Interacting Particle Systems with Applications in Finance

Jan M. Swart

Lecture 1: Introduction

Interacting Particle Systems

- Interacting particle systems are mathematical models for *collective behavior*.
- Applications in physics (atoms & molecules), biology (organisms) & sociology, financial mathematics (people).
- Simple rules lead to complicated behavior.
- *Markovian* dynamics.
- Easy to simulate, but not always easy to prove; open problems.
- Rigorous methods lead to better understanding.

Poisson construction of Markov processes

Let S be a finite set. A *probability kernel* on S is a function $K : S^2 \to [0,1]$ such that $\sum_y K(x,y) = 1$. We calculate with kernels as with matrices:

$$KL(x,z) := \sum_y K(x,y)L(y,z) \quad \text{and} \quad Kf(x) := \sum_y K(x,y)f(y).$$

Let $X = (X_t)_{t\geq 0}$ be a stochastic process with values in S. By definition, X is a (time-homogeneous) *Markov process* if

$$\mathbb{P}\big[X_u \in \cdot \,\big|\, (X_s)_{0\leq s\leq t}\big] = P_{u-t}(X_t,\,\cdot\,) \quad \text{a.s.} \qquad (0 \leq t \leq u),$$

where the *transition kernels* $(P_t)_{t\geq 0}$ form a collection of probability kernels on S such

$$P_s P_t = P_{s+t} \quad \text{and} \quad \lim_{t\downarrow 0} P_t = P_0 = 1.$$

Poisson construction of Markov processes

Each such *Markov semigroup* $(P_t)_{t\geq 0}$ is of the form

$$P_t = e^{tG} := \sum_{n=0}^{\infty} \frac{1}{n!}(tG)^n,$$

where the *generator* G is a matrix of the form

$$G(x,y) \geq 0 \quad (x \neq y) \quad \text{and} \quad \sum_y G(x,y) = 0.$$

We interpret $G(x,y)$ $(x \neq y)$ as the *rate* of transitions $x \mapsto y$.
The process $X = (X_t)_{t\geq 0}$ arises as the limit of discrete-time Markov chains with transition kernel of the form

$$P_\varepsilon(x,y) = 1_{\{x=y\}} + \varepsilon G(x,y) + O(\varepsilon^2).$$

Poisson construction of Markov processes

Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space with σ-finite, nonatomic measure μ. Recall that a *Poisson point set* with *intensity* μ is a random subset $\omega \subset \Omega$ such that

$$\big|\omega \cap A\big| \quad \text{is Poisson distributed with mean} \quad \mu(A)$$

whenever $A \in \mathcal{F}$, $\mu(A) < \infty$, and

$$\big|\omega \cap A_1\big|, \ldots, \big|\omega \cap A_n\big| \quad \text{are independent} \tag{1}$$

whenever $A_1, \ldots, A_n$ are disjoint.
Since μ is nonatomic, for each $\varepsilon > 0$ we can find a countable partition $\{A_i^\varepsilon : i \in I\}$ of Ω such that $\mu(A_i^\varepsilon) \leq \varepsilon \ \forall i$. Then

$$\mathbb{P}\big[|\omega \cap A_i^\varepsilon| = 1\big] = \mu(A_i^\varepsilon) + O(\varepsilon^2) \quad \text{and} \quad \mathbb{P}\big[|\omega \cap A_i^\varepsilon| \geq 2\big] = O(\varepsilon^2). \tag{2}$$

Any sequence of random sets satisfying (1) and (2) converges as $\varepsilon \downarrow 0$ to a Poisson point set with intensity μ.

Poisson construction of Markov processes

Each generator G has a *random mapping representation*

$$Gf(x) = \sum_{m \in \mathcal{M}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\},$$

where $(r_m)_{m \in \mathcal{M}}$ are nonnegative rates and $\mathcal{M}$ is a collection of maps $m : S \to S$. Let ω be a Poisson point set on $\mathcal{M} \times \mathbb{R}$ with intensity

$$\mu\big(\{m\} \times A\big) = r_m\, \ell(A) \qquad \big(A \in \mathcal{B}(\mathbb{R})\big),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel-σ-field on $\mathbb{R}$ and ℓ denotes Lebesgue measure. We may order the elements of

$$\omega \cap \mathcal{M} \times (s, t] =: \omega_{s,t} = \big\{(m_1, t_1), \ldots, (m_n, t_n)\big\}$$

with $t_1 < \cdots < t_n$.

Poisson construction of Markov processes

Define random maps $\mathbf{X}_{s,t} : S \to S$ $(s \leq t)$ by

$$\mathbf{X}_{s,t} := m_n \circ \cdots \circ m_1.$$

(Poisson construction of Markov processes) *Define maps* $(\mathbf{X}_{s,t})_{s \leq t}$ *as above in terms of a Poisson point set* ω*. Let* X_0 *be an* S*-valued random variable, independent of* ω*. Then*

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

is a Markov process with generator G*.*

Remark The sample paths of X are cadlag, i.e., right-continuous with left limits. We get left-continuous paths by defining

$$\mathbf{X}_{s,t-} \quad \text{in terms of} \quad \omega_{s,t-} := \omega \cap \mathcal{M} \times (s,t).$$

Poisson construction of Markov processes

Example 1 Consider the Markov process with state space $\{0,1\}$ that jumps with the rates

$$G(0,1) := 2 \quad \text{and} \quad G(1,0) := 1.$$

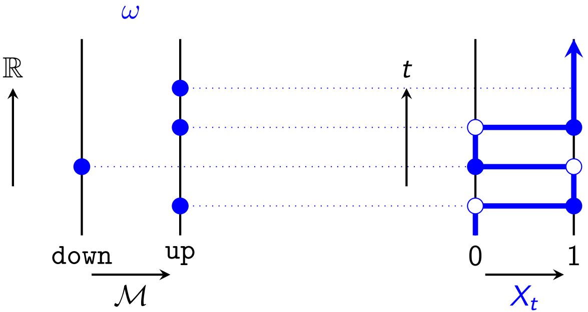
Consider $\mathcal{M} := \{\mathtt{up}, \mathtt{down}\}$, where

$$\mathtt{up}(x) := 1 \quad \text{and} \quad \mathtt{down}(x) := 0 \qquad \forall x = 0, 1.$$

Setting $r_{\mathtt{up}} := 2$ and $r_{\mathtt{down}} := 1$, we have

$$Gf(x) = r_{\mathtt{up}}\big\{f\big(\mathtt{up}(x)\big) - f\big(x\big)\big\} + r_{\mathtt{down}}\big\{f\big(\mathtt{down}(x)\big) - f\big(x\big)\big\}.$$

Poisson construction of Markov processes



Since it may happen that $m(X_t) = X_t$, not every time of the Poisson process corresponds to a jump of the Markov process.

Random mapping representations are not unique!

Example 2 The same Markov process, that jumps

$$0 \mapsto 1 \text{ with rate } 2 \quad \text{and} \quad 1 \mapsto 0 \text{ with rate } 1$$

can also be represented as

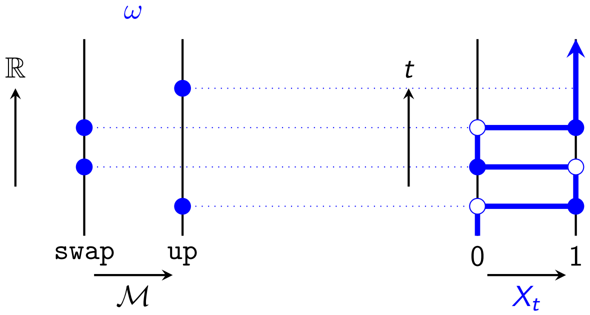
$$Gf(x) = r_{\mathtt{up}}\big\{f\big(\mathtt{up}(x)\big) - f\big(x\big)\big\} + r_{\mathtt{swap}}\big\{f\big(\mathtt{swap}(x)\big) - f\big(x\big)\big\},$$

with $r_{\mathtt{up}} := 1$, $r_{\mathtt{swap}} := 1$, and

$$\mathtt{up}(x) := 1 \quad \text{and} \quad \mathtt{swap}(x) := 1 - x \qquad (x = 0, 1).$$

Note that the total rate of jumps $0 \mapsto 1$ is $r_{\mathtt{up}} + r_{\mathtt{swap}} = 2$.

