Strong R-positivity

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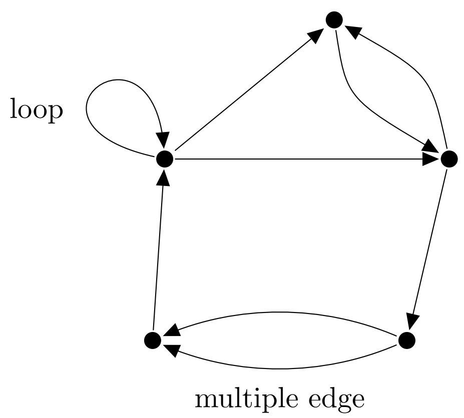
Lecture 1

Directed graphs

Directed graph $G = (S, E)$. S vertex set E edge set.
e^- starting vertex e^+ endvertex.
For us always: S, E countable.



Def simple graph = no multiple edges.

$$E_{x,\bullet} := \{e \in E : e^- = x\}, \quad E_{\bullet,y} := \{e \in E : e^+ = y\}, \quad E_{x,y} := E_{x,\bullet} \cap E_{\bullet,u}$$

$$[m:n] := \{k \in \mathbb{Z} : m \leq k \leq n\}, \quad (m:n] := \{k \in \mathbb{Z} : m < k \leq n\}.$$

A walk is a pair of functions

$$[m:n] \ni k \mapsto \omega_k \in S \quad \text{and} \quad (m:n] \ni k \mapsto \vec{\omega}_k \in E$$

with $\vec{\omega}_k^- = \omega_{k-1}$ and $\vec{\omega}_k^+ = \omega_k \ \forall \ k \in (m:n]$.
Starting and end- vertices $\omega^- := \omega_m \quad \text{and} \quad \omega^+ := \omega_n$.

$\ell_\omega := m - n \geq 0$ length of walk.
Set of walks $\Omega^{[m:n]} = \Omega^{[m:n]}(G)$, in particular $\Omega^n := \Omega^{[0:n]}$.
As before: $\Omega^n_{x,\bullet}$, $\Omega^n_{\bullet,y}$, $\Omega^n_{x,y}$.
$\Omega := \bigcup_{n=0}^\infty \Omega^n$.
Def $x \rightsquigarrow_G y \Leftrightarrow \Omega_{x,y} \neq \emptyset$.
Def communicating class = equivalence class of $\leftrightsquigarrow_G$.
Def period of x

$$\sup\{k \geq 1 : N_x \subset k\mathbb{N}\}, \quad \text{with} \quad N_x := \{n \geq 0 : \Omega^n_{x,x} \neq \emptyset\}.$$

The period is a class property i.e. constant on communicating classes.
Def x transitory if $N_x = \{0\}$.
All other communicating classes are irreducible classes.
Def $G = (S, E)$ irreducible $\Leftrightarrow$ S is an irreducible class.
Def G aperiodic $\Leftrightarrow$ all vertices have period 1.
Note: if x has period k in $G = (S, E)$, then x has period 1 in (S, Ω^k).

Nonnegative matrices

Def weight function on $G = (S, E)$ = function $\mathbb{A} : E \to [0, \infty]$.

$$\mathbb{A}(\omega) := \prod_{k=m+1}^{n} \mathbb{A}(\vec{\omega}_k) \qquad (\omega \in \Omega^{[m:n]}).$$

Associated nonnegative matrix

$$A(x, y) := \sum_{e \in E_{x,y}} \mathbb{A}(e) \qquad (x, y \in S).$$

If G simple, then A determines $\mathbb{A}$.
For $f : S \to [0, \infty]$, def

$$\begin{aligned} Af(x) &:= \sum_y A(x, y) f(y), \quad fA(x) := \sum_y f(y) A(y, x), \\ (AB)(x, z) &:= \sum_y A(x, y) B(y, z). \end{aligned}$$

$$A^n(x, y) = \sum_{\omega \in \Omega^n_{x,y}} \mathbb{A}(\omega) \qquad (n \geq 0),$$

with $A^0(x, y) := 1(x, y) = 1_{\{x=y\}}$ identity matrix.
Def A irreducible $\Leftrightarrow$ $\forall x, y\ \exists n \geq 1$ s.t. $A^n(x, y) > 0$.
Equivalent: $G^{\mathbb{A}} = (S, E^{\mathbb{A}})$ irreducible with $E^{\mathbb{A}} := \{e \in E : \mathbb{A}(e) > 0\}$.

Lemma (Local growth rate) If x has period k, then

$$\rho_x(A) := \lim_{n\in k\mathbb{N}} \big(A^n(x,x)\big)^{1/n} = \sup_{n\in k\mathbb{N}} \big(A^n(x,x)\big)^{1/n} \in (0,\infty].$$

If $x \leftrightsquigarrow_A y$, then $\rho_x(A) = \rho_y(A) = \lim_n \big(A^n(x,y)\big)^{1/n}$.

Proof Since $A^{n+m}(x,x) \geq A^n(x,x)A^m(x,x)$ for $n,m \geq 1$, the function $n \mapsto \log A^n(x,x)$ is superadditive. The first statement now follows from Fekete's lemma.
If $A^m(x,y) > 0$, then

$$\begin{aligned}&\liminf_n \frac{1}{n}\log A^n(x,y) \geq \liminf_n \frac{1}{n}\log\big(A^{n-m}(x,x)A^m(x,y)\big)\\&= \lim_n \frac{1}{n}\big[\log A^{n-m}(x,x) + \log A^m(x,y)\big] = \log\rho_x(A).\end{aligned}$$

and

$$\begin{aligned}&\limsup_n \frac{1}{n}\log A^n(y,x) = \limsup_n \frac{1}{n}\big[\log A^n(y,x) + \log A^m(x,y)\big]\\&\leq \lim_n \frac{1}{n}\log\big(A^{n+m}(y,y)\big) = \rho_y(A).\end{aligned}$$

∎

If A is irredicible, then $\rho(A) = \rho_x(A)$ spectral radius of A.

