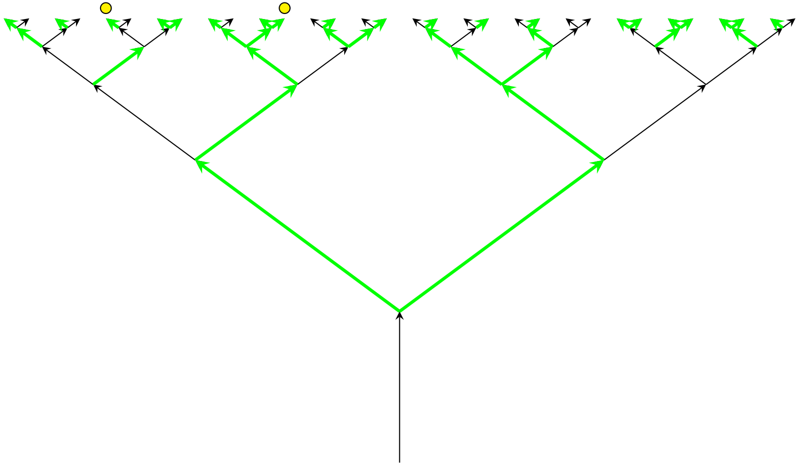
Frozen percolation on the oriented binary tree



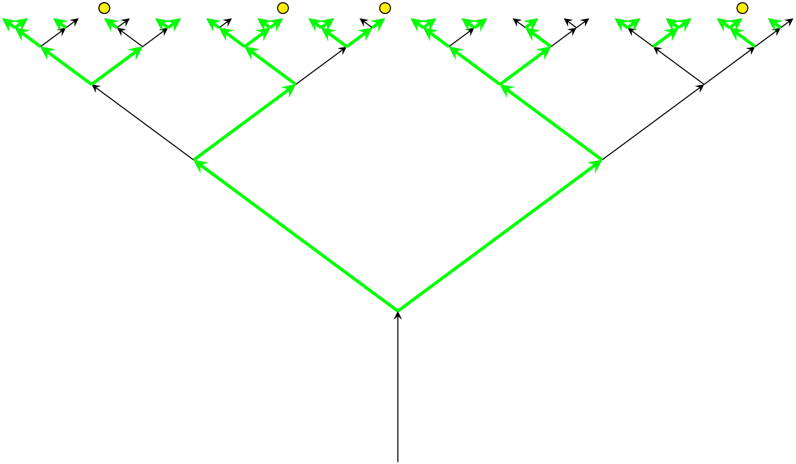
Frozen percolation on the oriented binary tree



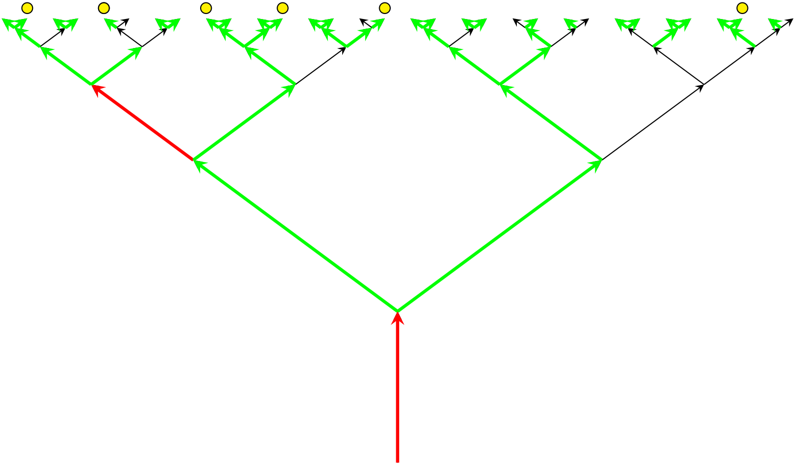
Frozen percolation on the oriented binary tree



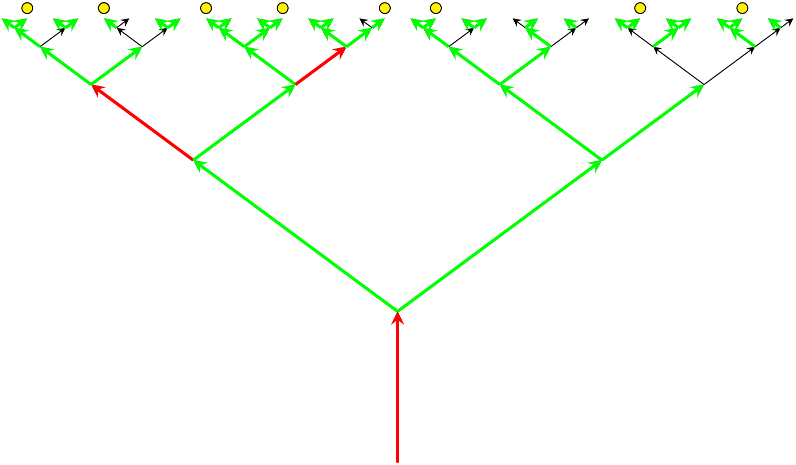
Frozen percolation on the oriented binary tree



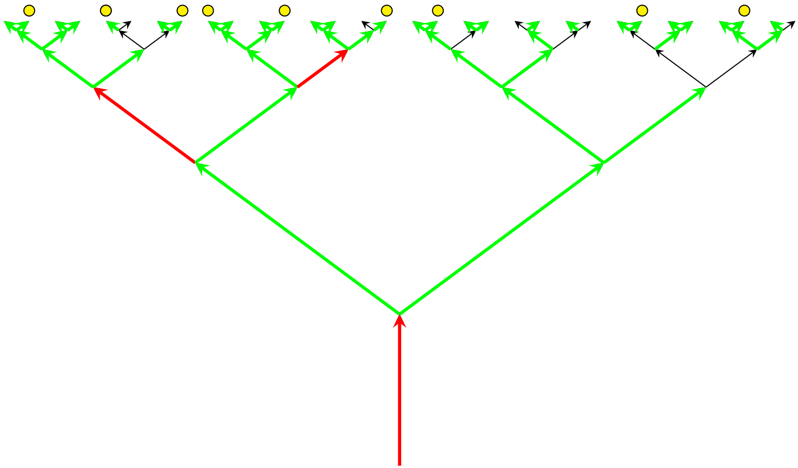
Frozen percolation on the oriented binary tree