Poisson construction of Markov processes



The representation of a generator in terms of maps is not unique.

An Ising model for collective decision making

Let Λ be a finite set, representing people. Each person $i \in \Lambda$ can be in two states $x(i) \in \{-1, 1\}$. The state of the whole system is an element $x = (x(i))_{i \in \Lambda}$ of the space $S := \{-1, +1\}^\Lambda$ of all functions $x : \Lambda \to \{-1, +1\}$.
A person chooses his/her state according to an utility function. Given that the system is at time t in the state x, the utility for the person i of being in the state $+1$ resp. -1 is described by the utility function

$$U_t^{\pm}(i, x) = \pm \tfrac{1}{2} J \sum_{j \in \mathcal{N}_i} x(j) \pm \tfrac{1}{2} W_t(i).$$

Here $\mathcal{N}_i \subset \Lambda$ is a *neighborhood* of i, and $W_t(i)$ is a random, logistically distributed term.

An Ising model for collective decision making

$$U_t^{\pm}(i,x) = \pm\tfrac{1}{2}J\sum_{j\in\mathcal{N}_i} x(j) \pm \tfrac{1}{2}W_t(i).$$

- For $J > 0$ it is advantageous to make the same choice as your neighbors.

An Ising model for collective decision making

$$U_t^{\pm}(i,x) = \pm\tfrac{1}{2}J\sum_{j\in\mathcal{N}_i} x(j) \pm \tfrac{1}{2}W_t(i).$$

- For $J > 0$ it is advantageous to make the same choice as your neighbors.
- $W_t(i)$ logistically distributed: $\mathbb{P}[W_t(i) \leq w] = (1 + e^{-\beta w})^{-1}$.

An Ising model for collective decision making

$$U_t^{\pm}(i,x) = \pm\tfrac{1}{2}J\sum_{j\in\mathcal{N}_i} x(j) \pm \tfrac{1}{2}W_t(i).$$

- For $J > 0$ it is advantageous to make the same choice as your neighbors.
- $W_t(i)$ logistically distributed: $\mathbb{P}[W_t(i) \leq w] = (1 + e^{-\beta w})^{-1}$.
- The noise terms $W_t(i)$ are independent for each person and are redrawn at times of a Poisson process with intensity one.

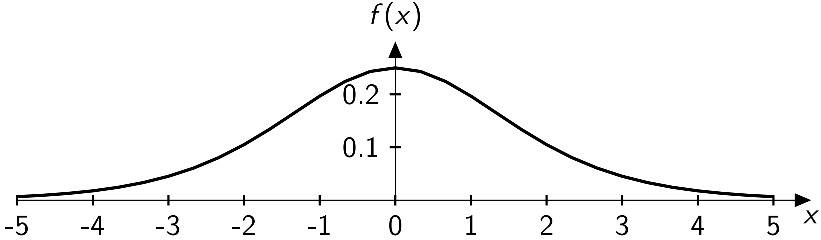
An Ising model for collective decision making

$$U_t^{\pm}(i,x) = \pm\tfrac{1}{2}J\sum_{j\in\mathcal{N}_i} x(j) \pm \tfrac{1}{2}W_t(i).$$

- For $J > 0$ it is advantageous to make the same choice as your neighbors.
- $W_t(i)$ logistically distributed: $\mathbb{P}[W_t(i) \leq w] = (1 + e^{-\beta w})^{-1}$.
- The noise terms $W_t(i)$ are independent for each person and are redrawn at times of a Poisson process with intensity one.
- After the noise is redrawn, each person immediately chooses his or her new state according to the highest utility.

An Ising model for collective decision making

Density $f(w) = \beta(e^{\beta w/2} + e^{-\beta w/2})^{-1}$ of the logistic distribution.



Probability that $U_t^+(i, X_t) > U_t^-(i, X_t)$ given that $\mathcal{N}_i$ contains $M_i(\pm)$ persons in the state ± 1 equals

$$\frac{e^{\beta J M_i(+)}}{e^{\beta J M_i(+)} + e^{\beta J M_i(-)}}.$$

Person i changes his/her state to $+1$ with this rate and to -1 with one minus this rate. Only the product βJ matters.

An Ising model for collective decision making

Possible choices for the “neighborhood” $\mathcal{N}_i$ of person i:

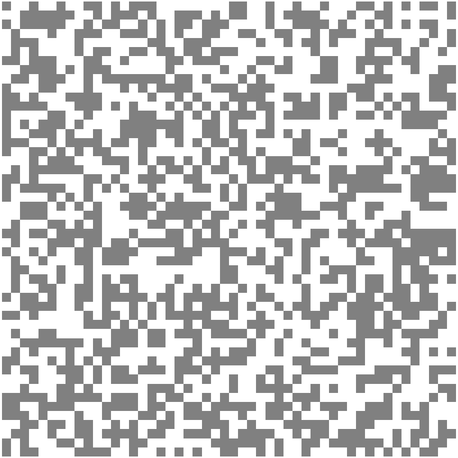
1. the whole set Λ,
2. points $j \in \mathbb{Z}^d$ with $|i - j| = 1$,
3. neighbors in any locally finite graph.

Choice 1. made by Brock & Durlauf (2001), who studied the invariant laws of this Markov chain (but not its time evolution).

Logistic distribution motivated by extreme-value theory, but mainly by the wish to obtain the *Ising model* (on $\mathbb{Z}^d$) or *Curie Weiss model* (on the complete graph), which are used in physics to describe systems of atoms at *inverse temperature* $1/\beta$ whose spin can be in two states ± 1 and that interact through magnetic forces.

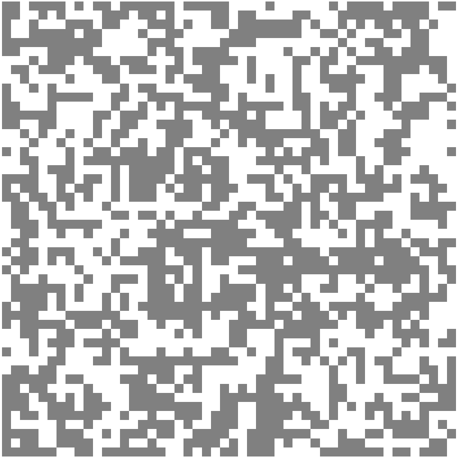
More precisely, our model is the Ising model with (continuous-time) *Glauber dynamics*.

The Ising model



$J\beta = 0.3$, time $t = 0$.

The Ising model

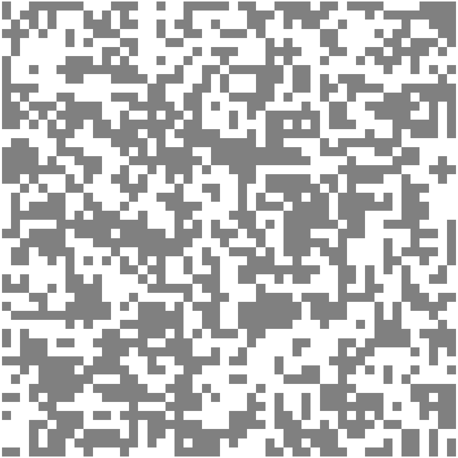


$J\beta = 0.3$, time $t = 1$.

The Ising model

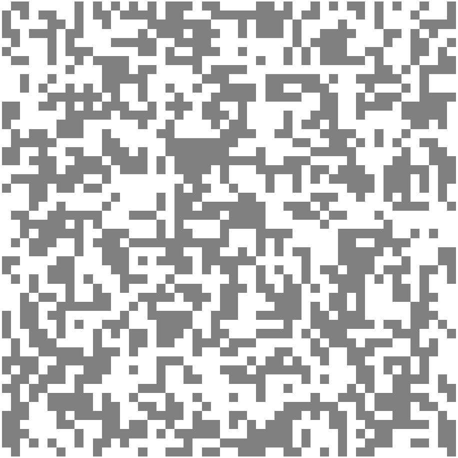
$J\beta = 0.3$, time $t = 2$.