One-dimensional Gibbs measures

If $0 < A^{n-m}(x,y) < \infty$, define Gibbs measure on $\Omega^{[m:n]}_{x,y}$ by

$$\overline{\mu}^{\mathbb{A},[m:n]}_{x,y}(\omega) := \frac{\mathbb{A}(\omega)}{A^{n-m}(x,y)}.$$

$\overline{\mu}^{\mathbb{A},n}_{x,y} = \overline{\mu}^{\mathbb{A},[0:n]}_{x,y}$.
We call A the transfer operator.
Let $c > 0$, $f : S \to (0,\infty)$.
Def $\mathbb{A} \sim_{c,f} \mathbb{B} \Leftrightarrow$

$$\mathbb{B}(e) = c^{-1}f(e^-)^{-1}\mathbb{A}(e)f(e^+) \qquad \forall e \in E.$$

Def $\mathbb{A} \sim_c \mathbb{B} \Leftrightarrow \mathbb{A} \sim_{c,f} \mathbb{B}$ for some f.
Def $\mathbb{A} \sim \mathbb{B} \Leftrightarrow \mathbb{A} \sim_c \mathbb{B}$ for some c.
Def $\mathbb{A}$ locally finite $\Leftrightarrow A^n(x,y) < \infty\ \forall x,y,n$.
Def $N_{x,y}(A) := \{n : A^n(x,y) > 0\}$.

Proposition (Equivalence of weight functions) Let $\mathbb{A}, \mathbb{B}$ irreducible, locally finite, and $N_{x,y}(A) = N_{x,y}(B)$ $(x, y \in S)$. Then the following are equivalent:

(i) $\overline{\mu}^{\mathbb{A},n} = \overline{\mu}^{\mathbb{B},n}$ $\forall x, y \in S$, $n \in N_{x,y}$.
(ii) $\mathbb{A} \sim_{c,f} \mathbb{B}$ for some $c > 0$ and $f : S \to (0, \infty)$.

In (ii), c is unique and f is unique up to scalar multiples.

Proof (ii)$\Rightarrow$(i): $\mathbb{A} \sim_{c,f} \mathbb{B}$ implies

$$\mathbb{B}(\omega) = c^{-\ell_\omega} f(\omega^-)^{-1} \mathbb{A}(\omega) f(\omega^+) \qquad \forall \omega \in \Omega.$$

Thus

$$\mathbb{B}(\omega) = C\mathbb{A}(\omega) \quad \forall \omega \in \Omega^n_{x,y} \text{with} \quad C := c^{-n} f(x)^{-1} f(y).$$

The constant C disappears in the normalization.
(i)$\Rightarrow$(ii) (sketch): Fix reference point z.
Def $g_m := \dfrac{A^m(z,z)}{B^m(z,z)}$. Then

$$\frac{A^m(z,z)A^n(z,z)}{A^{m+n}(z,z)} = \overline{\mu}^{\mathbb{A},m+n}_{z,z}[\omega_m = z] = \frac{B^m(z,z)B^n(z,z)}{B^{m+n}(z,z)},$$

hence

$$g_m g_n = \frac{A^m(z,z)}{B^m(z,z)} \frac{A^n(z,z)}{B^n(z,z)} = \frac{A^{m+n}(z,z)}{B^{m+n}(z,z)} = g_{m+n}.$$

It follows $g_m = c^m$ for some $c > 0$. Def

$$f(x) := c^{-\ell_\omega} \frac{\mathbb{A}(\omega)}{\mathbb{B}(\omega)} f(z) \qquad \omega \in \Omega_{x,z},\ \mathbb{B}(\omega) > 0.$$

Show def. does not dep. on choice of ω and $\mathbb{A} \sim_{c,f} \mathbb{B}$. ∎

Def A <u>probability kernel</u> if $\sum_y A(x,y) = 1$ $\forall x$.
Def $\mathbb{A}$ <u>Markovian</u> $\Leftrightarrow$ A probability kernel.
$\mathbb{A}$ Markovian $\Rightarrow$ $\overline{\mu}^{\mathbb{A},[m:n]}_{x,y}$ law of Markov chain with <u>transition function</u> $\mathbb{A}$ started in x and conditioned to end in y.

Lemma (Positive eigenfunction) $Af = cf$ if and only if $\mathbb{A} \sim_{c,f} \mathbb{P}$ for a Markovian weight function $\mathbb{P}$.

Proof

$$\begin{aligned}\sum_y P(x,y) &= \sum_{e \in E_{x,\bullet}} c^{-1} f(e^-)^{-1} \mathbb{A}(e) f(e^+) \\ &= \frac{\sum_y A(x,y) f(y)}{cf(x)} = \frac{Af(x)}{cf(x)}.\end{aligned}$$

∎

Theorem (Perron (1907) Frobenius (1912)) Let A be a finite irreducible nonnegative matrix. Then there exists a unique $c > 0$ and a $f : S \to (0, \infty)$ unique up to scalar multiples such that $Af = cf$. Moreover, $c = \rho(A)$.

Consequence: there exists a unique Markovian $\mathbb{P}$ s.t. $\mathbb{A} \sim \mathbb{P}$.
Let $\nu_x^{\mathbb{P}}$ denote law of Markov chain with initial state x and transition kernel $\mathbb{P}$. Then

$$\overline{\mu}_{x,y_n}^{\mathbb{A},n} \underset{n\to\infty}{\Longrightarrow} \nu_x^{\mathbb{P}} \qquad \forall y_n \in S.$$

For uncountable matrices:

- Limit need not exist.
- Limit may depend on choice of y_n.

Conseq: positive eigenfunctions may fail to exist or there can be many.
We will see:

- In general eigenvalue $c \geq \rho(A)$.
- There is at most one recurrent $\mathbb{P}$ s.t. $\mathbb{A} \sim \mathbb{P}$.
- If $\exists$ recurrent $\mathbb{P} \sim \mathbb{A}$, then $c = \rho(A)$ and $Af = \rho(A)f$ has unique sol.