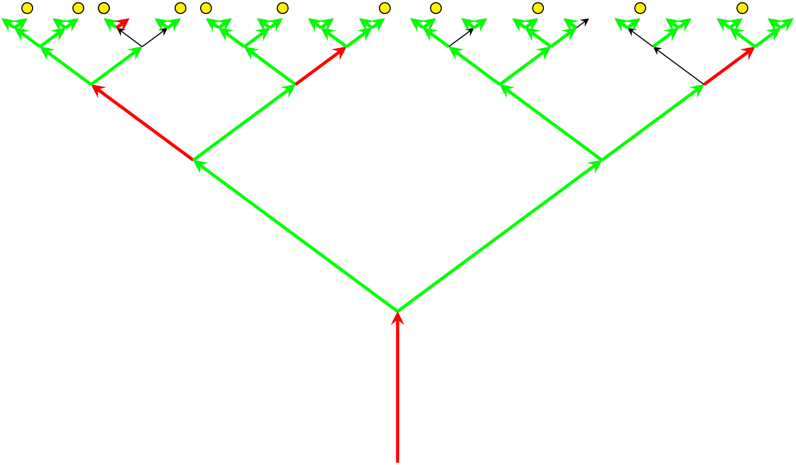
Frozen percolation on the oriented binary tree



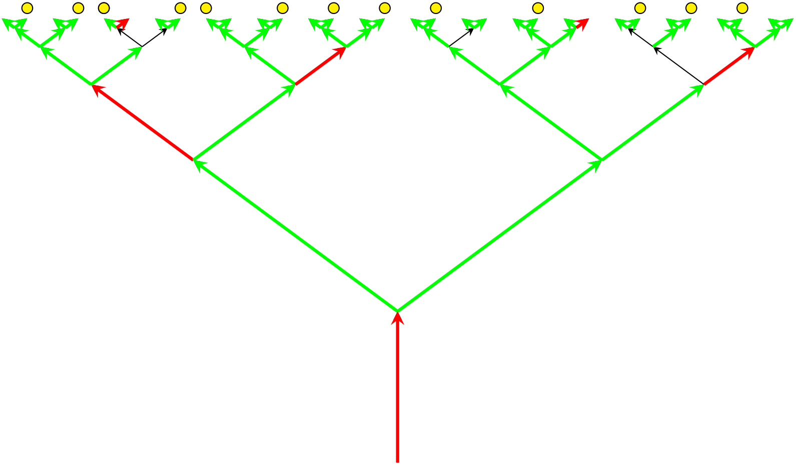
Frozen percolation on the oriented binary tree



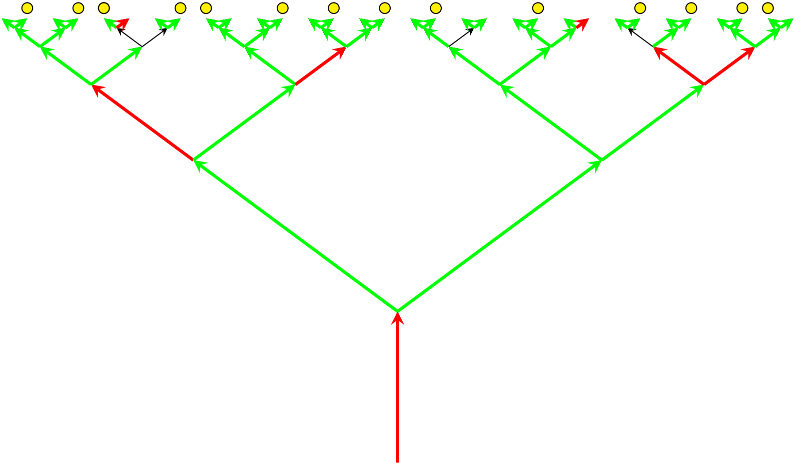
Frozen percolation on the oriented binary tree



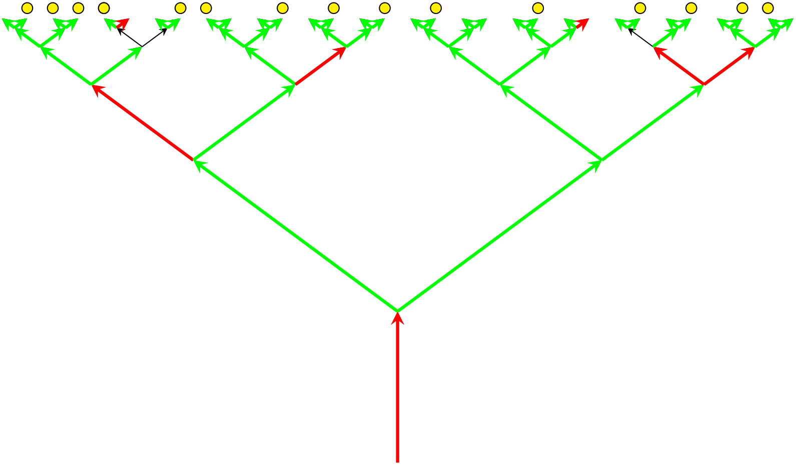
Frozen percolation on the oriented binary tree