The Ising model



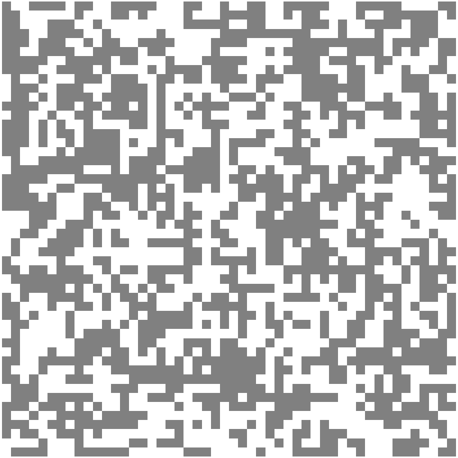
$J\beta = 0.3$, time $t = 4$.

The Ising model

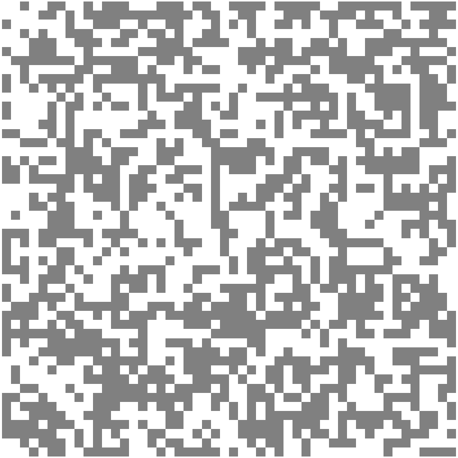


$J\beta = 0.3$, time $t = 8$.

The Ising model



$J\beta = 0.3$, time $t = 16$.

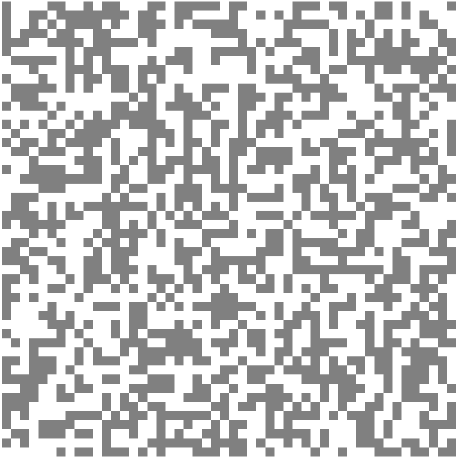


$J\beta = 0.3$, time $t = 32$.

The Ising model

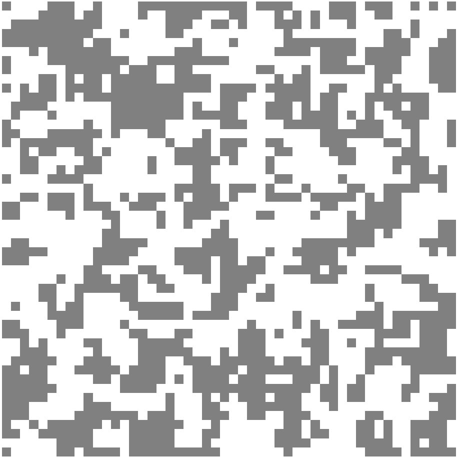
$J\beta = 0.3$, time $t = 64$.

The Ising model



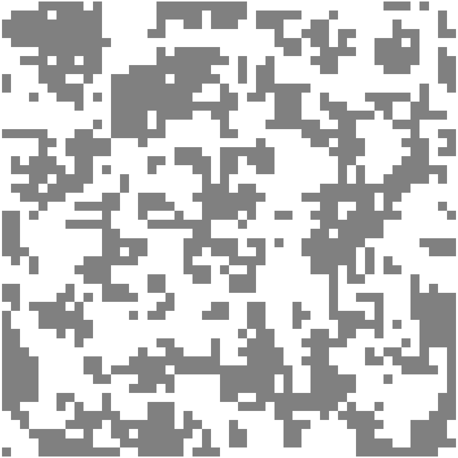
$J\beta = 0.7$, time $t = 0$.

The Ising model



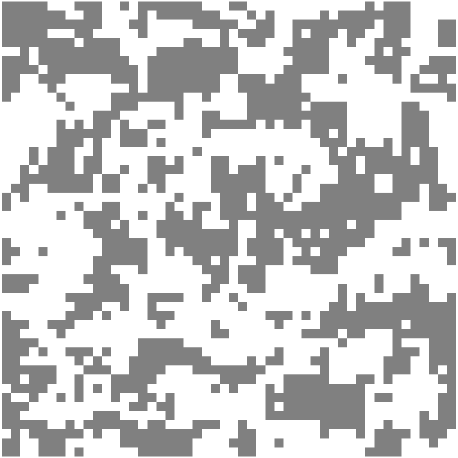
$J\beta = 0.7$, time $t = 1$.

The Ising model



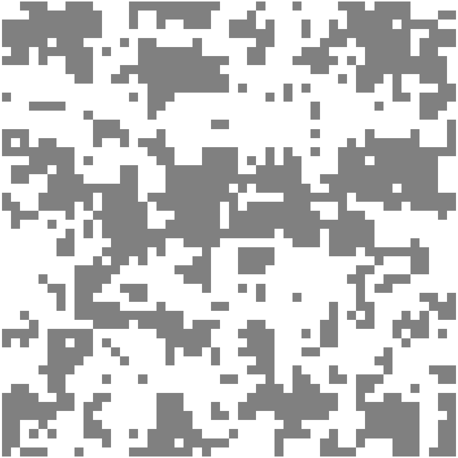
$J\beta = 0.7$, time $t = 2$.

The Ising model



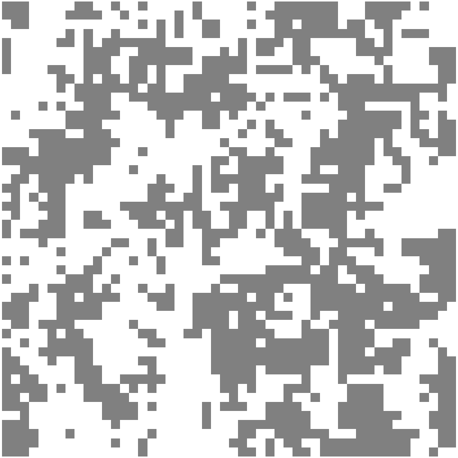
$J\beta = 0.7$, time $t = 4$.

The Ising model



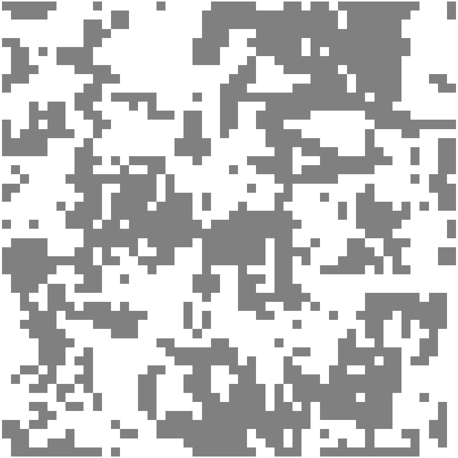
$J\beta = 0.7$, time $t = 8$.

The Ising model



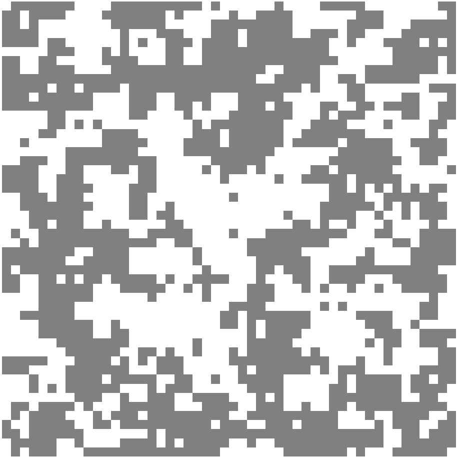
$J\beta = 0.7$, time $t = 16$.