Note: finite probability kernels are always positive recurrent.

Lecture 2

R-recurrence

Def <u>subprobability kernel</u> $\sum_y P(x,y) \le 1$.
Def <u>subMarkovian</u> weight function $\mathbb{P}$ <u>recurrent to</u> z if

$$\sum_{\substack{\omega \in \Omega_{x,z} \\ \omega_k \neq z \ \forall k < \ell_\omega}} \mathbb{P}(\omega) = 1 \qquad (x \neq z).$$

Theorem (Equivalent recurrent weight function) Let $\mathbb{A}$ be an irreducible weight function on a directed graph $G = (S, E)$. Then, for each $\rho(A) \le r < \infty$ and $z \in S$, there exists a unique subMarkovian weight function $\mathbb{P}_{r,z}$ such that

(i) $\mathbb{A} \sim_r \mathbb{P}_{r,z}$ (ii) $\mathbb{P}_{r,z}$ is recurrent to z.

Remark:

- $\mathbb{A} \sim_c \mathbb{B} \Rightarrow \rho(A) = c\rho(B)$.
- $\mathbb{P}$ subMarkovian $\Rightarrow \rho(P) \le 1$.
- $\mathbb{P}$ recurrent Markovian $\Rightarrow \rho(P) = 1$.

Hence:

- $\mathbb{A} \sim_r \mathbb{P}$ and $\mathbb{P}$ subMarkovian $\Rightarrow r \ge \rho(A)$.
- $\mathbb{A} \sim_r \mathbb{P}$ and $\mathbb{P}$ recurrent Markovian $\Rightarrow r = \rho(A)$.
- $\mathbb{P}_{r,z}$ Markovian $\Rightarrow r = \rho(A)$ and $\mathbb{P}_{r,z} = \mathbb{P}$ does not depend on z.
- At most one equivalent recurrent Markovian weight function.

Def

- A <u>R-transient</u> if $\mathbb{P}_z := \mathbb{P}_{\rho(A),z}$ not Markovian.
- A <u>R-recurrent</u> if $\mathbb{P} = \mathbb{P}_z$ Markovian.
- A <u>R-positive</u> if $\mathbb{P}$ positive recurrent.
- A <u>strongly R-positive</u> if $\mathbb{P}$ strongly positive recurrent.

Here

$$\sigma_z := \inf\{k \geq 1 : X_k = z\} \quad \underline{\text{first return time}}$$

Def Markov chain $(X_k)_{k\geq 0}$ <u>strongly positive recurrent</u> $\Leftrightarrow \mathbb{E}^z\big[e^{\varepsilon\sigma_z}\big] < \infty$ for some $\varepsilon > 0$ (class property).

> **Lemma (Alternative definitions)** Let $\mathbb{A}$ be irreducible and in (ii) also aperiodic. Fix $z \in S$.
>
> (i) A is R-recurrent $\Leftrightarrow \displaystyle\sum_k \rho(A)^{-k} A^k(z,z) = \infty$.
>
> (ii) If A is R-recurrent, then $\lim_k \rho(A)^{-k} A^k(z,z)$ exists and $= 0$ resp. > 0 when A is R-null recurrent resp. R-positive.

Proof $\rho(A)^{-k} A^k(z,z) = P_z^k(z,z)$ with

- $\displaystyle\sum_k P_z^k(z,z) =$ expected # returns $< \infty$ iff P_z subMarkovian.
- $\mathbb{P}^k(z,z) \to \pi(z)$ invariant law if $\mathbb{P}$ pos. rec. and $\to 0$ if null rec.

∎

Using the alternative defs, Vere-Jones (1967) proved:

> **Theorem (Unique eigenfunction)** A R-recurrent $\Rightarrow\ \exists\, f : S \to (0,\infty)$ s.t. $Af = \rho(A) f$. If $g : S \to [0,\infty)$ solves $Ag \leq \rho(A) g$, then $g = \lambda f$ for some $\lambda \geq 0$.

Proof Existence of f by arguments above. $Ag \leq \rho(A)g \Rightarrow g \equiv 0$ or $g > 0$. Def $\mathbb{A} \sim_{\rho(A),g} \mathbb{P}$. Then $Ag \leq \rho(A)g \Rightarrow \mathbb{P}$ subMarkovian. Also $\displaystyle\sum_k \rho(A)^{-k} A^k(z,z) = \infty \Rightarrow \sum_k P_z^k(z,z) = \infty$ so $\mathbb{P}$ recurrent Markovian, hence $\mathbb{P}$ unique and $g = \lambda f$. ∎

> **Lemma (Upper bound on spectral radius)** Let A irreducible. Then $\rho(A) \leq r \Leftrightarrow \exists f : S \to (0,\infty)$ s.t. $Af \leq rf$.

Proof $Af \leq rf \Rightarrow A^n f \leq r^n f \Rightarrow \rho(A) \leq r$.
Conversely, $\rho(A) \leq r \Rightarrow \mathbb{A} \sim_{r,f} \mathbb{P}_{r,z}$ with $Af \leq rf$. ∎

Excursions

Let $\mathbb{A}$ irreducible, fix $z \in S$.
Def $\widehat{\Omega}_z := \{\omega \in \Omega : \ell_\omega \geq 1,\ \omega^- = \omega^+ = z,\ \omega_k \neq z\ \forall 0 < k < \ell_\omega\}$.
Def $\nu_{\lambda,z}(\omega) := e^{\lambda \ell_\omega}\mathbb{A}(\omega)$ $(\omega \in \widehat{\Omega}_z)$.
Normalize $\hat{\nu}_{\lambda,z} := e^{-\psi_z(\lambda)}\nu_{\lambda,z}$ with

$$\psi_z(\lambda) := \log\Big(\sum_{\omega\in\widehat{\Omega}_z} e^{\lambda\ell_\omega}\,\mathbb{A}(\omega)\Big)$$

logarithmic moment generating function.
Different parametrization $r =: e^{-\lambda}$. $\mathbb{P}_{(\lambda),z} = \mathbb{P}_{e^{-\lambda},z}$
Assume $\mathbb{P}_{(\lambda),z}$ subMarkovian,

$$\text{(i) } \mathbb{A} \sim_{e^{-\lambda}} \mathbb{P}_{(\lambda),z} \qquad \text{(ii) } \mathbb{P}_{(\lambda),z} \text{ is recurrent to } z.$$

Let X be the Markov chain with transition kernel $\mathbb{P}_{(\lambda),z}$ and possibly finite lifetime, started in z.