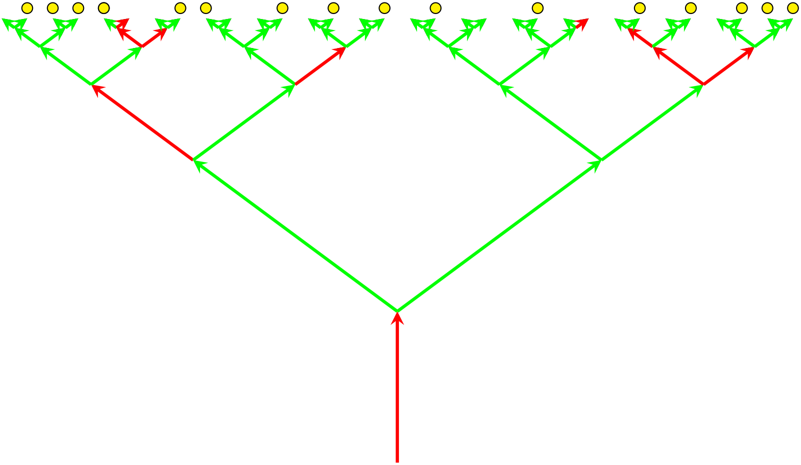
Frozen percolation on the oriented binary tree



Frozen percolation on the oriented binary tree



Frozen percolation on the oriented binary tree



A Recursive Tree Process

Consider the *Recursive Distributional Equation* (RDE)

$$X \stackrel{\mathrm{d}}{=} \Phi[\tau](X_1 \wedge X_2),$$

where X has law ν, X_1, X_2 are i.i.d. copies of X, and τ is independent uniform $[0,1]$-valued.

If a probability law ν on $I := [0,1] \cup \{\infty\}$ solves RDE, then by Kolmogorov's extension theorem, there exists a *Recursive Tree Process* (RTP) $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, unique in law, such that

(i) For each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}$ are i.i.d. with common law ν and independent of $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$.

(ii) $X_{\mathbf{i}} = \Phi[\tau_{\mathbf{i}}](X_{\mathbf{i}1} \wedge X_{\mathbf{i}2}) \qquad (\mathbf{i} \in \mathbb{T})$.

A Recursive Tree Process

Given an RTP $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, define

$$\mathbb{F} := \big\{\mathbf{i} \in \mathbb{T} : \tau_{\mathbf{i}} \geq X_{\mathbf{i}1} \wedge X_{\mathbf{i}2}\big\},$$

$$X^{\uparrow}_{\mathbf{i}} := \inf\big\{t \in [0,1] : \mathbf{i} \xrightarrow{\mathbb{T}_t \backslash \mathbb{F}} \infty\big\}.$$

We call $X_{\mathbf{i}}$ the *burning time* and $X^{\uparrow}_{\mathbf{i}}$ the *percolation time*. One has $X^{\uparrow}_{\mathbf{i}} \leq X_{\mathbf{i}}$.

Theorem $X^{\uparrow}_{\mathbf{i}} = X_{\mathbf{i}}$ a.s. if and only if the solution to RDE is

$$\nu(\mathrm{d}x) := \frac{\mathrm{d}x}{2x^2} 1_{[\frac{1}{2},1]}(x) \qquad \nu\big(\{\infty\}\big) := \tfrac{1}{2}.$$

"If" part of theorem proved by Aldous (2000).

Endogeny

Def The RTP is *endogenous* if $X_\varnothing$ is measurable w.r.t. the σ-field generated by $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$.

Def *bivariate map*

$$T^{(2)}(\mu^{(2)}) := \text{ the law of } \big(\Phi[\tau](X_1 \wedge X_2), \Phi[\tau](X_1' \wedge X_2')\big),$$

where $(X_1, X_1'), (X_2, X_2')$ are i.i.d. with law $\mu^{(2)}$ and τ is independent uniform $[0,1]$-valued.

Let $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to ν.
Let $(X_{\mathbf{i}}')_{\mathbf{i}\in\mathbb{T}}$ be a copy of $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, conditionally independent given $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. Then

$$\underline{\nu}^{(2)} := \mathbb{P}\big[(X_\varnothing, X_\varnothing') \in \cdot\,\big],$$
$$\overline{\nu}^{(2)} := \mathbb{P}\big[(X_\varnothing, X_\varnothing) \in \cdot\,\big],$$

solve the *bivariate RDE* $T^{(2)}(\nu^{(2)}) = \nu^{(2)}$.

Endogeny

Def $\mathcal{P}(I^2)_\nu =$ space of probability laws on I^2 whose one-dimensional marginals are given by ν.

Theorem (Aldous & Bandyopadhyay 2005)
The following statements are equivalent:

(i) The RTP $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is endogenous.

(ii) $\underline{\nu}^{(2)} = \overline{\nu}^{(2)}$.

(iii) The bivariate map $T^{(2)}$ has a unique fixed point in $\mathcal{P}(I^2)_\nu$.

(iv) $(T^{(2)})^n(\mu^{(2)}) \underset{n\to\infty}{\Longrightarrow} \overline{\nu}^{(2)}$ for all $\mu^{(2)} \in \mathcal{P}(I^2)_\nu$.

Moreover, $(T^{(2)})^n(\nu\otimes\nu) \underset{n\to\infty}{\Longrightarrow} \underline{\nu}^{(2)}$.

Reformulation of the problem To prove that frozen percolation is *not* a.s. unique, it suffices to find a nontrivial solution $\nu^{(2)} \neq \overline{\nu}^{(2)}$ to the bivariate RDE.

Endogeny

History of the problem

Aldous (2000) conjectured a.s. uniqueness (i.e., endogeny).

Bandyopahyay (2004), arXiv:math/0407175 announced a false proof.

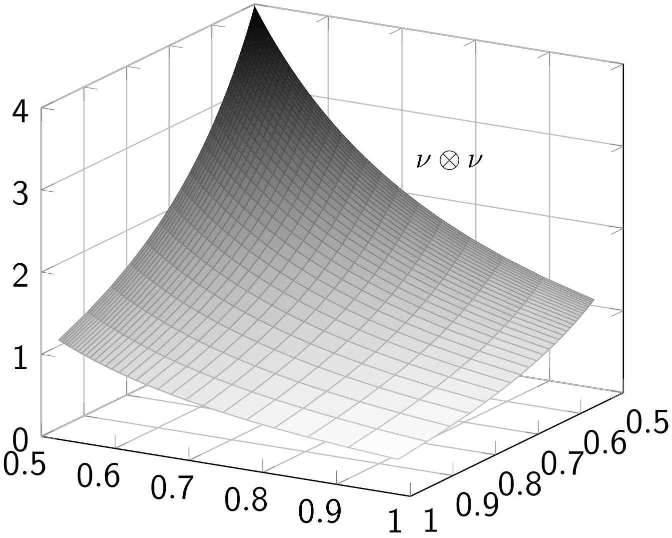
Bandyopahyay (2005) numerical simulations
$(T^{(2)})^n(\nu \otimes \nu) \underset{n\to\infty}{\Longrightarrow} \underline{\nu}^{(2)} \neq \overline{\nu}^{(2)}$.

