On rebellious voter models

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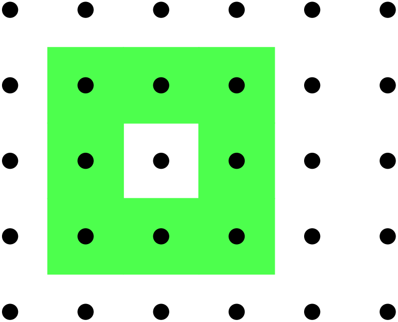
joint with Anja Sturm and Karel Vrbenský

The Neuhauser-Pacala model

Denote a point in $\mathbb{Z}^d$ by $i = (i_1, \ldots, i_d)$.

Def *neighborhood* of a site $\mathcal{N}_i := \{j \in \mathbb{Z}^d : 0 < \|i - j\|_\infty \leq R\}$.



(Here $R = 1$, $d = 2$).

The Neuhauser-Pacala model

Def *local frequency* $f_\tau(i) := |\mathcal{N}_i|^{-1}|\{j \in \mathcal{N}_i : x(j) = \tau\}|$.

1	0	1	1	0	0
1	1	0	1	1	1
1	1	0	0	1	1
0	0	1	1	1	0
1	0	1	1	1	0

Here $f_0(i) = 3/8$, $f_1(i) = 5/8$.

Fix rates $\alpha_{01}, \alpha_{10} \geq 0$.

1	0	1	1	0	0
1	1	0	1	1	1
1	1	†	0	1	1
0	0	1	1	1	0
1	0	1	1	1	0

With rate $f_0 + \alpha_{01} f_1$ an organism of type 0 dies...

The Neuhauser-Pacala model

1	0	1	1	0	0
1	1	0	1	1	1
1	1	1	0	1	1
0	0	1	1	1	0
1	0	1	1	1	0

. . . and is replaced by a random type from the neighborhood.

The Neuhauser-Pacala model

Neuhauser & Pacala (1999): Markov process in the space $\{0,1\}^{\mathbb{Z}^d}$ of spin configurations $x = (x(i))_{i\in\mathbb{Z}^d}$, where spin $x(i)$ flips:

$$
\begin{aligned}
0 \mapsto 1 &\text{ with rate } f_1(f_0 + \alpha_{01} f_1),\\
1 \mapsto 0 &\text{ with rate } f_0(f_1 + \alpha_{10} f_0),
\end{aligned}
$$

with

$$
f_\tau(i) := \frac{|\{j \in \mathcal{N}_i : x(j) = \tau\}|}{|\mathcal{N}_i|} \quad \mathcal{N}_i := \{j : 0 < \|i-j\|_\infty \le R\}.
$$

the local frequency of type $\tau = 0,1$.

Interpretation: *Interspecific competition rates* α_{01}, α_{10}. Organism of type 0 dies with rate $f_0 + \alpha_{01} f_1$ and is replaced by type sampled at random from distance $\le R$.

The Neuhauser-Pacala model

Parameter α_{01} measures the strength of competition felt by type 0 from type 1 (compared to strength 1 from its own type).
If $\alpha_{01} < 1$, then type 0 dies *less* often due to competition from type 1 than from competition with its own type: *balancing selection*.
If $\alpha_{01} > 1$, then type 0 dies *more* often due to competition from type 1 than from competition with its own type, i.e., type 1 is an *agressive species*.

By definition, type 0 *survives* if starting from a single organism of type 0 and all other organisms of type 1, there is a positive probability that the organisms of type 0 never die out.

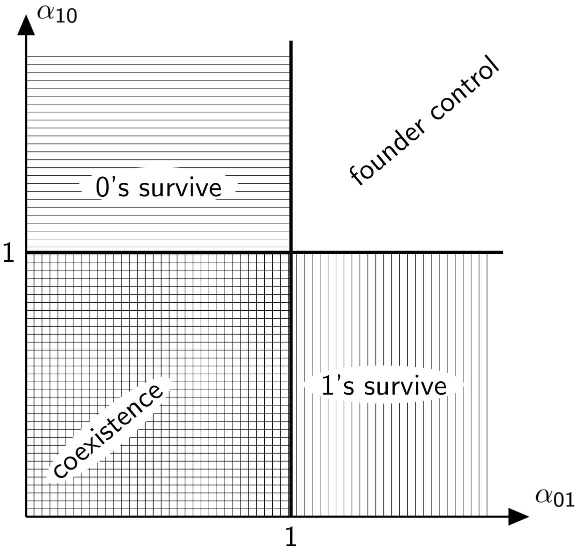
By definition, one has *coexistence* if there exists an invariant law concentrated on states where both types are present.