> **Proposition (Excursion decomposition)** Assumptions $\Rightarrow$
>
> $\psi_z(\lambda) \leq 0$ and X can be written as a finite or infinite concatenation
>
> $$X = V_1 \circ V_2 \circ \cdots \circ V_K \quad \text{or} \quad X = V_1 \circ V_2 \circ \cdots,$$
>
> where $V_1, V_2, \ldots$ are i.i.d. with law $\hat{\nu}_{\lambda,z}$ and K is geometrically distributed with
>
> $$\mathbb{P}[K = k] = e^{k\psi_z(\lambda)}\big(1 - e^{\psi_z(\lambda)}\big) \qquad (k \geq 0).$$

Proof $\mathbb{P}_{(\lambda),z}(\omega) = e^{\lambda\ell_\omega} f(z)^{-1}\mathbb{A}(\omega) f(z) \qquad (\omega \in \widehat{\Omega}_z)$.
$\Rightarrow \psi_z(\lambda) \leq 0$. At each visit to z, X either killed with probab. $e^{\psi_z(\lambda)}$ or makes excursion with law $\hat{\nu}_{\lambda,z}$. ∎

Consequence: at most one $\mathbb{P}_{(\lambda),z}$ satisfies (i) and (ii).

The Green's function

Recall:

> **Theorem (Equivalent recurrent weight function)** Let $\mathbb{A}$ be an irreducible weight function on a directed graph $G = (S, E)$. Then, for each $\rho(A) \leq e^{-\lambda} < \infty$ and $z \in S$, there exists a unique subMarkovian weight function $\mathbb{P}_{(\lambda),z}$ such that
>
> $$\text{(i) } \mathbb{A} \sim_{e^{-\lambda}} \mathbb{P}_{(\lambda),z} \qquad \text{(ii) } \mathbb{P}_{(\lambda),z} \text{ is recurrent to } z.$$

Uniqueness proved. Next step: existence when $\psi_z(\lambda) < 0$.
Def Green's function

$$G_\lambda(x,y) := \sum_{k=0}^{\infty} e^{\lambda k} A^k(x,y).$$

Proposition (Green's function) If $\psi_z(\lambda) < 0$, then

(a) $0 < G_\lambda(x,y) < \infty \ \forall x,y \in S$.

(b) $(1 - e^\lambda A) G_\lambda(\,\cdot\,, y)(x) = 1_{\{x=y\}}$.

(c) $G_\lambda(z,z) = (1 - e^{\psi_z(\lambda)})^{-1}$.

(d) Setting $f(x) := G_\lambda(x,z)$ and $\mathbb{A} \sim_{e^{-\lambda},f} \mathbb{P}_{(\lambda),z}$ defines a sub-Markovian $\mathbb{P}_{(\lambda),z}$ that is recurrent to z.

If $\psi_z(\lambda) \geq 0$, then $G_\lambda(x,y) = \infty \ \forall x,y \in S$.

Proof

$$\begin{aligned} G_\lambda(z,z) &= \sum_{n=0}^{\infty} e^{\lambda n} A^n(z,z) = \sum_{n=0}^{\infty} \sum_{\omega \in \Omega^n_{z,z}} e^{\lambda \ell_\omega} \mathbb{A}(\omega) \\ &= \sum_{m=0}^{\infty} \prod_{i=1}^{m} \Big(\sum_{\omega^{(i)} \in \widehat{\Omega}_z} e^{\lambda \ell_{\omega^{(i)}}} \mathbb{A}(\omega^{(i)}) \Big) = \sum_{m=0}^{\infty} e^{m \psi_z(\lambda)} = (1 - e^{\psi_z(\lambda)})^{-1} \end{aligned}$$

if $\psi_z(\lambda) < 0$ and $= \infty$ otherwise.

To be continued next time.

Lecture 3

Continuation of the proof of the proposition.

By irreducibility, either $0 < G_\lambda(x,y) < \infty$ $\forall x, y \in S$, or $G_\lambda(x,y) = \infty$ $\forall x, y \in S$. This proves (a) and (c). Part (b) follows from

$$e^\lambda A G_\lambda = \sum_{k=0}^{\infty} e^{\lambda(k+1)} A^{k+1} = G_\lambda - 1.$$

To prove (d), observe that by (b)

