Interacting Particle Systems with Applications in Finance

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Lecture 0: Markov Chain Preliminaries

Probability Kernels

Let S be a finite set. A *probability kernel* on S is a function $K : S \times S \to [0,1]$ such that

- $K(x,y) \geq 0$ $(x, y \in S)$,
- $\sum_{y \in S} K(x,y) = 1$ $(x \in S)$.

We calculate with kernels as with matrices, so we define the product of two kernels as:

$$(KL)(x,z) := \sum_{y \in S} K(x,y)L(y,z).$$

If $f : S \to \mathbb{R}$ is a real function on S, then we can let a kernel act on this function to the left or to the right:

$$(fK)(y) := \sum_{x \in S} f(x)K(x,y) \quad \text{and} \quad (Kf)(x) := \sum_{y \in S} K(x,y)f(y).$$

In particular, if $\mu : S \to [0,1]$ satisfies $\sum_{x\in S} \mu(x) = 1$, i.e., μ is a probability law on S, then also

$$\mu K \quad \text{is a probability law on } S.$$

For any probability law μ on S and function $f : S \to \mathbb{R}$ we define

$$\mu f := \sum_{x\in S} \mu(x) f(x),$$

which is the expectation of f under μ.

Markov Chains

The *Markov chain* with *transition kernel* P and *initial law* μ is the discrete-time stochastic process $X = (X_k)_{k\geq 0}$ whose finite-dimensional distributions are given by

$$\mathbb{P}\big[X_0 = x_0, \ldots, X_n = x_n\big] = \mu(x_0)P(x_0, x_1)\cdots P(x_{n-1}, x_n).$$

We observe that the law of X_1 is given by

$$\begin{aligned}\mathbb{P}\big[X_1 = x_1\big] &= \sum_{x_0} \mathbb{P}\big[X_0 = x_0,\ X_1 = x_1\big] \\ &= \sum_{x_0} \mu(x_0)P(x_0, x_1) = (\mu P)(x_1).\end{aligned}$$

Similarly,

$$\mathbb{P}\big[X_2 = x_2\big] = \sum_{x_0}\sum_{x_1} \mu(x_0)P(x_0, x_1)P(x_1, x_2) = (\mu P^2)(x_1).$$

and

$$\mathbb{P}\big[X_n = x_n \,\big|\, X_0 = x_0\big] = P^n(x_0, x_n).$$

Markov Chains

If $f : S \to \mathbb{R}$ is a function, then

$$\mathbb{E}\big[f(X_n) \,\big|\, X_0 = x_0\big] = \sum_{x_n} \mathbb{P}\big[X_n = x_n \,\big|\, X_0 = x_0\big] f(x_n)$$
$$= \sum_{x_n} P^n(x_0, x_n) f(x_n) = (Pf)(x_0).$$

Similarly, for the process with initial law μ,

$$\mathbb{E}\big[f(X_n)\big] = \mu P^n f.$$

Invariant Laws

Let P be the transition kernel of a Markov chain.
A probability measure ν on S is an invariant law for this Markov chain if $\nu P = \nu$, or equivalently, if the Markov chain $(X_k)_{k\geq 0}$ with initial law ν is stationary.

By definition, P is *irreducible* if for each $x, y \in S$ there exists an $n \geq 1$ such that $P^n(x, y) > 0$: every state can be reached from every state.

By definition, P is *aperiodic* if for some (ans hence for all) $x \in S$, the largest common divisor of $\{n \geq 1 : P^n(x, x) > 0\}$ is 1.

If P is irreducible, then it has a unique invariant law.
If P is moreover aperiodic, then the Markov chain started in any initial law μ satisfies

$$\mathbb{P}[X_n = x] = \mu P^n(x) \underset{n\to\infty}{\longrightarrow} \nu(x) \qquad (x \in S).$$

Traps

A point $z \in S$ is a *trap* if $P(z, z) = 1$.

Lemma 1 *Assume that there exists a subset $T \subset S$ such that:*

- *Each $z \in T$ is a trap.*
- *For each $x \notin T$ there is some $z \in T$ and $n \geq 1$ such that $P^n(x, z) > 0$.*

Then the Markov chain started in any initial law satisfies

$$\mathbb{P}\big[X_n \in T \text{ for some } n \geq 0\big] = 1.$$

Continuous Time

Let S be a finite set and let $G : S \times S \to \mathbb{R}$ satisfy

- $G(x,y) \geq 0 \ \forall x \neq y$,
- $\sum_y G(x,y) = 0$.