Antar Bandyopahyay, Tamás Terpai, and especially Balázs Ráth pursued the problem for many years...

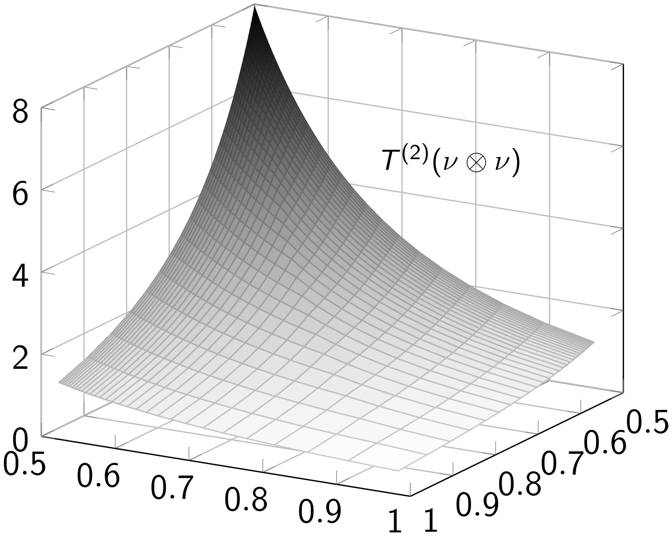
Theorem (2019) Endogeny does *not hold*.

Proof The problem can be translated into frozen percolation on the MBBT, which is easier to handle.