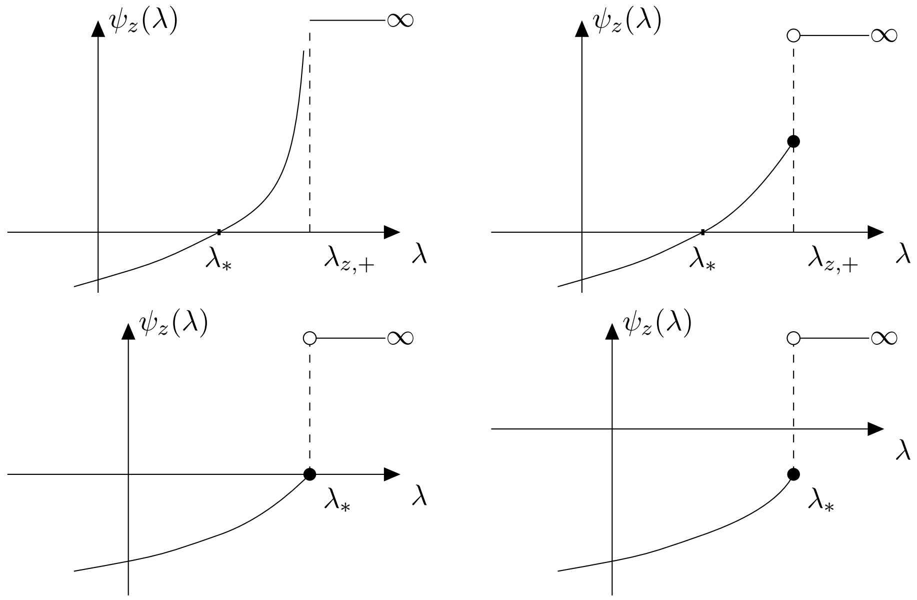
$$\begin{aligned}\sum_y P_{(\lambda),z}(x,y) &= e^\lambda \sum_y G_\lambda(x,z)^{-1} A(x,y) G_\lambda(y,z)\\ &= G_\lambda(x,z)^{-1} e^\lambda A G_\lambda(x,z)\\ &= G_\lambda(x,z)^{-1}\big(G_\lambda(x,z) - 1_{\{x=z\}}\big) = 1 - 1_{\{x=z\}} G_\lambda(z,z)^{-1}.\end{aligned}$$

Starting from z, the Markov process with kernel $P_{(\lambda),z}$ is eventually killed at z with probability

$$\sum_{k=0}^{\infty} P^k_{(\lambda),z}(z,z) G_\lambda(z,z)^{-1} = G_\lambda(z,z)^{-1} \sum_{k=0}^{\infty} e^{\lambda k} A^k(z,z) = 1.$$

$\Rightarrow$ recurrent to z. ∎

The logarithmic moment generating function



Recall

$$\psi_z(\lambda) = \log\Big(\sum_{m=1}^{\infty} e^{\lambda m} C_z(m)\Big) \quad \text{where} \quad C_z(m) := \sum_{\omega \in \widehat{\Omega}_z^m} \mathbb{A}(\omega).$$

Def

$$\begin{aligned}\lambda_{z,+} &:= \sup\{\lambda \in \mathbb{R} : \psi_z(\lambda) < \infty\},\\ \lambda_{z,*} &:= \sup\{\lambda \in \mathbb{R} : \psi_z(\lambda) < 0\}.\end{aligned}$$

Proposition (Logarithmic moment generating function)
Assume $\mathbb{A}$ irreducible, $0 < \rho(A) < \infty$. Fix $z \in S$. Then:

(i) ψ_z is convex.

(ii) ψ_z is lower semi-continuous.

(iii) $\lambda_* = -\log \rho(A)$ and $-\infty < \lambda_* \le \lambda_{z,+} \le \infty$.

(iv) ψ_z is infinitely differentiable on $(-\infty, \lambda_{z,+})$.

(v) ψ_z is strictly increasing on $(-\infty, \lambda_{z,+})$.

(vi) $\lim_{\lambda \to \pm\infty} \psi_z(\lambda) = \pm\infty$.

Let $V_{\lambda,z}$ have law $\hat{\nu}_{\lambda,z}$. Then, for all $\lambda < \lambda_{z,+}$,

(vii) $\frac{\partial}{\partial \lambda}\psi_z(\lambda) = \mathbb{E}[\ell_{V_{\lambda,z}}]$.

(viii) $\frac{\partial^2}{\partial \lambda^2}\psi_z(\lambda) = \mathrm{Var}(\ell_{V_{\lambda,z}})$.

If $\psi_z(\lambda_{z,+}) < \infty$, then moreover

(ix) $\lim\limits_{\lambda \uparrow \lambda_{z,+}} \frac{\partial}{\partial \lambda}\psi_z(\lambda) = \lim\limits_{\varepsilon \downarrow 0} \varepsilon^{-1}\big(\psi_z(\lambda_{z,+}) - \psi_z(\lambda_{z,+} - \varepsilon)\big)$
$= \mathbb{E}[\ell_{V_{\lambda_{z,+},z}}]$.

Proof

$$\frac{\partial}{\partial \lambda}\psi_z(\lambda) = \frac{\partial}{\partial \lambda}\log\Big(\sum_{m=1}^{\infty} e^{\lambda m} C_z(m)\Big) = \frac{\sum_{m=1}^{\infty} m e^{\lambda m} C_z(m)}{\sum_{m=1}^{\infty} e^{\lambda m} C_z(m)} = \mathbb{E}[\ell_{V_{\lambda,z}}].$$

Similarly

$$\frac{\partial^2}{\partial \lambda^2}\psi_z(\lambda) = \mathrm{Var}(\ell_{V_{\lambda,z}})$$

so ψ_z is convex and strictly increasing. Since

$$A^k(z,z) = e^{k \log \rho(A) + o(k)} \quad \text{as } k \to \infty,$$

we have $G_\lambda(z,z) < \infty$ for $\lambda > -\log \rho(A)$ and $G_\lambda(z,z) = \infty$ for $\lambda < -\log \rho(A)$. By earlier results $G_\lambda(z,z) < \infty \Leftrightarrow \psi_z(\lambda) < 0$, hence $\lambda_* = -\log \rho(A)$. ∎

R-recurrence: proof of basic theorem

Recall:

> **Theorem (Equivalent recurrent weight function)** Let $\mathbb{A}$ be an irreducible weight function on a directed graph $G = (S, E)$. Then, for each $\rho(A) \leq e^{-\lambda} < \infty$ and $z \in S$, there exists a unique subMarkovian weight function $\mathbb{P}_{(\lambda),z}$ such that
>
> $$\text{(i) } \mathbb{A} \sim_{e^{-\lambda}} \mathbb{P}_{(\lambda),z} \qquad \text{(ii) } \mathbb{P}_{(\lambda),z} \text{ is recurrent to } z.$$

Uniqueness proved. Existence proved for $\psi_z(\lambda) < 0$.