Mean field model

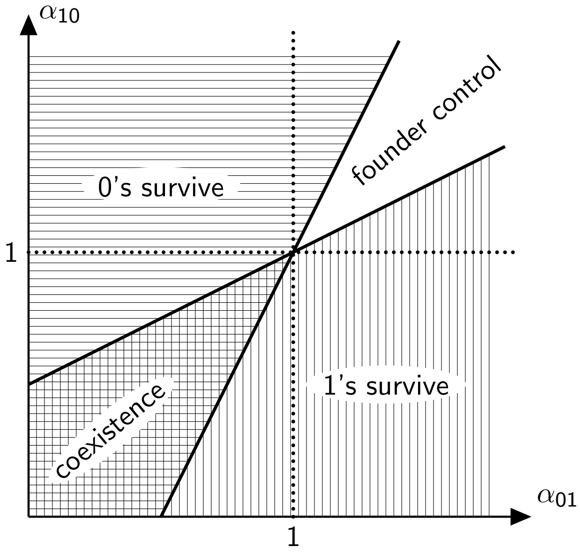
In the *mean field model*, the lattice $\mathbb{Z}^d$ is replaced by a complete graph with N vertices. The neighborhood $\mathcal{N}_i$ of a vertex i is simply all sites except i.
In the limit $N \to \infty$, the frequencies $F_\tau(t)$ of type $\tau = 0, 1$ satisfy a differential equation:

$$\frac{\partial}{\partial t} F_0(t) = F_1(t)\big(F_0(t) + \alpha_{01} F_1(t)\big) - F_0(t)\big(F_1(t) + \alpha_{10} F_0(t)\big),$$
$$\frac{\partial}{\partial t} F_1(t) = -\frac{\partial}{\partial t} F_0(t).$$