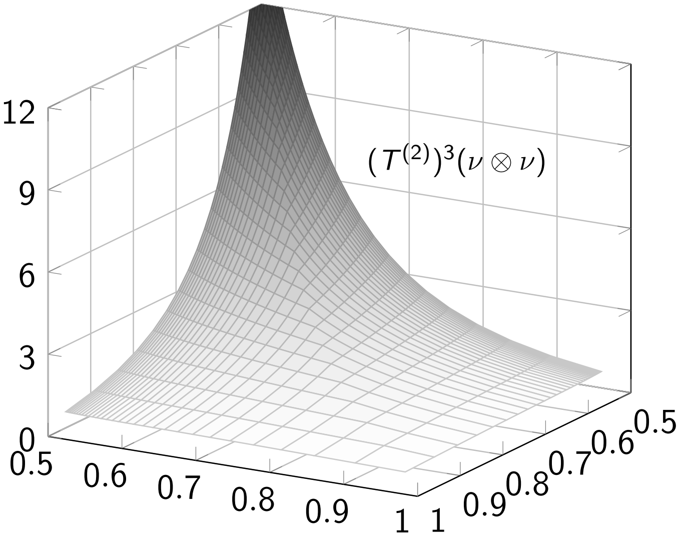
Endogeny



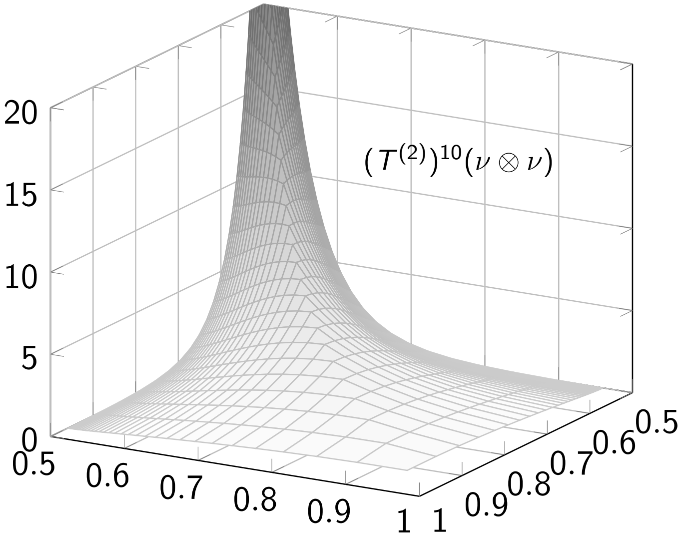
Endogeny



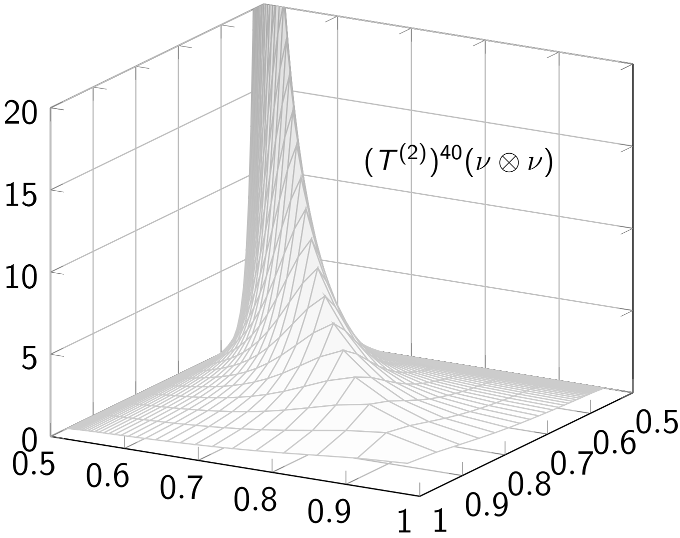
Endogeny



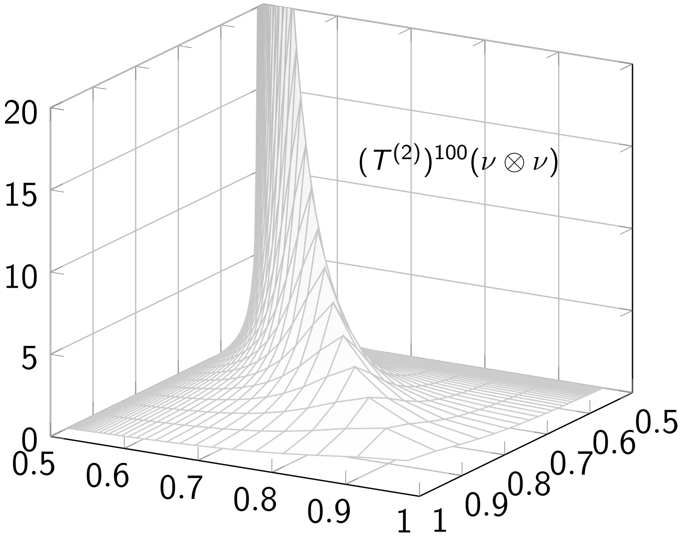
Endogeny



Endogeny



Endogeny



The Marked Binary Branching Tree

Def The *Marked Binary Branching Tree* (MBBT) is a pair $(\mathcal{T}, \Pi)$ with:

- $\mathcal{T}$ is the family tree of a rate one continuous-time binary branching process.
- Π is a rate one Poisson process on $\mathcal{T} \times [0,1]$.

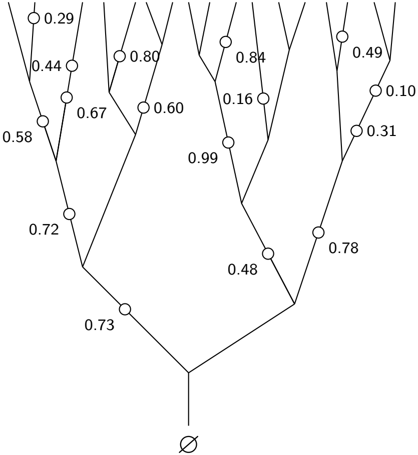
Def $\Pi_t := \{(z,\tau) \in \Pi : \tau > t\}$.

Equivalently, $\Pi = \{(z,\tau_z) : z \in \Pi_0\}$, where:

- Π_0 is a rate one Poisson process on $\mathcal{T}$,
- $(\tau_z)_{z\in\Pi_0}$ are i.i.d. uniform $[0,1]$-valued.

Interpretation Initially, points in Π_0 are closed. At time τ_z, the point z opens. Π_t set of closed points at time t.

The Marked Binary Branching Tree



The Marked Binary Branching Tree

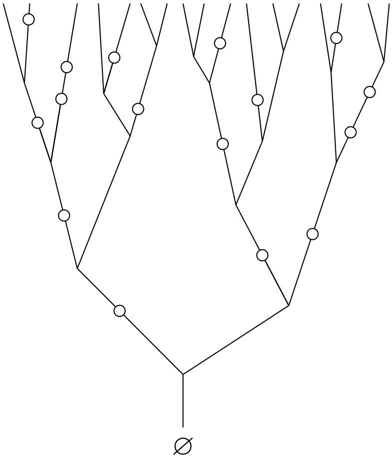
The MBBT is the *universal scaling limit* of near-critical percolation on trees.

Related to this, the MBBT itself enjoys a form of *scale invariance*: Write $z \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} \infty$ if at time t there is an open upward path starting at z.
Then

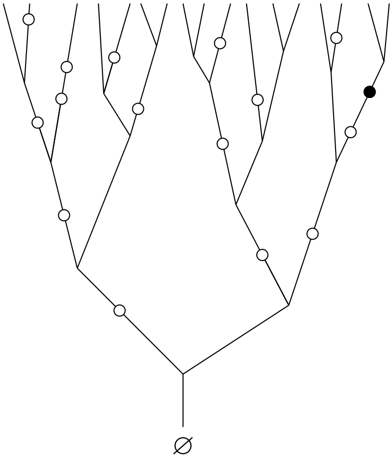
$$\mathcal{T}' := \{z \in \mathcal{T} : \varnothing \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} z \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} \infty\}$$

is the family tree of a rate t binary branching process.
Moreover, $\Pi' := \big\{(z, \tau_z) \in \Pi : z \in \mathcal{T}'\big\}$ is a rate one Poisson process on $\mathcal{T}' \times [0, t]$.