Proof of existence in remaining case When $\psi_z(\lambda) \leq 0$, define

$$X^\lambda := V_1 \circ V_2 \circ \cdots \circ V_K \quad \text{or} \quad X^\lambda := V_1 \circ V_2 \circ \cdots ,$$

where $V_1, V_2, \ldots$ are i.i.d. with law $\hat{\nu}_{\lambda,z}$ and K is geometrically distributed with

$$\mathbb{P}[K = k] = e^{k\psi_z(\lambda)}\big(1 - e^{\psi_z(\lambda)}\big) \qquad (k \geq 0).$$

If $\psi_z(\lambda) < 0$, then for $e \in E$, $\omega \in \Omega_{z,e^-}$

$$\mathbb{P}_{(\lambda),z}(e) = \mathbb{P}[\vec{X}^\lambda_{n+1} = e \,|\, X^\lambda_{[0:n]} = \omega] = \frac{\mathbb{P}[X^\lambda_{[0:n+1]} = \omega \circ e]}{\mathbb{P}[X^\lambda_{[0:n]} = \omega]}.$$

If $\psi_z(\lambda) = 0$, then $\lambda = -\log \rho(A) = \lambda_*$. Now

$$\mathbb{P}_{(\lambda),z}(e) \underset{\lambda\uparrow\lambda_*}{\longrightarrow} \mathbb{P}_{(\lambda_*),z}(e)$$

and X^{λ_*} is a Markov chain with transition function $\mathbb{P}_{(\lambda_*),z}$. Since

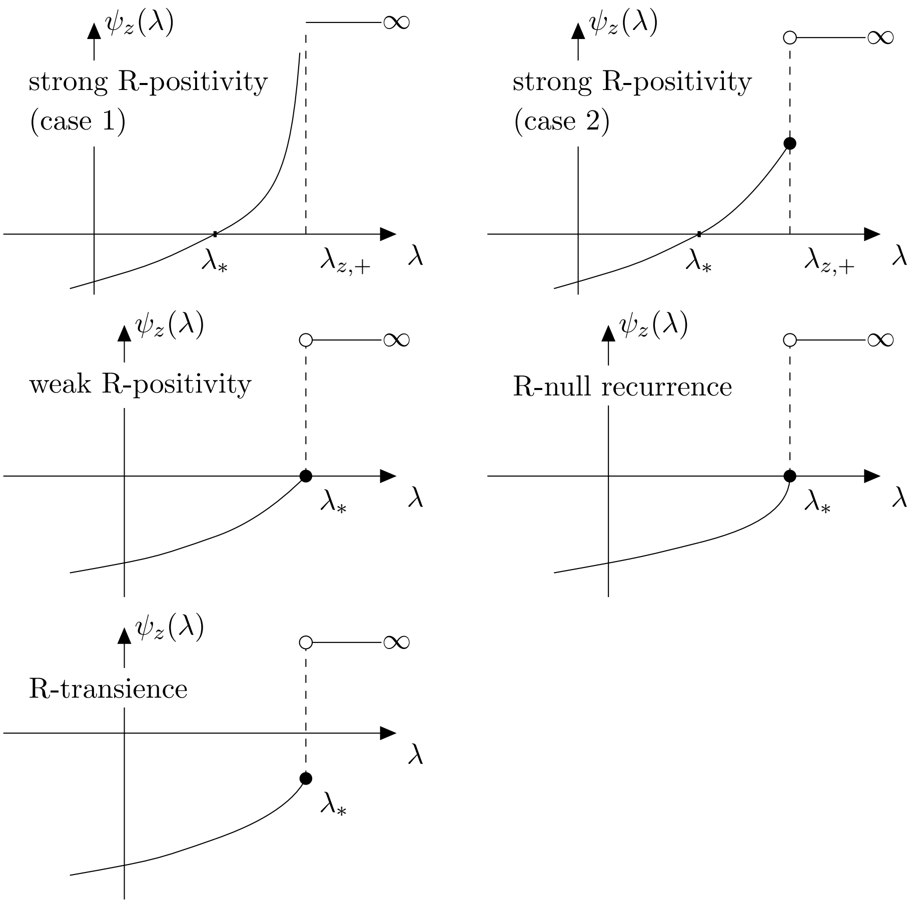
$$X^{\lambda_*} = V_1 \circ V_2 \circ \cdots$$

$\mathbb{P}_{(\lambda_*),z}$ is recurrent. Letting $\lambda \uparrow \lambda_*$ in

$$\mathbb{P}_{(\lambda),z}(\omega) = e^{\lambda \ell_\omega} \mathbb{A}(\omega) \qquad (\omega \in \Omega_{z,z}),$$

using earlier results, we obtain $\mathbb{A} \sim_{e^{-\lambda_*}} \mathbb{P}_{(\lambda_*),z}$. ∎

The logarithmic moment generating function



Proposition (Characterization in terms of ψ_z) Let $\mathbb{A}$ be a weight function on a directed graph $G = (S, E)$ and let $z \in S$ satisfy $0 < \rho_z(A) < \infty$. Then

(a) z is R-recurrent if and only if $\psi_z(\lambda_*) = 0$.

(b) z is R-positive if and only if $\lim\limits_{\varepsilon \downarrow 0} \varepsilon^{-1}\big(0 - \psi_z(\lambda_* - \varepsilon)\big) < \infty$.

(c) z is strongly R-positive if and only if $\lambda_* < \lambda_{z,+}$.