The Ising model



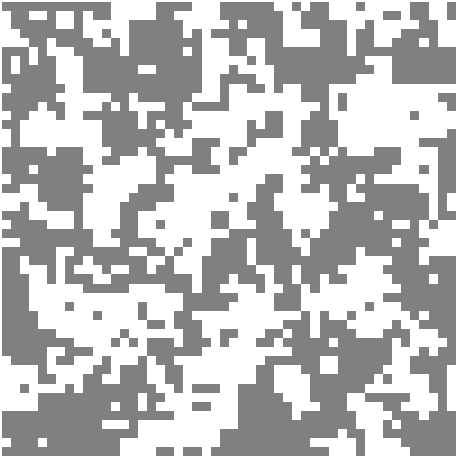
$J\beta = 0.7$, time $t = 32$.

The Ising model



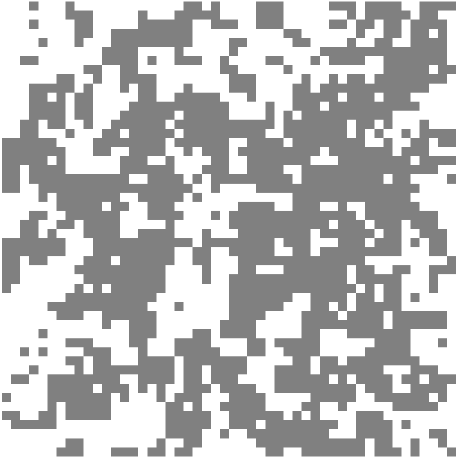
$J\beta = 0.7$, time $t = 64$.

The Ising model



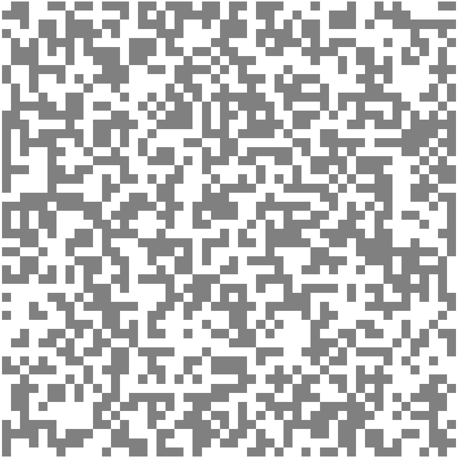
$J\beta = 0.7$, time $t = 125$.

The Ising model



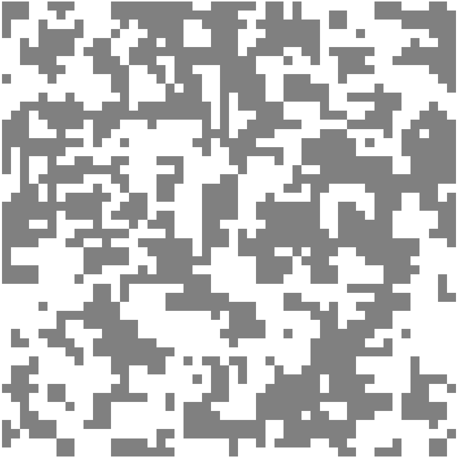
$J\beta = 0.7$, time $t = 250$.

The Ising model



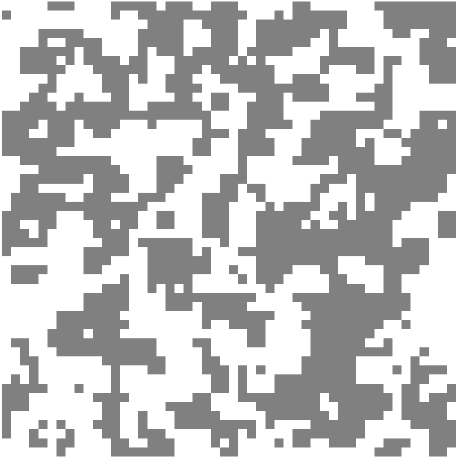
$J\beta = 1$, time $t = 0$.

The Ising model



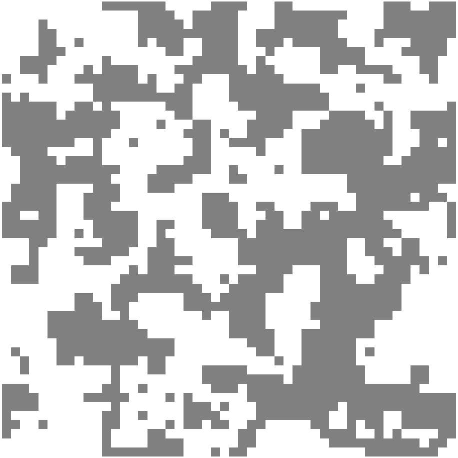
$J\beta = 1$, time $t = 1$.

The Ising model



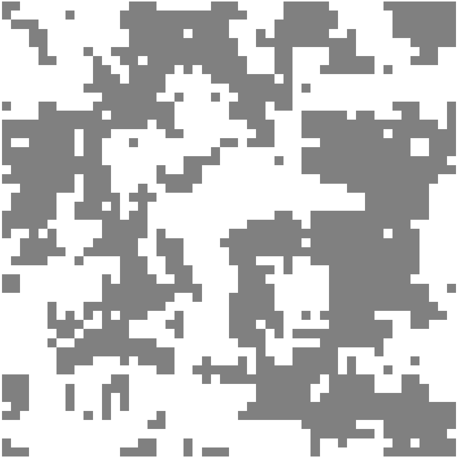
$J\beta = 1$, time $t = 2$.

The Ising model



$J\beta = 1$, time $t = 4$.

The Ising model

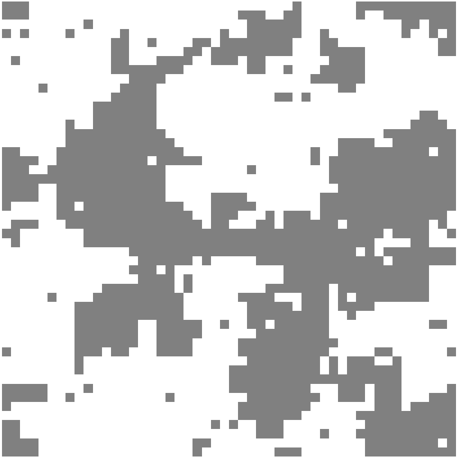


$J\beta = 1$, time $t = 8$.

The Ising model

$J\beta = 1$, time $t = 16$.

The Ising model

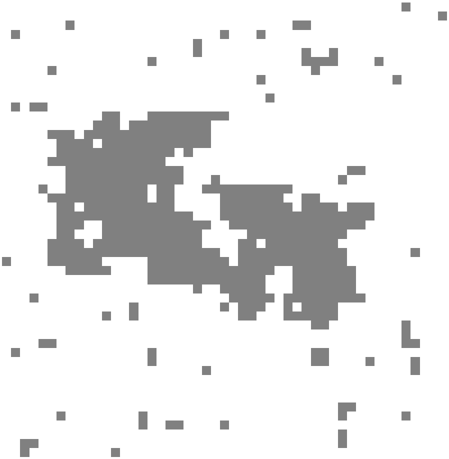


$J\beta = 1$, time $t = 32$.

The Ising model

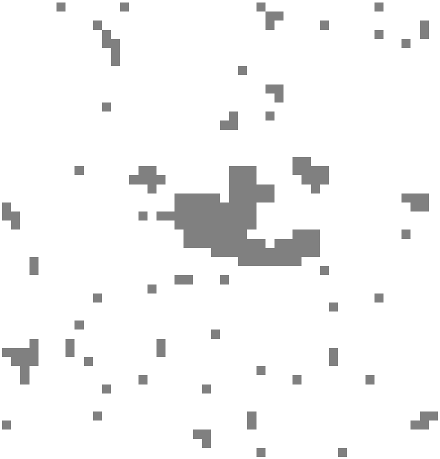
$J\beta = 1$, time $t = 64$.