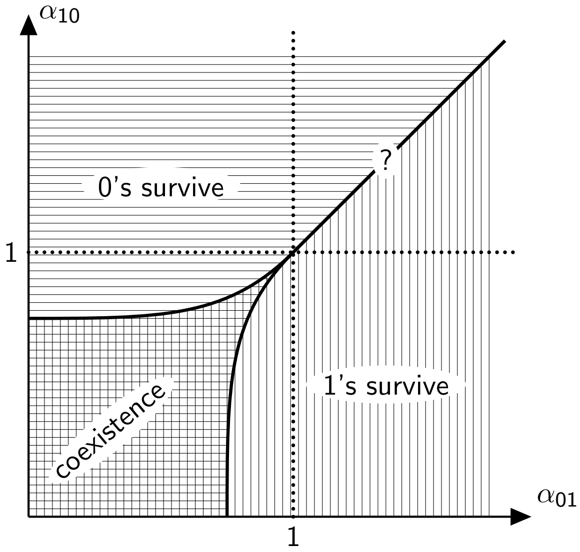
Mean field model



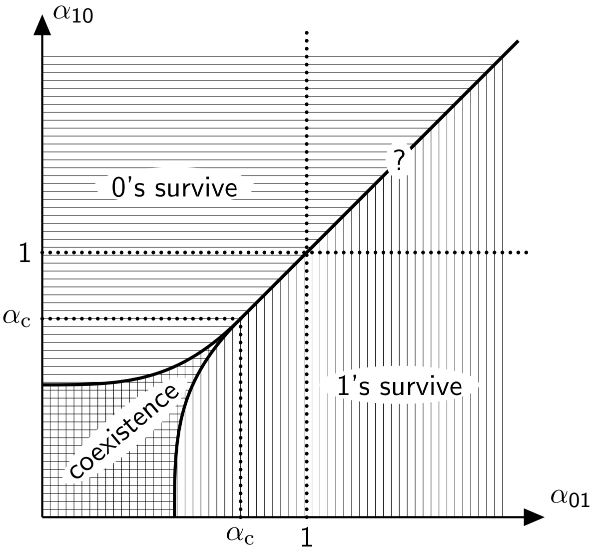
Dimension $d \geq 3$



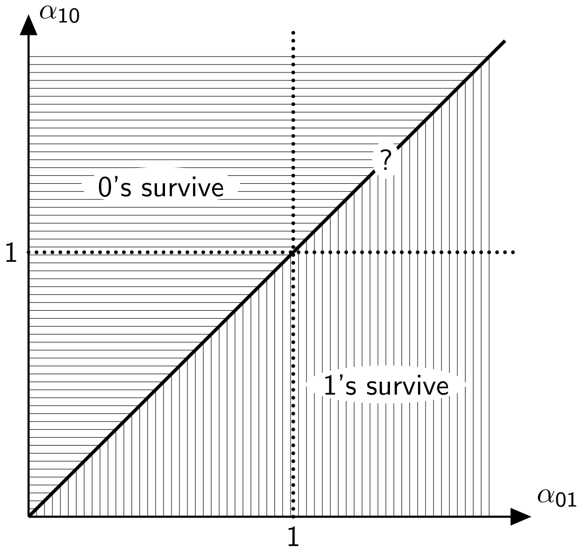
Dimension $d = 2$



Dimension $d = 1$, range $R \geq 2$



Dimension $d = 1$, range $R = 1$



Rigorous results

Sudbury, AOP, 1990

Neuhauser & Pacala, AAP, 1999

Cox & Perkins, AOP, 2005

Cox & Perkins, PTRF, 2007

Cox & Perkins, AAP, 2008

Sturm & S., AAP, 2008

Sturm & S., ECP, 2008

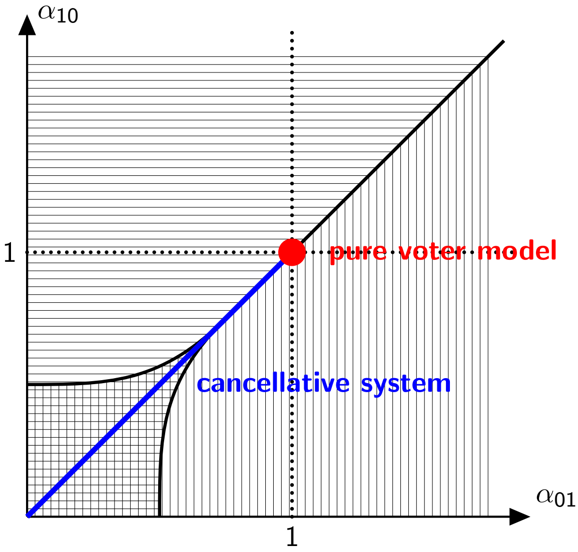
Cox, Merle, & Perkins, EJP, 2010

S., ECP, 2013

Cox, Durrett, & Perkins, Astérisque, 2013

Cox & Perkins, AAP, 2014

Special models



Cancellative systems

Equip $\{0,1\}$ with the usual product and with addition modulo 2, denoted as $\oplus$. Then $\{0,1\}$ is a *finite field*. We may view $\{0,1\}^{\mathbb{Z}^d}$ (equipped with $\oplus$) as a *linear space* over $\{0,1\}$.

A *cancellative system* $X = (X_t)_{t\geq 0}$ is a *linear system* w.r.t. to the finite field $\{0,1\}$, that evolves as

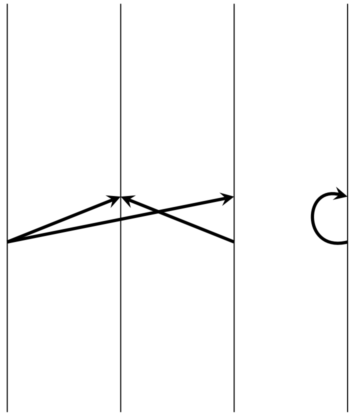
$$x \mapsto x \oplus Ax \quad \text{with rate } r(A) \geq 0,$$

where

$$Ax(i) := \bigoplus_{j\in\mathbb{Z}^d} A(i,j)x(j)$$

with $A(i,j) = 1$ for finitely many i,j and $A(i,j) = 0$ otherwise.