Then, by the finiteness of S, for $\varepsilon > 0$ small enough, setting

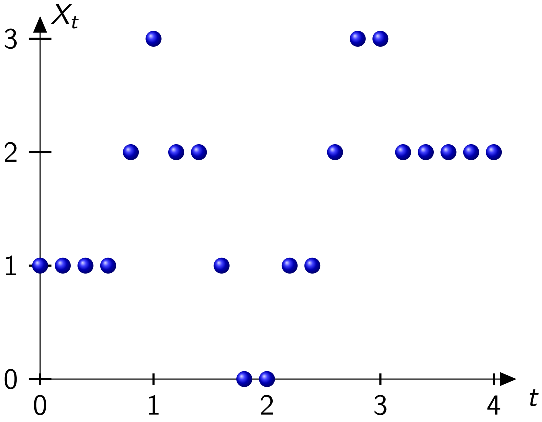
$$P_\varepsilon(x,y) := 1_{\{x=y\}} + \varepsilon G(x,y) \qquad (x,y \in S)$$

defines a probability kernel on S. We can use this kernel to construct a Markov chain

$$X_0, X_\varepsilon, X_{2\varepsilon}, \ldots$$

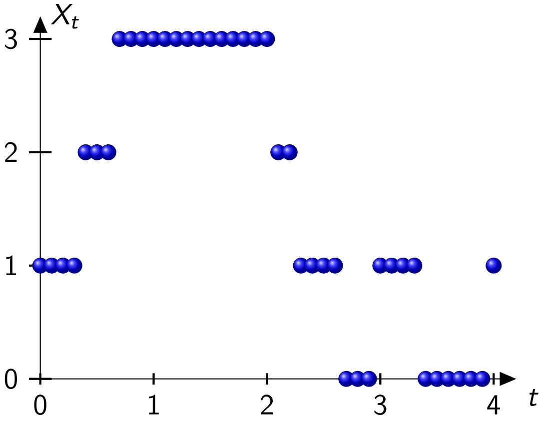
where we measure time in steps of size ε.

Continuous Time



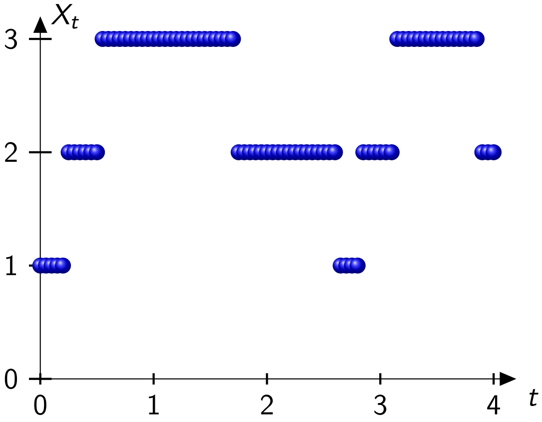
Time step $\varepsilon = 0.2$.

Continuous Time



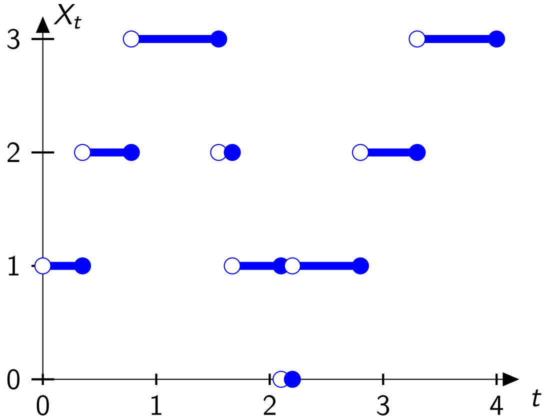
Time step $\varepsilon = 0.1$.

Continuous Time



Time step $\varepsilon = 0.05$.

Continuous Time



By convention, we take the limiting process right-continuous.

Continuous Time

In the limit, we obtain a continuous-time process $(X_t)_{t\geq 0}$ and a continuous family of transition probabilities $(P_t)_{t\geq 0}$. Here

$$P_t = e^{tG} := \sum_{n=0}^{\infty} \frac{1}{n!}(tG)^n,$$

satisfy

$$P_s P_t = P_{s+t} \quad \text{and} \quad \lim_{t\downarrow 0} P_t = P_0 = 1.$$

One has

$$P_t(x,y) = 1_{\{x=y\}} + tG(x,y) + O(t^2).$$

We interpret $G(x,y)$ as the *rate* of jumps from x to y.