The Ising model



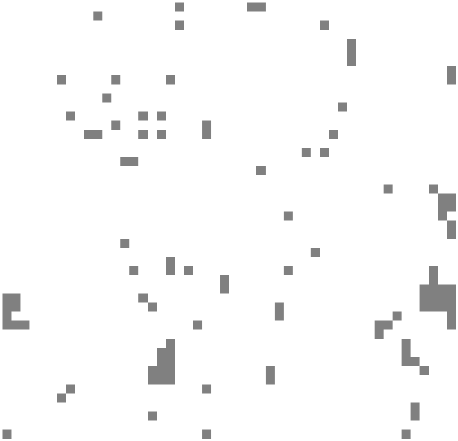
$J\beta = 1$, time $t = 125$.

The Ising model



$J\beta = 1$, time $t = 250$.

The Ising model



$J\beta = 1$, time $t = 500$.

A Potts model for collective decision making

Instead of allowing only two states $-1, +1$, we can more generally allow $q \geq 2$ states $1, \ldots, q$.

Each person i chooses a new state at times of a Poisson process with rate 1.

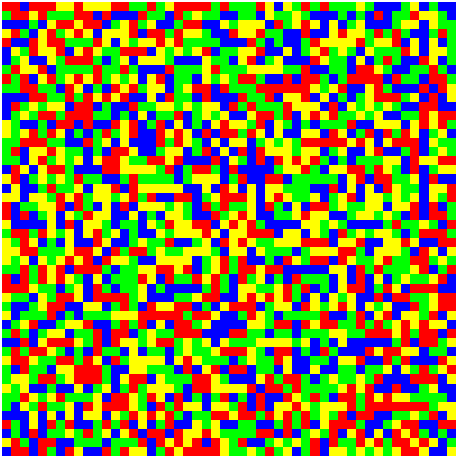
The probability that the newly chosen state is $k \in \{1, \ldots, q\}$ equals

$$\frac{e^{\beta J M_i(k)}}{\sum_{m=1}^{q} e^{\beta J M_i(m)}},$$

where $M_i(k)$ denotes the number of neighbors of i that are in the state k.

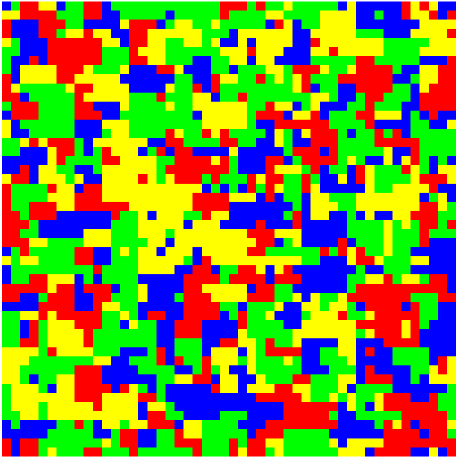
Special case $q = 2$ amounts to the Ising model.

The Potts model



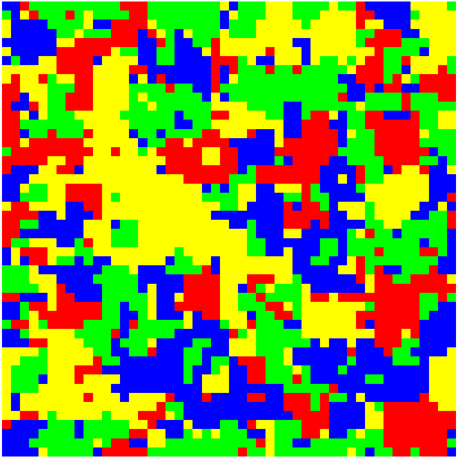
$J\beta = 1.2$, time $t = 0$.

The Potts model



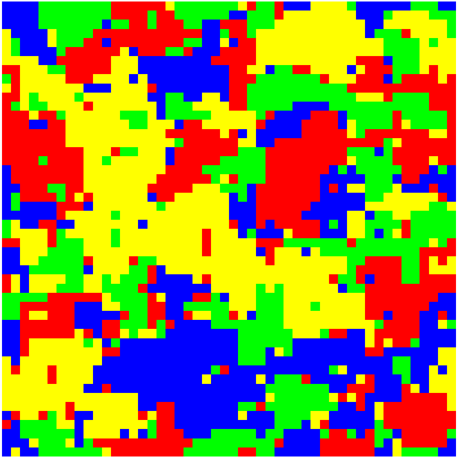
$J\beta = 1.2$, time $t = 1$.

The Potts model



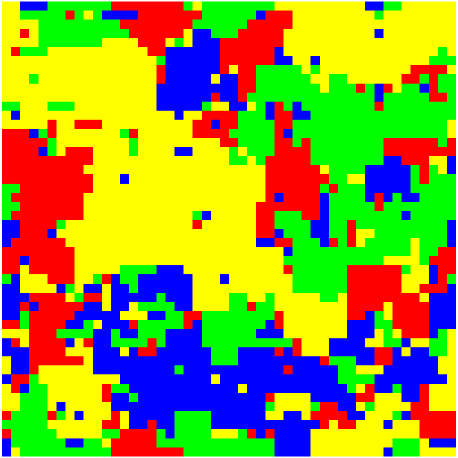
$J\beta = 1.2$, time $t = 2$.

The Potts model



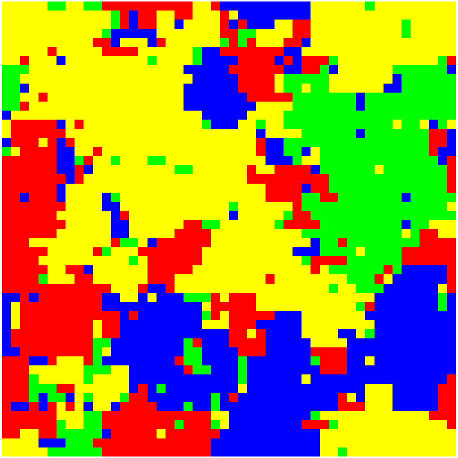
$J\beta = 1.2$, time $t = 4$.

The Potts model



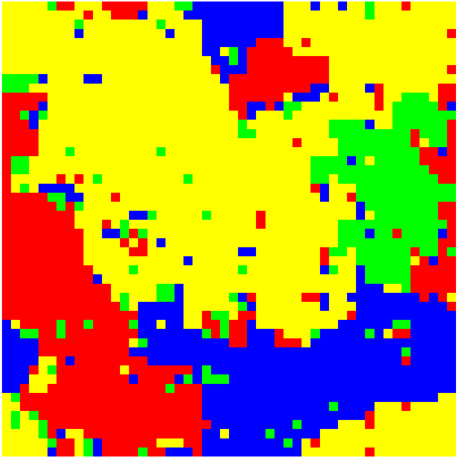
$J\beta = 1.2$, time $t = 8$.

The Potts model



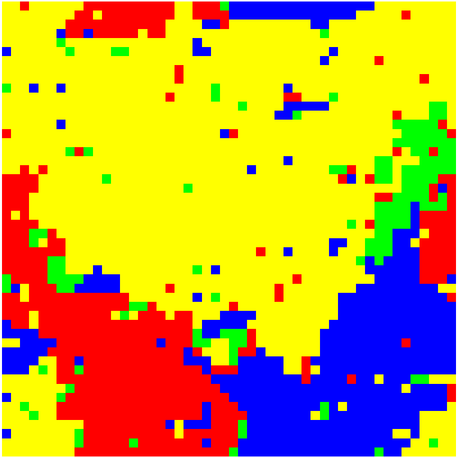
$J\beta = 1.2$, time $t = 16$.

The Potts model



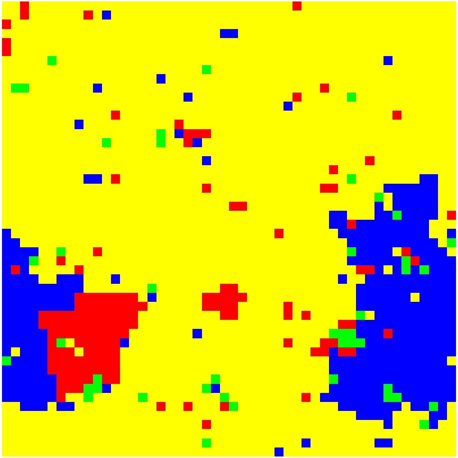
$J\beta = 1.2$, time $t = 32$.