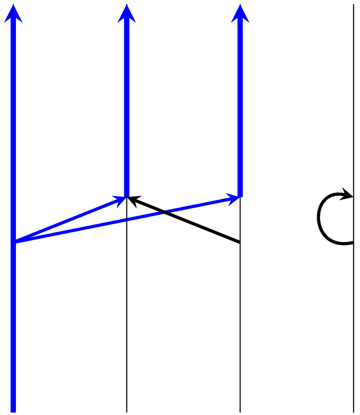
Graphical representation



Draw an arrow $i \to j$ whenever $A(j,i) = 1$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Graphical representation



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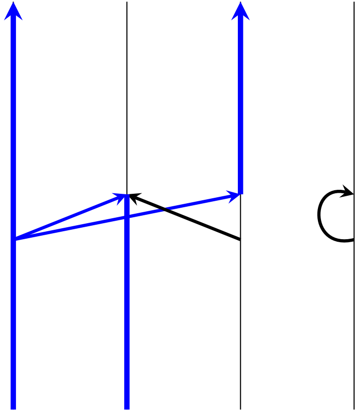

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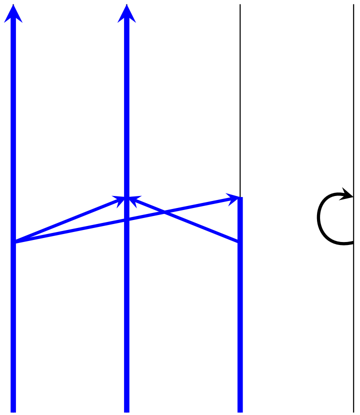

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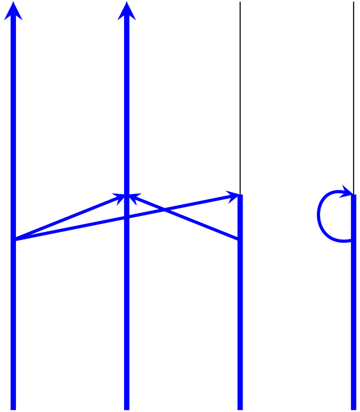


Draw an arrow $i \to j$ whenever $A(j,i) = 1$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The symmetric model

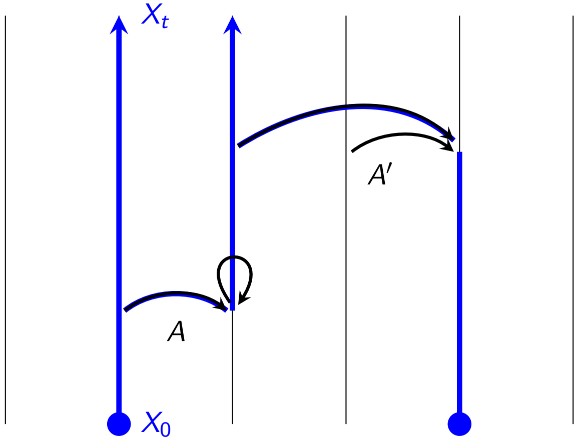
Claim The symmetric Neuhauser-Pacala model with $\alpha := \alpha_{01} = \alpha_{10} \leq 1$ is cancellative.

Proof For each i:

- With rate α, choose uniform $j \in \mathcal{N}_i$, draw two arrows $i \mapsto i$ and $j \mapsto i$.
- With rate $1 - \alpha$, choose uniform, independent $j, k \in \mathcal{N}_i$. If $j = k$, do nothing; if $j \neq k$, draw two arrows $j \mapsto i$ and $k \mapsto i$.