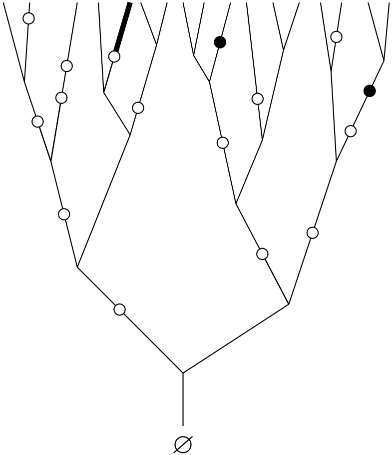
The Marked Binary Branching Tree



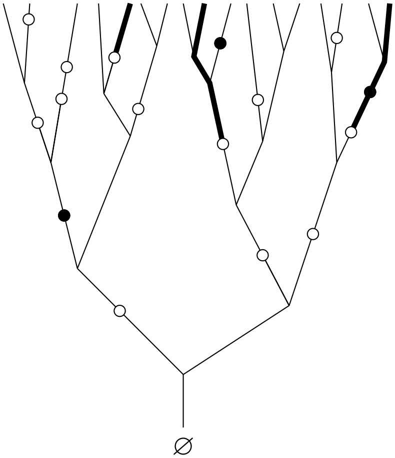
The Marked Binary Branching Tree



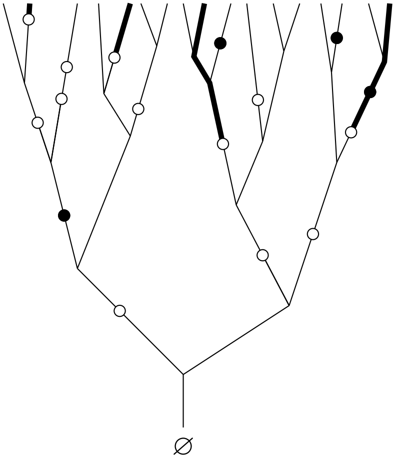
The Marked Binary Branching Tree



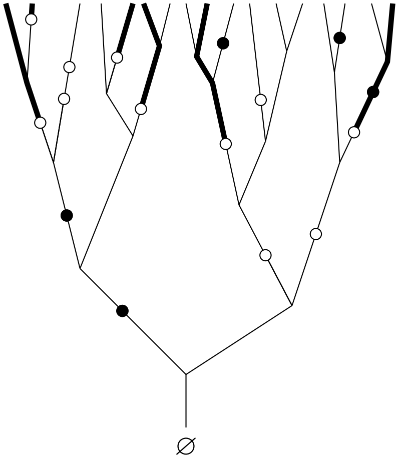
The Marked Binary Branching Tree



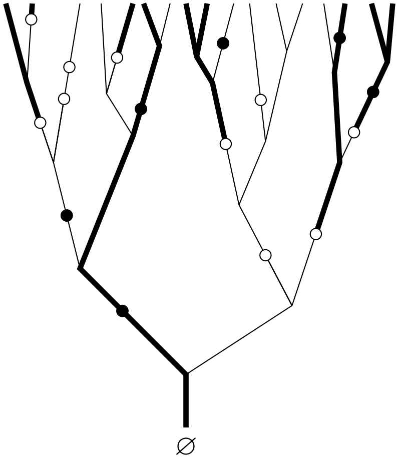
The Marked Binary Branching Tree



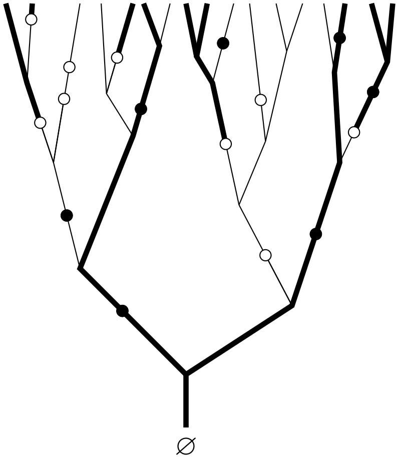
The Marked Binary Branching Tree



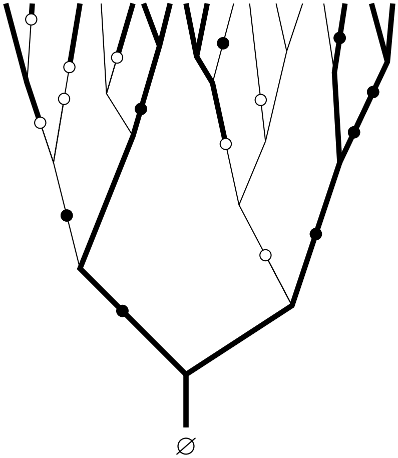
The Marked Binary Branching Tree



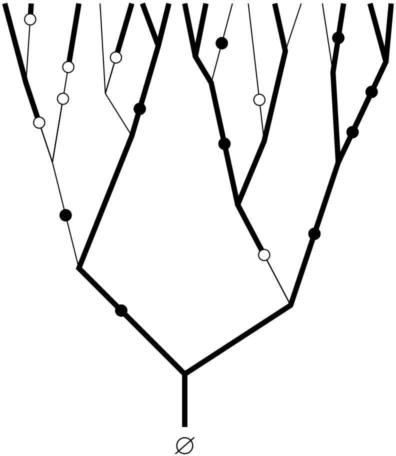
The Marked Binary Branching Tree



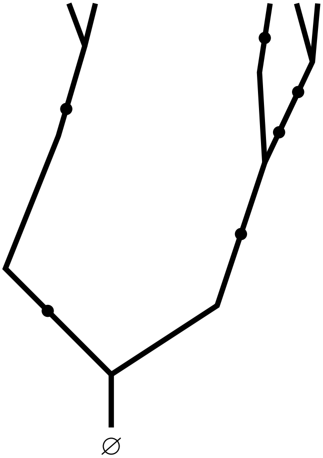
The Marked Binary Branching Tree



The Marked Binary Branching Tree



The Marked Binary Branching Tree



It is possible to construct frozen percolation on the MBBT such that:

At time $t = \tau_z$, the point z opens unless $z \overset{\mathcal{T}\backslash\Pi_t}{\longrightarrow} \infty$.

Let $Y_\varnothing := \inf\big\{t \in [0,1] : \varnothing \overset{\mathcal{T}\backslash\Pi_t}{\longrightarrow} \infty\big\}$ and $:= \infty$ if this never happens.
Then

$$\rho\big([0,t]\big) := \mathbb{P}[Y_\varnothing \leq t] = \tfrac{1}{2}t \qquad \big(t \in [0,1]\big).$$

Lemma The corresponding $\underline{\rho}^{(2)}$ has the scaling property

$$\mathbb{P}\big[(Y_\varnothing, Y'_\varnothing) \in [0,tr] \times [0,ts]\big] = t\mathbb{P}\big[(Y_\varnothing, Y'_\varnothing) \in [0,r] \times [0,s]\big]$$

$(0 \leq r,s,t \leq 1)$.