Proof (a): z is R-recurrent $\Leftrightarrow$ $\mathbb{P}_{\rho_z(A),z}$ is Markovian $\Leftrightarrow$ $\psi_z(\lambda_*) = 0$.
Assume R-recurrence. Let X Markov chain with transition function $\mathbb{P} = \mathbb{P}_{\rho_z(A),z}$ started in z and $\sigma_z := \inf\{k \geq 1 : X_k = z\}$ first return time. Then $\sigma_z \stackrel{\mathrm{d}}{=} \ell_{V_{\lambda_*,z}}$ where $V_{\lambda_*,z}$ has law $\hat{\nu}_{\lambda_*,z}(\omega) := e^{\lambda_* \ell_\omega} \mathbb{A}(\omega)$ $(\omega \in \widehat{\Omega}_z)$.
(b): $\frac{\partial}{\partial \lambda} \psi_z(\lambda) = \mathbb{E}[\ell_{V_{\lambda,z}}]$.

(c): $\mathbb{E}^z\big[e^{\varepsilon\sigma_z}\big] = \sum_{\omega\in\widehat{\Omega}_z} e^{\lambda_*\ell_\omega}\mathbb{A}(\omega)e^{\varepsilon\ell_\omega} = \psi_z(\lambda_*+\varepsilon).$ ∎

Finite matrices

View $G = V\cup E$ as the disjoint union of its vertex and edge set.
$F\subset V\cup E$ is a <u>subgraph</u> if

$$e^-, e^+ \in F\cap S \quad \forall e\in F\cap E.$$

Set of <u>excursions away from</u> F

$$\widehat{\Omega}(F) := \big\{\omega\in\Omega(G) : \ell_\omega \geq 1,\ \omega^- = \omega^+ = z,\ \omega_k \neq z\ \forall 0<k<\ell_\omega,\\ \vec{\omega}_k \neq z\ \forall 0<k\leq \ell_\omega\big\}.$$

Generalize:

$$\psi^F_{x,y}(\lambda) := \log\phi^F_{x,y}(\lambda) \quad \text{with} \quad \phi^F_{x,y}(\lambda) := \sum_{\omega\in\widehat{\Omega}_{x,y}(F)} e^{\lambda\ell_\omega}\mathbb{A}(\omega)$$

Proposition (Continuity of the l.m.g.f.) Let G be a finite directed graph and $\mathbb{A}: E\to[0,\infty)$ a weight function. Then for each subgraph F and vertices $x,y\in F$, the function $\psi^F_{x,y}:\mathbb{R}\to[0,\infty]$ is continuous.

Corollary (Finite matrices) Let $A: S^2\to[0,\infty)$ be an irreducible nonnegative matrix indexed by a finite set S. Then A is strongly R-positive.

Further implications:

- Perron-Frobenius
- Finite irreducible probab. kernel is strongly pos. rec.

Proof of the proposition By induction, using two lemmas. ∎

Lemma (Removal of an edge) Assume F a subgraph, $e\in F\cap E$, and $F' := F\backslash\{e\}$. Then

$$\phi^{F'}_{x,y}(\lambda) = \begin{cases} \phi^F_{x,y}(\lambda) + e^\lambda\mathbb{A}(e) & \text{if } e^- = x,\ e^+ = y,\\ \phi^F_{x,y}(\lambda) & \text{otherwise}\end{cases}$$

$(x,y\in F\cap V,\ \lambda\in\mathbb{R})$.

Proof $\widehat{\Omega}(F') = \widehat{\Omega}(F) \cup \{e\}$. ∎

Lemma (Removal of an isolated vertex) Assume F, F' subgraphs, $x \in F \cap V$, and $F' := F \backslash \{z\}$. Then

$$\phi^{F'}_{x,y}(\lambda) = \phi^F_{x,y}(\lambda) + \phi^F_{x,z}(\lambda)\phi^F_{z,y}(\lambda)\big(1 - \phi^F_{z,z}(\lambda)\big)^{-1}$$

$(x, y \in F' \cap V,\ \lambda \in \mathbb{R})$.

Proof Distinguishing excursions away from F' according to how often they visit the vertex z, we have

$$\begin{aligned}\phi^{F'}_{x,y}(\lambda) = &\sum_{\omega_{x,y}} e^{\lambda \ell_{\omega_{x,y}}} \mathbb{A}(\omega_{x,y}) \\ &+ \sum_{k=0}^{\infty} \sum_{\omega_{x,z}} \sum_{\omega_{z,y}} \sum_{\omega^1_{z,z}} \cdots \sum_{\omega^k_{z,z}} e^{\lambda(\ell_{\omega_{x,z}} + \ell_{\omega_{z,y}} + \ell_{\omega^1_{z,z}} + \cdots + \ell_{\omega^k_{z,z}})} \\ &\qquad \times \mathbb{A}(\omega_{x,z})\mathbb{A}(\omega_{z,y})\mathbb{A}(\omega^1_{z,z}) \cdots \mathbb{A}(\omega^k_{z,z}),\end{aligned}$$

where we sum over $\omega_{x,y} \in \widehat{\Omega}_{x,y}(F)$ etc. Rewriting gives

$$\begin{aligned}\phi^{F'}_{x,y}(\lambda) = &\sum_{\omega_{x,y}} e^{\lambda \ell_{\omega_{x,y}}} \mathbb{A}(\omega_{x,y}) \\ &+ \big(\sum_{\omega_{x,z}} e^{\lambda \ell_{\omega_{x,z}}} \mathbb{A}(\omega_{x,z})\big)\big(\sum_{\omega_{z,y}} e^{\lambda \ell_{\omega_{z,y}}} \mathbb{A}(\omega_{z,y})\big) \sum_{k=0}^{\infty} \big(\sum_{\omega_{z,z}} e^{\lambda \ell_{\omega_{z,z}}} \mathbb{A}(\omega_{z,z})\big)^k.\end{aligned}$$

∎

Finite modifications

Def $\mathcal{E}_{\mathbb{A},r} := \{e \in E : \mathbb{A}(e) = r\}$
Def $\mathbb{A}, \mathbb{B}$ finite modifications of each other $\Leftrightarrow$

(i) $\mathcal{E}_{\mathbb{A},0} = \mathcal{E}_{\mathbb{B},0}$ and $\mathcal{E}_{\mathbb{A},\infty} = \mathcal{E}_{\mathbb{B},\infty}$.