The Potts model



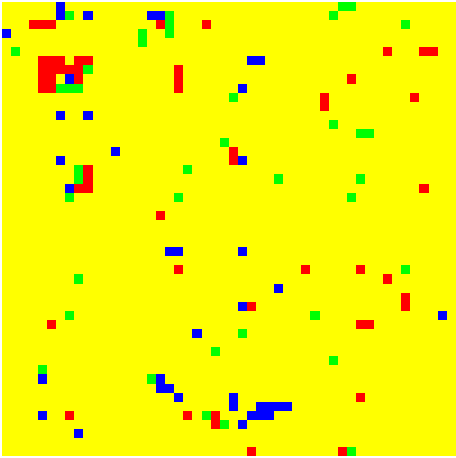
$J\beta = 1.2$, time $t = 64$.

The Potts model



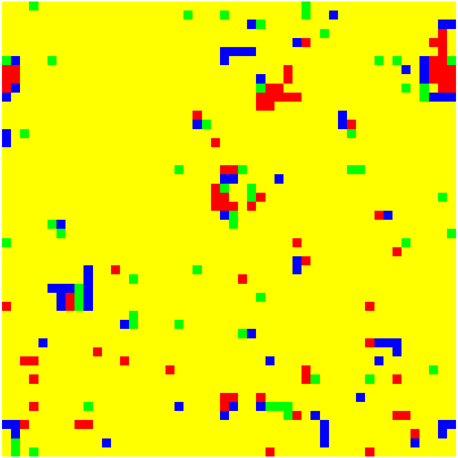
$J\beta = 1.2$, time $t = 125$.

The Potts model



$J\beta = 1.2$, time $t = 250$.

The Potts model



$J\beta = 1.2$, time $t = 500$.

The voter model

In the *voter model*, persons can have $q \geq 2$ states.

Each person i chooses a new state at times of a Poisson process with rate 1.

The probability that the newly chosen state is $k \in \{1, \ldots, q\}$ equals

$$\frac{1}{|\mathcal{N}_i|} M_i(k)$$

where $M_i(k)$ denotes the number of neighbors of i that are in the state k.

Contrary to the Potts model, types, once extinct, cannot reappear.

Used to model *credit contagion* in Gieseck & Weber (2002).

Also used to model *voting behavior*, or the spread of *neutral genetic types* in population biology.

The voter model

Using the *voter map*

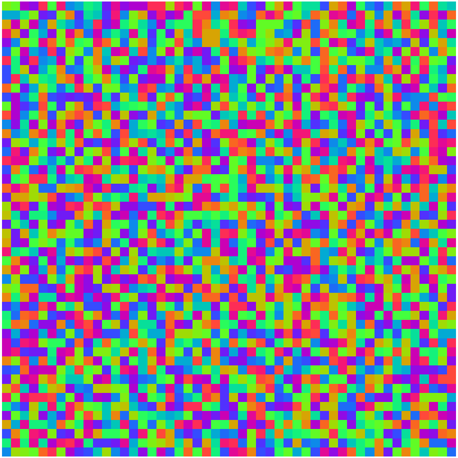
$$\mathtt{vot}_{ij}(x) := \begin{cases} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

we can give the following random mapping representation of the generator:

$$Gf(x) = \sum_{i \in \Lambda} \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \big\{ f\big(\mathtt{vot}_{i,j} x\big) - f\big(x\big) \big\}.$$

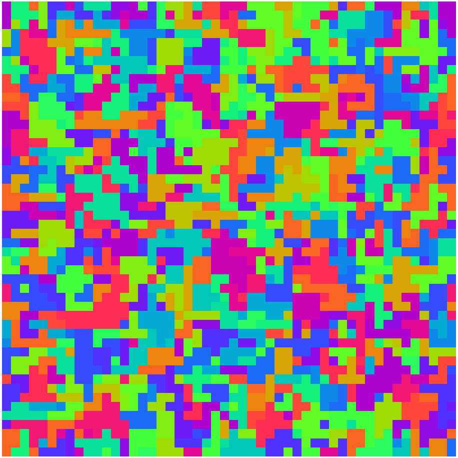
Interpretation: each person copies with rate 1 the type of a uniformly chosen random person of $\mathcal{N}_i$.

The voter model

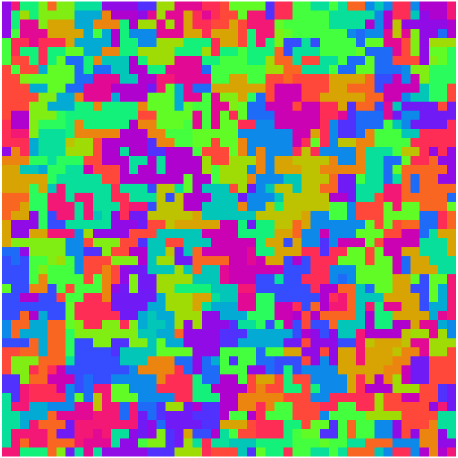


Time $t = 0$.

The voter model

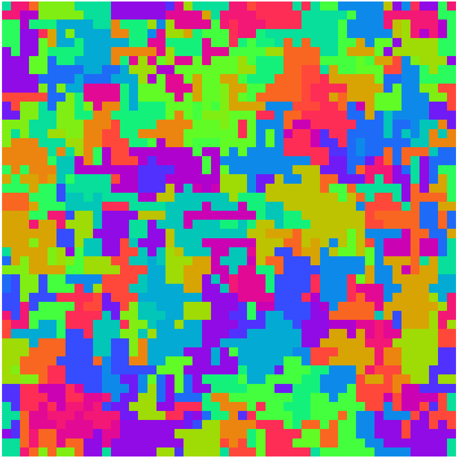


Time $t = 0.25$.



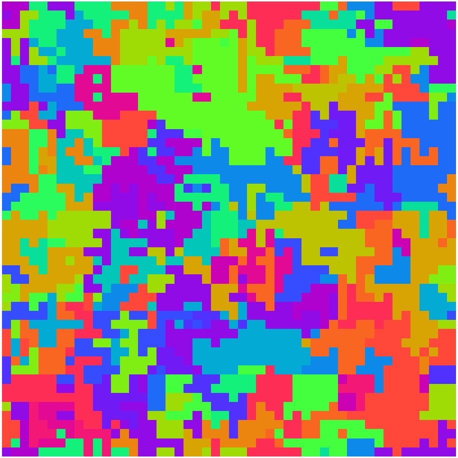
Time $t = 0.5$.

The voter model



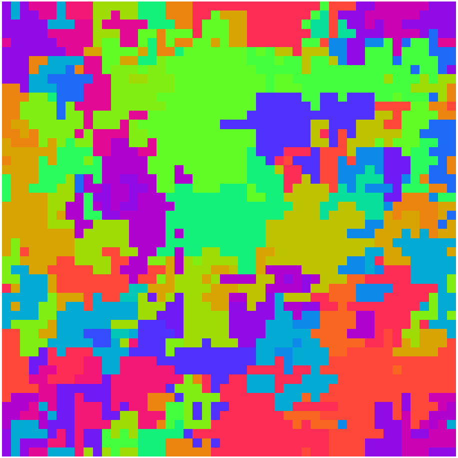
Time $t = 1$.

The voter model



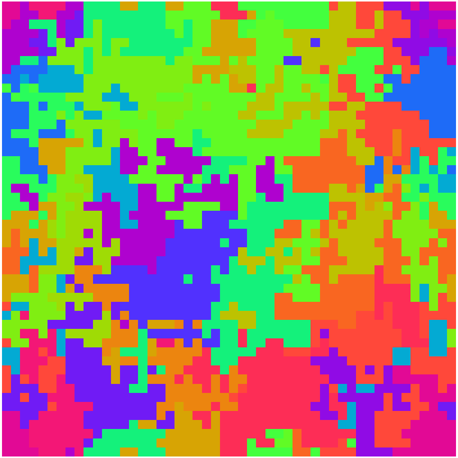
Time $t = 2$.