Theorem For frozen percolation on the MBBT, the bivariate RDE has precisely two symmetric scale-invariant fixed points.
A symmetric scale invariant law $\rho^{(2)}$ on I^2 solves the bivariate RDE if and only if the function

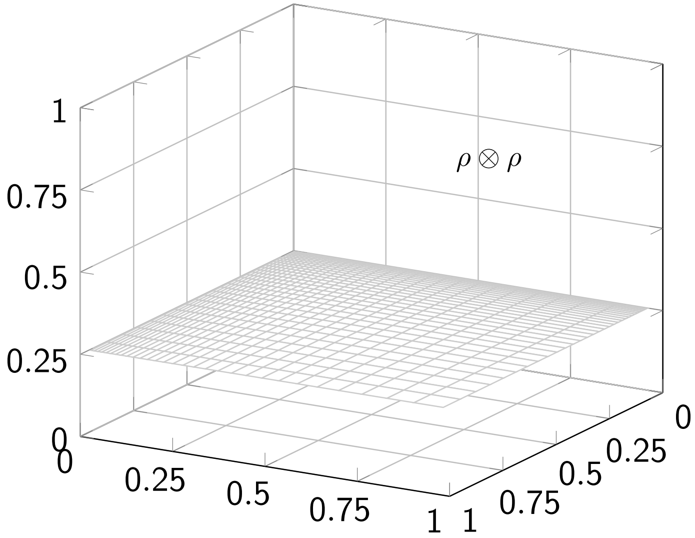
$$f(r) := \rho^{(2)}\big(\{(y_1, y_2) \in I^2 : y_1 \leq r \text{ or } y_2 \leq 1\}\big) \qquad (0 \leq r \leq 1)$$

solves the differential equation

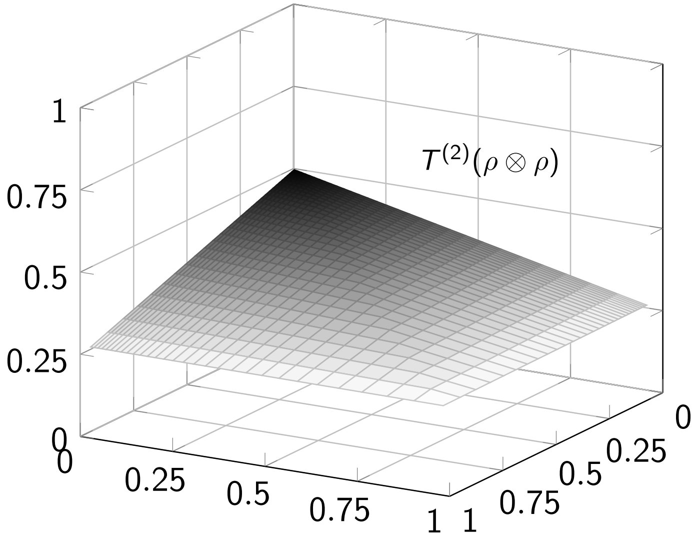
$$\text{(i) } \frac{\partial}{\partial r} f(r) = \frac{cr}{f(r) - r/2} \qquad \big(r \in [0,1)\big),$$
$$\text{(ii) } f(0) = \tfrac{1}{2}, \qquad \text{(iii) } f(1)^2 - \tfrac{1}{2} f(1) = 2c$$

for some $c \geq 0$. There are two values $0 = \overline{c} < \underline{c} < \frac{1}{4}$ for which this equation has a solution, corresponding to $\overline{\rho}^{(2)}$ and $\underline{\rho}^{(2)}$.