(ii) $\{e \in E : A(e) \neq \mathbb{B}(e)\}$ is finite.

If A not irred., def $\rho(A) := \sup_z \rho_z(A)$.

Theorem (Strong R-positivity)
Assume $\mathbb{A}$ irreducible, $\rho(A) < \infty$, $\mathbb{A}' \leq \mathbb{A}$, $\mathbb{A}' \neq \mathbb{A}$.

(a) A strongly R-positive $\Rightarrow \rho(A') < \rho(A)$.

(b) $\rho(A') < \rho(A)$, A' fin. modif. of $\mathbb{A} \Rightarrow A$ strongly R-positive.

$$\lambda^F_{x,y,+} := \sup\{\lambda \in \mathbb{R} : \psi^F_{x,y}(\lambda) < \infty\},$$
$$\lambda^F_{x,y,*} := \sup\{\lambda \in \mathbb{R} : \psi^F_{x,y}(\lambda) < 0\}.$$

Proposition (Exponential moments of excursions)
Assume $\mathbb{P}$ subMarkovian irreducible. If

$$\lambda^F_{x,y,+} > 0 \ \forall x, y \in F \cap S$$

holds for some finite subgraph F, then it holds for all finite subgraphs.

Proof (sketch) It suffices to prove the statement for two subgraphs that differ by a single edge or vertex. Now use the lemmas before. ∎

Proof of the theorem Def ψ_z, ψ'_z in terms of $\mathbb{A}, \mathbb{A}'$.
(a): $\psi'_z < \psi_z$ on $\{\lambda : \psi'_z(\lambda) < \infty\}$. A strongly R-positive $\Rightarrow \lambda_* < \lambda_{z,+}$. Now $\lambda'_* > \lambda_* \Rightarrow \rho(A') < \rho(A)$.
(b): $\mathbb{A} \sim_{\rho(A),f} \mathbb{P}$ and $\mathbb{A}' \sim_{\rho(A),f} \mathbb{P}'$ for subMarkovian $\mathbb{P}, \mathbb{P}'$ with $\rho(\mathbb{P}) = 1$.
Suffices to prove $\rho(\mathbb{P}') < 1 \Rightarrow \mathbb{P}$ strongly R-positive.
$\rho(\mathbb{P}') = 1 \Leftrightarrow \lambda'_* = 0 \Leftrightarrow \lambda'_{z,+} \leq 0$.
Choose finite F such that $\mathbb{A} = \mathbb{A}'$ outside F. Then

$$\lambda_{z,+} = 0 \quad \Leftrightarrow \quad \lambda^F_{x,y,+} \leq 0 \text{ for some } x, y \in F \cap S \quad \Leftrightarrow \quad \lambda'_{z,+} \leq 0.$$

A strongly R-positive $\Leftrightarrow \lambda'_{z,+} > 0$. ∎

R-transience

Theorem (R-transience) Assume $\mathbb{A}$ irreducible, $\mathbb{A} \leq \mathbb{A}'$, $\mathbb{A} \neq \mathbb{A}'$. Let $E' \subset E$ finite. Def $\mathbb{A}_\varepsilon := \mathbb{A} + \varepsilon 1_{E'}$.

(a) A R-transient $\Rightarrow \rho(A_\varepsilon) = \rho(A)$ for some $\varepsilon > 0$.
(b) $\rho(A) = \rho(A') \Rightarrow A$ R-transient.

Proof (a): Set $S' := \{e^- : e \in E'\}$. Then $\mathbb{A} \sim_{\rho(A),f} \mathbb{P}$ with $\mathbb{P}$ subMarkovian and $\sum_y P(x,y) < 1$ $(x \in S')$. Then $\mathbb{P} + \varepsilon 1_{E'}$ still subMarkovian $\Rightarrow \rho(\mathbb{P}') = 1 = \rho(\mathbb{P})$.
(b): $\mathbb{A} \neq \mathbb{A}' \Rightarrow \psi_z(\lambda) < \psi'_z(\lambda)$ on $(-\infty, \lambda_{z,+}]$ and $\rho(A) = \rho(A') \Rightarrow \lambda_* = \lambda'_*$. Now $\psi_z(\lambda_*) < \psi'_z(\lambda'_*) \leq 0 \Rightarrow A$ R-transient. ∎

Bibliographical notes

The Perron-Frobenius theorem was proved for strictly positive matrices by Perron in [Per07] and then generalized to irreducible matrices by Frobenius in [Fro12]. Kreĭn and Rutman [KR48] proved a generalization of the Perron-Frobenius theorem for Banach spaces of real functions. The basic facts about R-recurrence were proved by Vere-Jones in [Ver62, Ver67]. His proof is based on generating functions and does not mention Gibbs measures. For finite, strictly positive matrices, equivalence is defined in [Geo88, formula (11.5)] and it is proved there that two matrices are equivalent if and only if they define the same Gibbs measures. In [Num84], the theory of R-recurrence is extended to Markov chains with uncountable state space. After 1985, the study of R-recurrence was largely forgotten by probabilists, but was taken up by people working in ergodic theory. Salama [Sal88] proved the characterization of strong R-positivity in terms of finite modifications for matrices that can only take the values 0 and 1. His proof contained errors, which were corrected in [Rue03]. As far as I am aware, the only published proof of this theorem for general nonnegative matrices is in a survey paper of Gurevich and Savchenko [GS98, Thm 3.15]. I am indebted to Sergey Savchenko for pointing out the references to the literature in ergodic theory. There is even a version of this sort of results for Gibbs measures that do not have nearest-neighbor interactions [CS09]. I was not aware of the ergodic theory literature when I wrote down my own proof of the characterization of strong R-positivity in terms of finite modifications in [Swa17].

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