The voter model



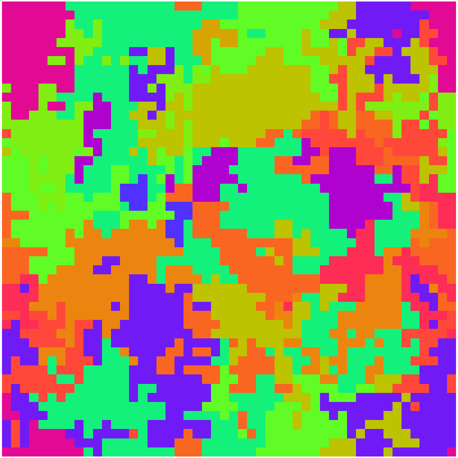
Time $t = 4$.

The voter model



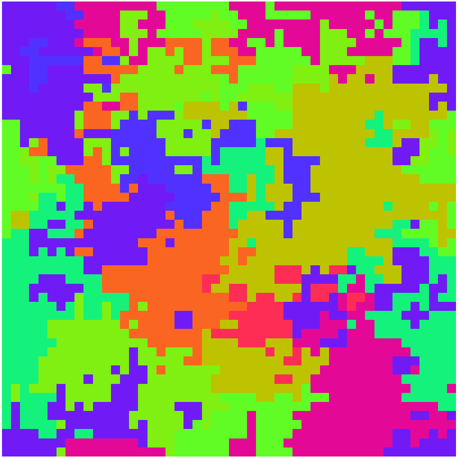
Time $t = 8$.

The voter model



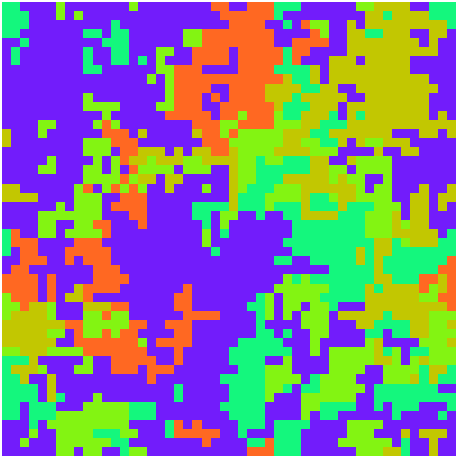
Time $t = 16$.

The voter model



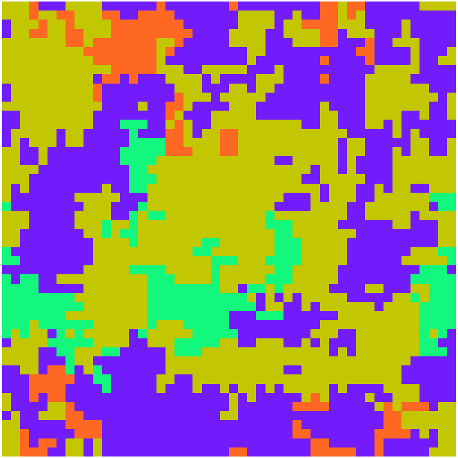
Time $t = 31.25$.

The voter model



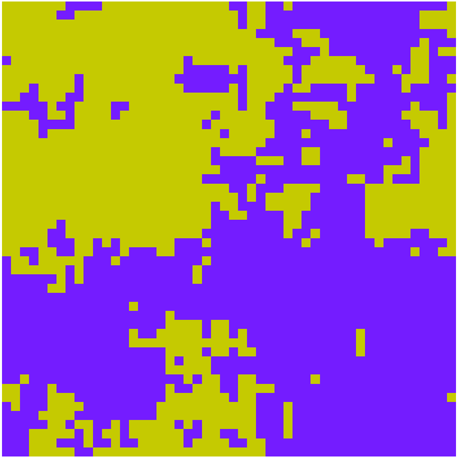
Time $t = 62.5$.

The voter model



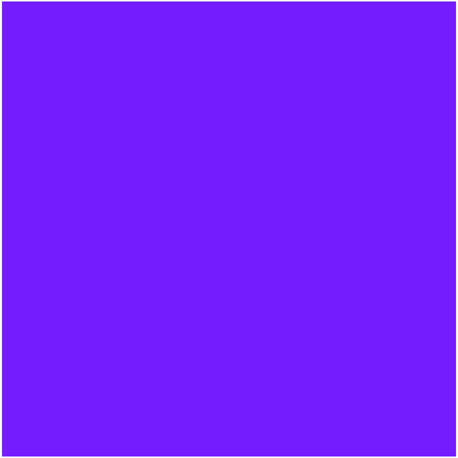
Time $t = 125$.

The voter model



Time $t = 250$.

The voter model



Time $t = 500$.

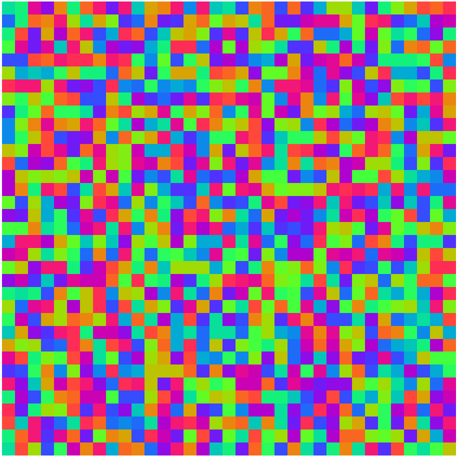
The voter model

The behavior of the voter model strongly depends on the dimension.

Clustering in dimensions $d = 1, 2$.

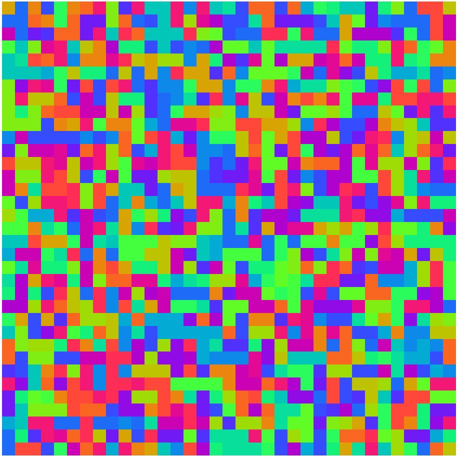
Stable behavior in dimensions $d \geq 3$.

The voter model



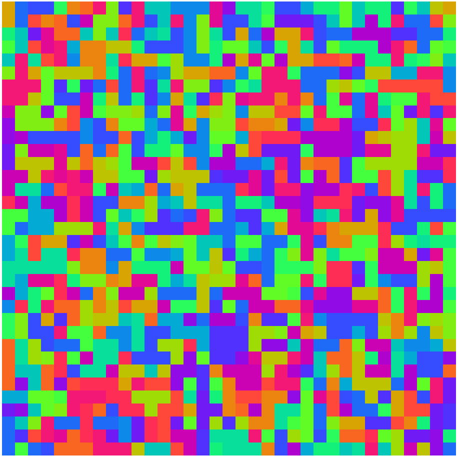
Cut of 3-dimensional model, time $t = 0$.

The voter model



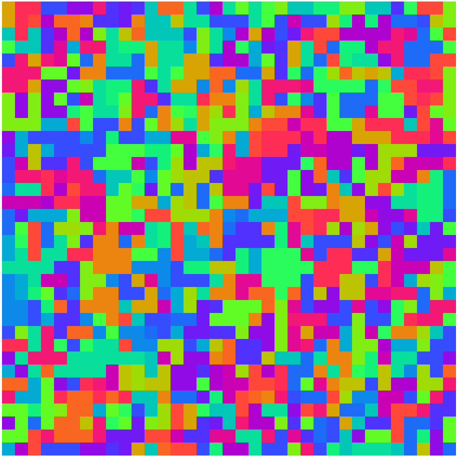
Cut of 3-dimensional model, time $t = 1$.