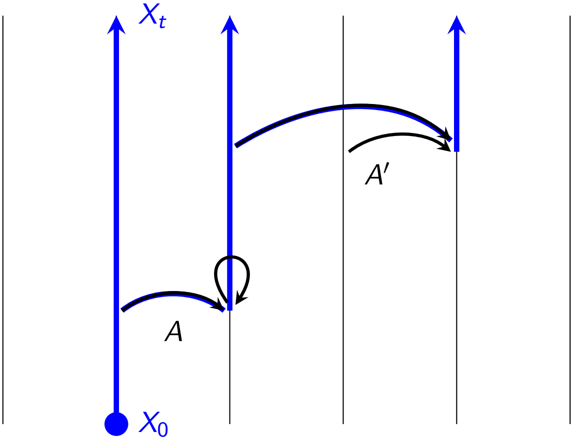
Check that this yields the desired flip rates.

Graphical representation



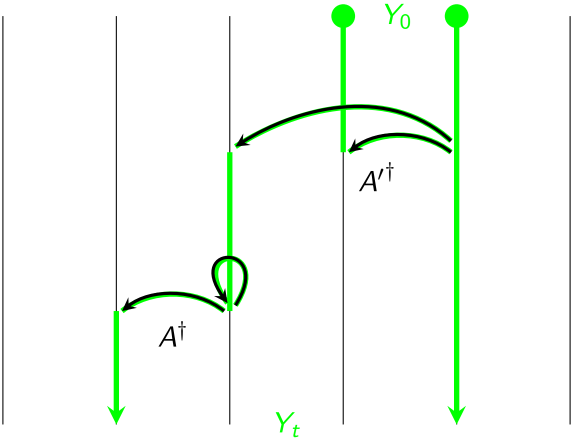
$X_t(i) = 1$ iff there is a odd number of paths from X_0 to (i, t).

Graphical representation



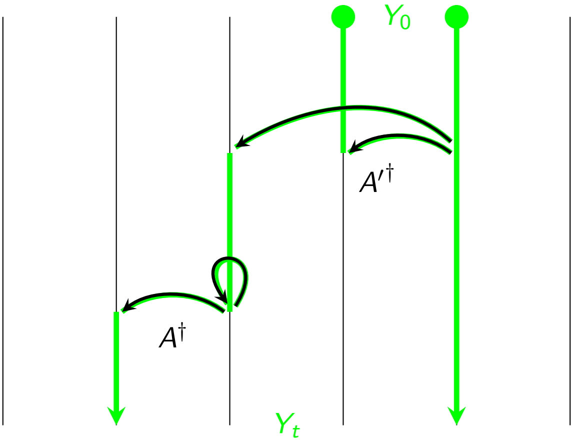
$X_t(i) = 1$ iff there is a odd number of paths from X_0 to (i, t).

Graphical representation of dual model



Time runs backwards and all arrows are reversed.

Dual of the Neuhauser-Pacala model



The dual Y is a *parity preserving* system of *double branching* and *annihilating* random walks.

Cancellative system duality

Rates of the dual model:

$$r_Y(A^\dagger) = r_X(A),$$

where $A^\dagger(i,j) = A(j,i)$ denotes the *adjoint* of A.

Duality:

$$\mathbb{P}\big[|X_t Y_0| \text{ is odd}\big] = \mathbb{P}\big[|X_0 Y_t| \text{ is odd}\big] \qquad (t \geq 0)$$

whenever X and Y are independent, where

$$|x| := \textstyle\sum_i x(i) \quad \text{and} \quad xy(i) := x(i)y(i).$$

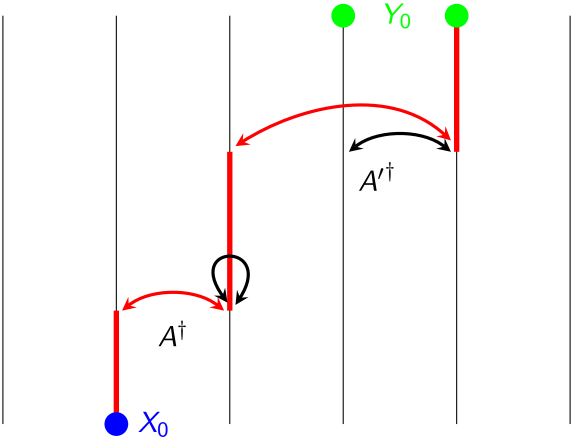
Alternative formulation

$$\mathbb{E}\big[\|X_t Y_0\|\big] = \mathbb{P}\big[\|X_0 Y_t\|\big] \qquad (t \geq 0),$$