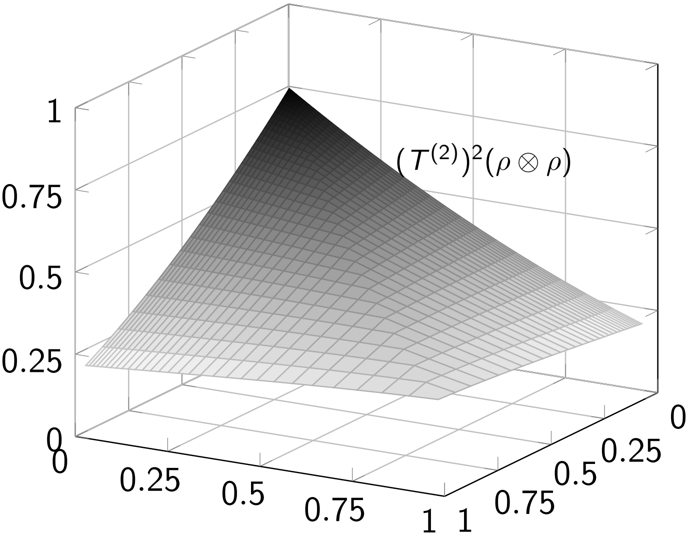
The bivariate map for the MBBT



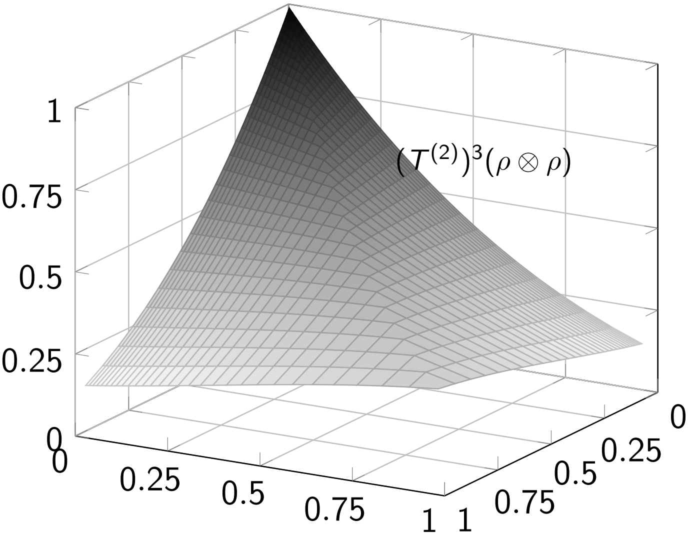
The bivariate map for the MBBT



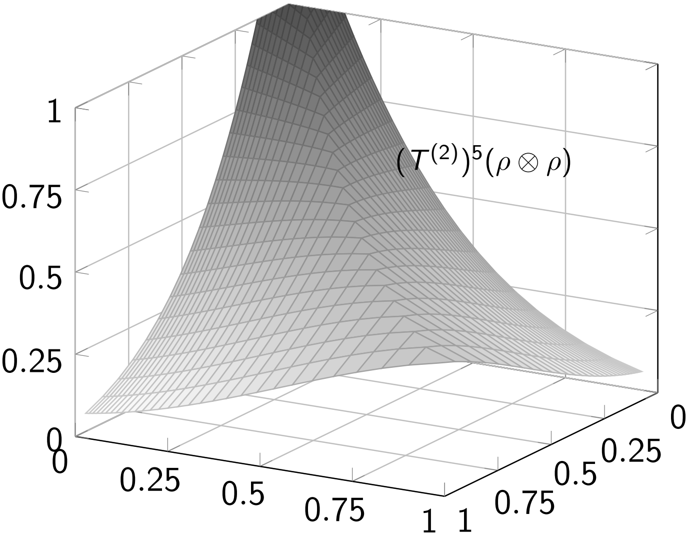
The bivariate map for the MBBT



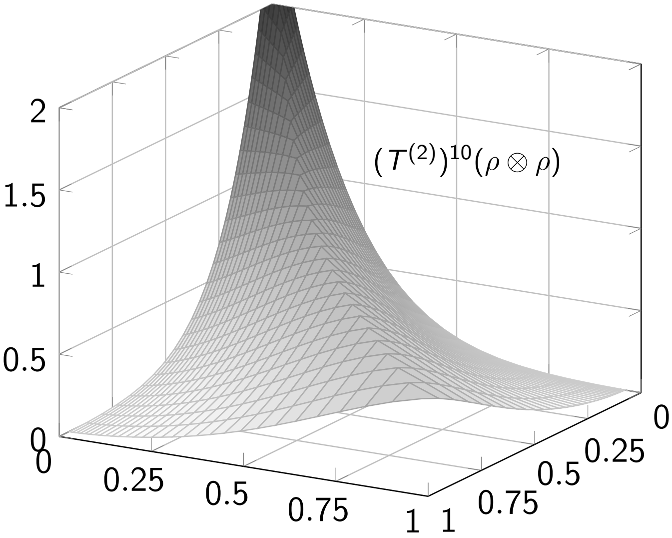
The bivariate map for the MBBT



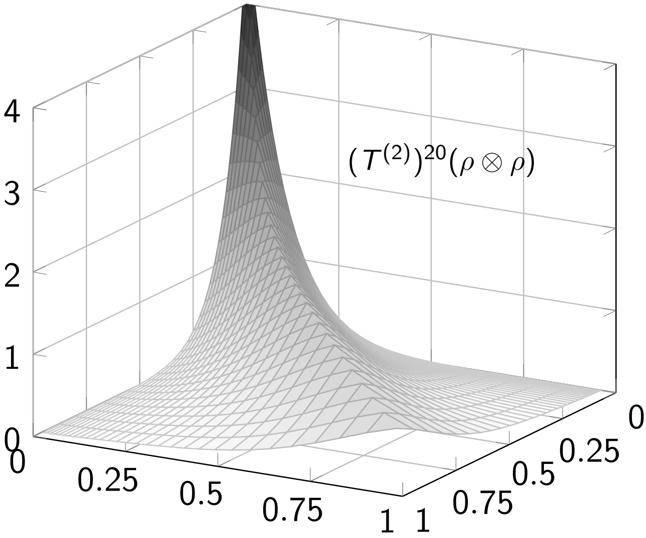
The bivariate map for the MBBT



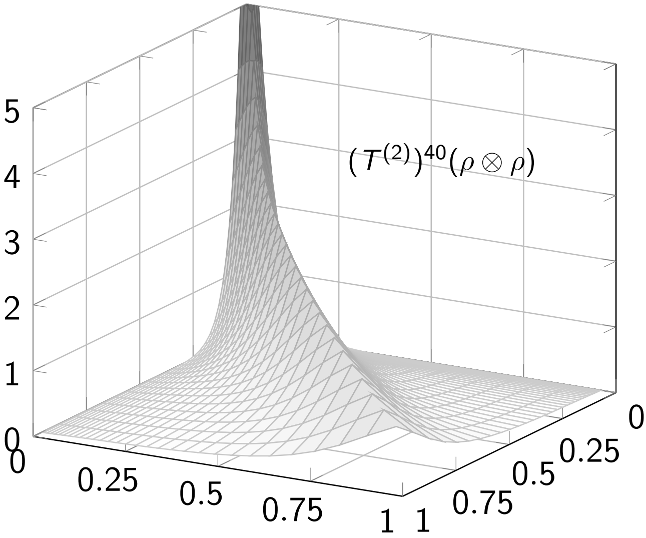
The bivariate map for the MBBT



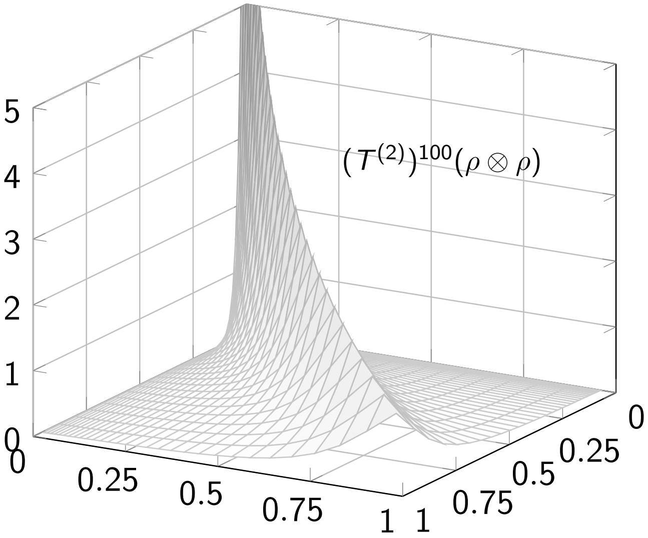
The bivariate map for the MBBT



The bivariate map for the MBBT



The bivariate map for the MBBT



The bivariate map for the MBBT