Poisson construction of Markov processes

Each generator G has a *random mapping representation*

$$Gf(x) = \sum_{m \in \mathcal{M}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\},$$

where $(r_m)_{m \in \mathcal{M}}$ are nonnegative rates and $\mathcal{M}$ is a collection of maps $m : S \to S$. Let ω be a Poisson point set on $\mathcal{M} \times \mathbb{R}$ with intensity

$$\mu\big(\{m\} \times A\big) = r_m \, \ell(A) \qquad \big(A \in \mathcal{B}(\mathbb{R})\big),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel-σ-field on $\mathbb{R}$ and ℓ denotes Lebesgue measure. We may order the elements of

$$\omega \cap \mathcal{M} \times (s, t] =: \omega_{s,t} = \big\{(m_1, t_1), \ldots, (m_n, t_n)\big\}$$

with $t_1 < \cdots < t_n$.

Poisson construction of Markov processes

Define random maps $\mathbf{X}_{s,t} : S \to S$ $(s \leq t)$ by

$$\mathbf{X}_{s,t} := m_n \circ \cdots \circ m_1.$$

(Poisson construction of Markov processes) *Define maps $(\mathbf{X}_{s,t})_{s \leq t}$ as above in terms of a Poisson point set ω. Let X_0 be an S-valued random variable, independent of ω. Then*

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

is a Markov process with generator G.

Remark The sample paths of X are cadlag, i.e., right-continuous with left limits. We get left-continuous paths by defining

$$\mathbf{X}_{s,t-} \quad \text{in terms of} \quad \omega_{s,t-} := \omega \cap \mathcal{M} \times (s,t).$$

Example 1 Consider the Markov process with state space $\{0, 1\}$ that jumps with the rates

$$G(0, 1) := 2 \quad \text{and} \quad G(1, 0) := 1.$$

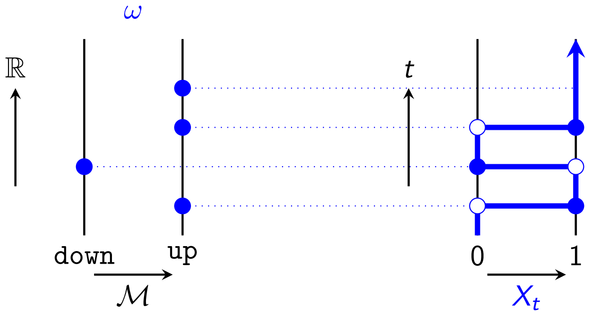
Consider $\mathcal{M} := \{\mathtt{up}, \mathtt{down}\}$, where

$$\mathtt{up}(x) := 1 \quad \text{and} \quad \mathtt{down}(x) := 0 \qquad \forall x = 0, 1.$$

Setting $r_{\mathtt{up}} := 2$ and $r_{\mathtt{down}} := 1$, we have

$$Gf(x) = r_{\mathtt{up}}\big\{f\big(\mathtt{up}(x)\big) - f\big(x\big)\big\} + r_{\mathtt{down}}\big\{f\big(\mathtt{down}(x)\big) - f\big(x\big)\big\}.$$

Poisson construction of Markov processes



Since it may happen that $m(X_t) = X_t$, not every time of the Poisson process corresponds to a jump of the Markov process.

Random mapping representations are not unique!

Example 2 The same Markov process, that jumps

$$0 \mapsto 1 \text{ with rate } 2 \quad \text{and} \quad 1 \mapsto 0 \text{ with rate } 1$$

can also be represented as

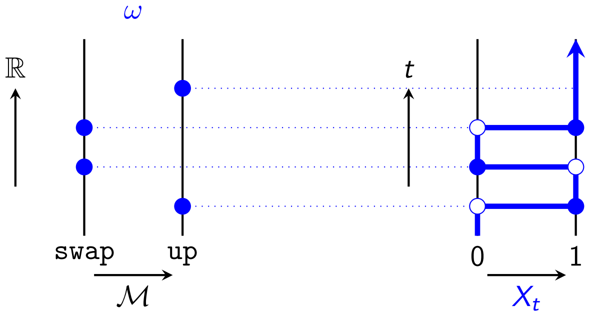
$$Gf(x) = r_{\mathtt{up}}\big\{f\big(\mathtt{up}(x)\big) - f\big(x\big)\big\} + r_{\mathtt{swap}}\big\{f\big(\mathtt{swap}(x)\big) - f\big(x\big)\big\},$$

with $r_{\mathtt{up}} := 1$, $r_{\mathtt{swap}} := 1$, and

$$\mathtt{up}(x) := 1 \quad \text{and} \quad \mathtt{swap}(x) := 1 - x \qquad (x = 0, 1).$$

Note that the total rate of jumps $0 \mapsto 1$ is $r_{\mathtt{up}} + r_{\mathtt{swap}} = 2$.

Poisson construction of Markov processes



The representation of a generator in terms of maps is not unique.