where

$$\|xy\| := \textstyle\bigoplus_i x(i)y(i) = 1_{\{|xy| \text{ is odd}\}}.$$

Duality



$|X_0 Y_t|$ is odd $\Leftrightarrow$ # paths from X_0 to Y_0 is odd $\Leftrightarrow$ $|X_t Y_0|$ is odd.

Duality in general

Def A cancellative system X is *type symmetric* if the transition $x \mapsto x'$ has the same rate as $(1-x) \mapsto (1-x')$.

Def A cancellative system X is *parity preserving* if a.s. $|X_t|$ is odd iff $|X_0|$ is odd $(t \geq 0)$.

- X type symmetric iff jumps only for A with $|\{j : A(i,j) = 1\}|$ even for all i. *Even number of incoming arrows* at each site.
- X parity preserving iff jumps only for A with $|\{j : A(j,i) = 1\}|$ even for all i. *Even number of outgoing arrows* at each site.

Consequence X type symmetric $\Leftrightarrow$ dual Y is parity preserving.

Interfaces

From now on, restrict to one dimension.

Let $\mathbb{Z} + \frac{1}{2} := \{k + \frac{1}{2} : k \in \mathbb{Z}\}$ and let $\mathbb{I} = \mathbb{Z}$ or $= \mathbb{Z} + \frac{1}{2}$.
Define a gradient operator $\nabla : \{0,1\}^{\mathbb{I}} \to \{0,1\}^{\mathbb{I}+\frac{1}{2}}$ by

$$\nabla x(i) := x(i - \tfrac{1}{2}) \oplus x(i + \tfrac{1}{2}).$$

If $(X_t)_{t\geq 0}$ is type symmetric, then $(\nabla X_t)_{t\geq 0}$ is a Markov process: the *interface model* of X.

$$\begin{array}{ccccccccccccccccc} X & 0 & & 1 & & 1 & & 1 & & 0 & & 0 & & 1 & & 0 \\ \nabla X & & 1 & & 0 & & 0 & & 1 & & 0 & & 1 & & 1 & \end{array}$$

Interface models are always parity preserving.

Interfaces and duality

[S. '13] The interface model of a type symmetric cancellative spin system is a parity preserving cancellative spin system. Conversely, every parity preserving cancellative spin system is the interface model of a unique type symmetric cancellative spin system. Moreover, the following commutative diagram holds:

$$\begin{array}{ccc} X & \xrightarrow{\text{interface}} & Y \\ \text{dual}\updownarrow & & \updownarrow\text{dual} \\ Y' & \xleftarrow[\text{interface}]{} & X' \end{array}$$

Here X, X' are type symmetric and Y, Y' are parity preserving.

Proof (sketch) Recall the duality function

$$\|xy\| = \bigoplus_i x(i)y(i) = 1_{\{|xy| \text{ is odd}\}}.$$

Then

$$\|x\nabla y\| = \|(\nabla x)y\| \qquad (x \in \{0,1\}^{\mathbb{I}},\ y \in \{0,1\}^{I+\frac{1}{2}}).$$

If A is type symmetric, then $A^\dagger$ is the dual action and $\nabla A \nabla^{-1}$ is the corresponding action on interfaces. Now

$$(\nabla A \nabla^{-1})^\dagger = \nabla^{-1} A^\dagger \nabla$$

correspond to the dual of the interface model resp. the model whose interface model is the dual.

Classification of behavior

Let Y be parity preserving.

Def Y *persists* if there exists an invariant law that is concentrated on states other than $\underline{0}$ (all zero).

Def Y *survives* if $\mathbb{P}^y[Y_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some *even* initial state y.

If $|Y_0|$ is finite and odd, then let $l_t := \inf\{i \in \mathbb{Z} + \frac{1}{2} : Y_t(i) = 1\}$ denote the left-most one and let

$$\hat{Y}_t(i) := Y(l_t + i) \qquad (t \geq 0,\ i \in \mathbb{N})$$

denote the process Y *viewed from the left-most one*.

Def Y is *stable* if $\hat{Y}$ is positively recurrent.

Def Y is *strongly stable* if $\hat{Y}$ is stable and $\mathbb{E}\big[|\hat{Y}_\infty|\big] < \infty$ in equilibrium.

Classification of behavior

Let X be type symmetric.

Def X exhibits *coexistence* if there exists an invariant law that is concentrated on states other than $\underline{0}$ and $\underline{1}$.

Def X *survives* if $\mathbb{P}^y[X_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some *finite* initial state y.

Def X exhibits *(strong) interface tightness* if its interface model is (strongly) stable.

Interface tightness introduced for the contact process by Cox & Durrett (1995) and studied by Belhaouari, Mountford & Valle (2007) and & Sturm & S. (2008).

Abstract results

$$\begin{array}{ccc} X & \xrightarrow{\text{interface}} & Y \\ \text{dual} \updownarrow & & \updownarrow \text{dual} \\ Y' & \xleftarrow[\text{interface}]{} & X' \end{array}$$

Claim

interface model Y persists $\Leftrightarrow$ X coexists $\Leftrightarrow$ dual Y' survives.

Proof of second claim

Start X in product measure with intensity $1/2$. Then
$\mathbb{P}\big[X_t(i) \neq X_t(j)\big] = \mathbb{P}\big[|X_t(\delta_i + \delta_j)| \text{ is odd}\big] =$
$\mathbb{P}^{\delta_i+\delta_j}\big[|X_0 Y'_t| \text{ is odd}\big] = \frac{1}{2}\mathbb{P}^{\delta_i+\delta_j}\big[Y'_t \neq \underline{0}\big]$
$\underset{t\to\infty}{\longrightarrow} \frac{1}{2}\mathbb{P}^{\delta_i+\delta_j}\big[Y'_t \neq \underline{0}\ \forall t \geq 0\big]$. *Odd upper invariant law.*

Claim X survives $\Leftrightarrow$ dual Y' persists. (Similar.)

Thm [S. '13] Strong interface tightness implies noncoexistence.

Strong interface tightness implies noncoexistence

Lemma Assume that strong interface tightness holds for X. Then

$$h(x) := \sum_{i \in \mathbb{Z}+\frac{1}{2}} \mathbb{E}\big[\|(\hat{Y}_\infty + i)x\|\big]$$

is a harmonic function for the process X' (dual of interface model of X). Moreover, there exist constants $0 < c \leq C < \infty$ s.t.

$$c|x| \leq h(x) \leq C|x|.$$

Proof of Thm (sketch) By martingale convergence, $h(X'_t)$ converges a.s., which implies that X' dies out a.s. The same holds for its interface model Y' which is dual to X, so by duality X exhibits noncoexistence.

Facts and conjectures

Symmetric Neuhauser-Pacala model in $d = 1$ with range $R \geq 2$.

Conjecture There exists an $0 < \alpha_{\rm c} < 1$ such that

- Survival, coexistence, no interface tightness for $0 \leq \alpha < \alpha_{\rm c}$.
- Extinction, noncoexistence, no interface tightness at $\alpha = \alpha_{\rm c}$.
- Strong interface tightness and (hence) noncoexistence for $\alpha > \alpha_{\rm c}$.

Known Survival, coexistence, no interface tightness for α small, strong interface tightness and noncoexistence at $\alpha = 1$.

Open problems Monotonicity in α, (strong) interface tightness & coexistence at any $\alpha < 1$, equivalence of survival and coexistence.

The rebellious voter model

In the *rebellious voter model* the type at i flips with rate

$$\begin{aligned}&\alpha\big[\tfrac{1}{2}1_{\{x(i-1)\neq x(i)\}}+\tfrac{1}{2}1_{\{x(i)\neq x(i+1)\}}\big]\\ +&(1-\alpha)\big[\tfrac{1}{2}1_{\{x(i-2)\neq x(i-1)\}}+\tfrac{1}{2}1_{\{x(i+1)\neq x(i+2)\}}\big].\end{aligned}$$

This model is *self-dual* in the sense that $X=X'$, i.e.,

$$\begin{tikzcd} X \arrow[d, bend right, leftrightarrow, "\text{dual}"'] \arrow[d, bend left, "\text{interface}"] \\ Y \end{tikzcd}$$

Consequence Survival equivalent to coexistence.

The disagreement voter model

The $d = 1$ Neuhauser-Pacala model X with range $R = 1$ is up to reparametrization equal to the *disagreement voter model*, in which the type at i flips with rate

$$\alpha\big[\tfrac{1}{2}1_{\{x(i-1)\neq x(i)\}} + \tfrac{1}{2}1_{\{x(i)\neq x(i+1)\}}\big]$$
$$+(1-\alpha)1_{\{x(i-1)\neq x(i+1)\}}.$$

The interface model Y of this is a mixture of annihilating random walk and exclusion dynamics.

Clearly Y dies out for all $\alpha > 0$ hence X exhibits noncoexistence.

The exclusion process

Recall that in the symmetric, nearest-neighbor exclusion process, pairs of neighboring 0's and 1's make the transitions $01 \leftrightarrow 10$ at rate one. This model is both type symmetric and parity preserving. It is part of a commutative diagram where:

X = pure disagreement dynamics

Y = exclusion process

Z = double branching annihilating process

$$
\begin{tikzcd}
X \arrow[r, "\text{interface}"] \arrow[d, leftrightarrow, "\text{dual}"'] & Y \arrow[r, "\text{interface}"] \arrow[d, leftrightarrow, "\text{dual}"'] & Z \arrow[d, leftrightarrow, "\text{dual}"'] \\
Z & Y \arrow[l, "\text{interface}"] & X \arrow[l, "\text{interface}"]
\end{tikzcd}
$$

Ergodic results

[Sturm & S. '08] A symmetric Neuhauser-Pacala or rebellious voter model have at most one spatially homogeneous coexisting invariant law. If moreover $\alpha > 0$ and the dual model Y' is not stable, then this is the long-time limit law started from any spatially homogeneous coexisting initial law.

[Sturm & S. '08] For the rebellious voter model with α sufficiently close to zero, there is a unique coexisting invariant law ν and one has *complete convergence*

$$\mathbb{P}[X_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \rho_0\delta_{\underline{0}} + \rho_1\delta_{\underline{1}} + (1-\rho_0-\rho_1)\nu,$$

where $\rho_\tau := \mathbb{P}[X_t = \underline{\tau} \text{ for some } t \geq 0]$.

[Cox & Perkins '14] There exists some $\alpha' < 1$ such that the symmetric Neuhauser-Pacala model in dimensions $d \geq 2$ exhibits complete convergence for $\alpha \in (\alpha', 1)$.

Ergodic results

Idea of proof Recall that if law of X_0 is product measure with intensity $1/2$, then

$$\mathbb{P}\big[|X_t y| \text{ is odd}\big] = \mathbb{P}^y\big[|X_0 Y'_t| \text{ is odd}\big] = \tfrac{1}{2}\mathbb{P}^y\big[Y'_t \neq \underline{0}\big].$$

As a consequence, $\mathbb{P}[X_t \in \cdot\,]$ converges weakly to $\nu_{1/2} := \mathbb{P}[X_\infty \in \cdot\,]$ characterized by

$$\mathbb{P}\big[|X_\infty y| \text{ is odd}\big] = \tfrac{1}{2}\mathbb{P}^y\big[Y'_t \neq \underline{0}\ \forall t \geq 0\big].$$

For more general initial laws, convergence will follow if

$$\mathbb{P}^y\big[|X_0 Y'_t| \text{ is odd}\big] \approx \tfrac{1}{2}\mathbb{P}^y\big[Y'_t \neq \underline{0}\big] \quad \text{as } t \to \infty.$$

This requires one to show that conditional on survival, Y'_t is large and sufficiently random so that $|X_0 Y'_t|$ is odd with probab. $\approx 1/2$.