The voter model



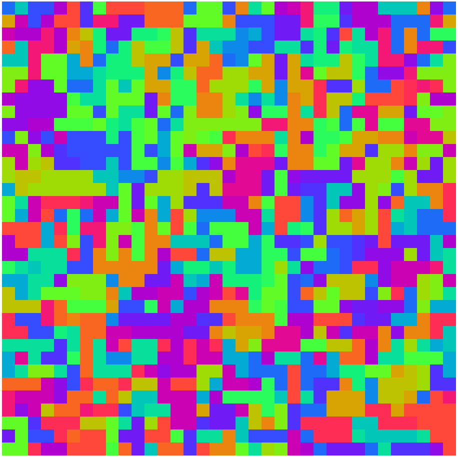
Cut of 3-dimensional model, time $t = 2$.

The voter model



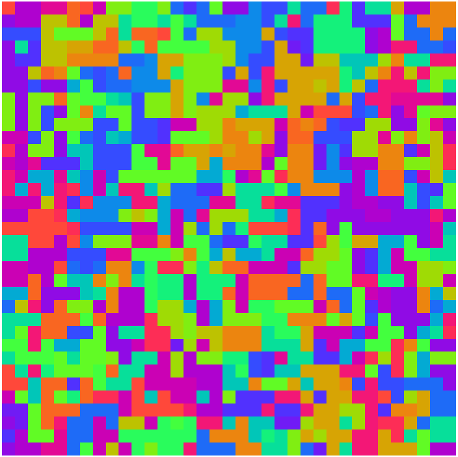
Cut of 3-dimensional model, time $t = 4$.

The voter model

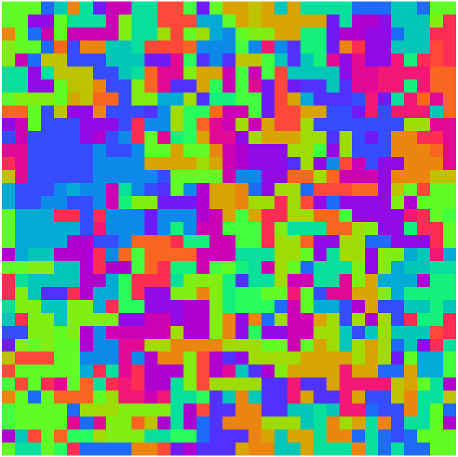


Cut of 3-dimensional model, time $t = 8$.

The voter model

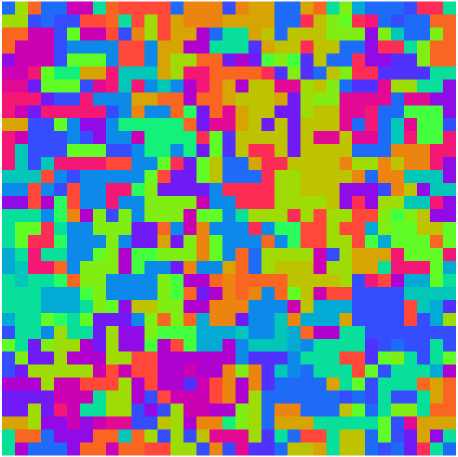


Cut of 3-dimensional model, time $t = 16$.



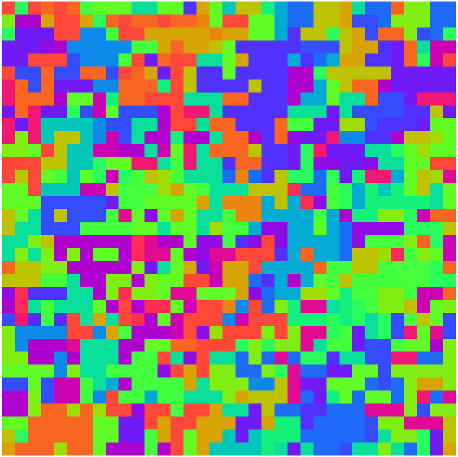
Cut of 3-dimensional model, time $t = 32$.

The voter model



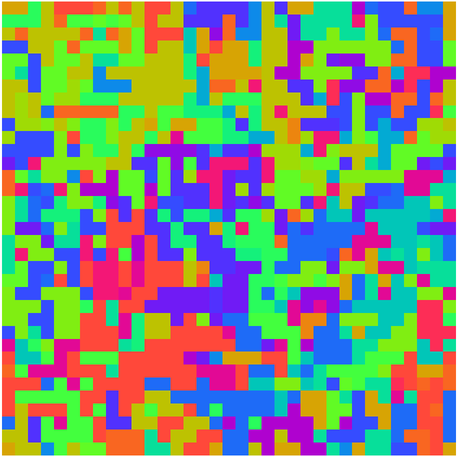
Cut of 3-dimensional model, time $t = 64$.

The voter model



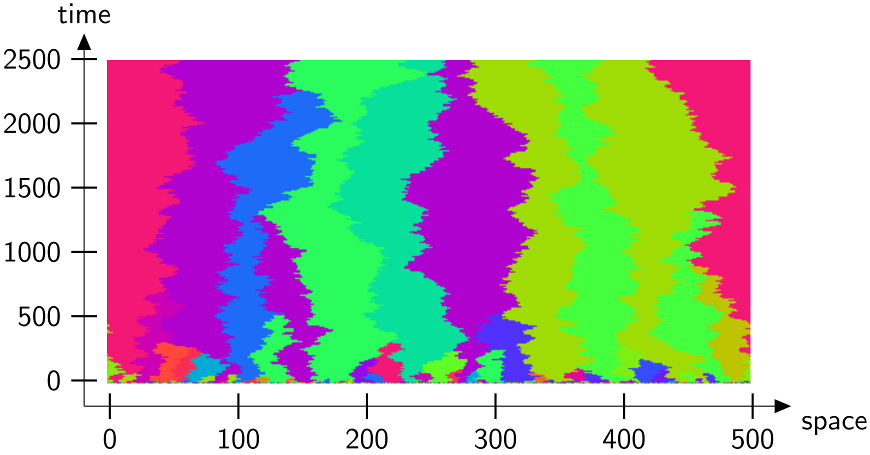
Cut of 3-dimensional model, time $t = 125$.

The voter model



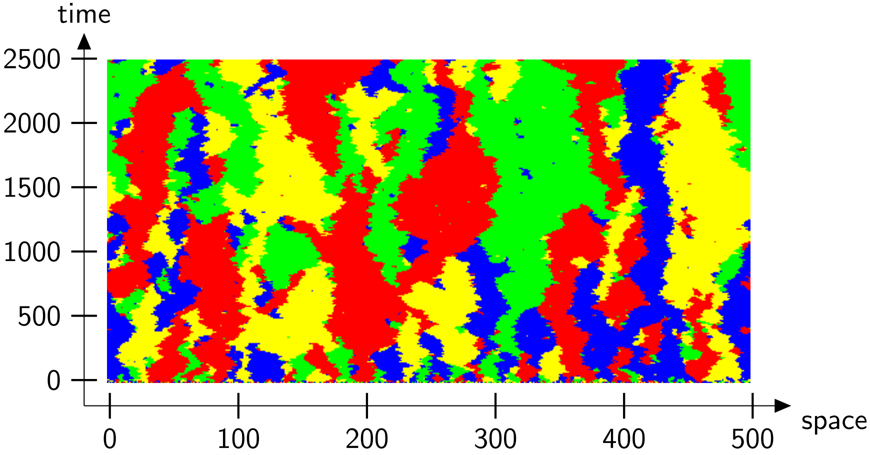
Cut of 3-dimensional model, time $t = 250$.

The voter model



A one-dimensional voter model.

A one-dimensional Potts model



In one-dimensional Potts models, the cluster size remains bounded in time even at very high β ($=$ low temperature).

The biased voter model

In the *biased* voter model with two states $\{0, 1\}$, each person i changes its type $X_t(i)$ with the rates

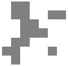
$$\begin{aligned} 0 \mapsto 1 &\qquad \text{with rate } (1+s)\cdot\text{fraction of type 1 neighbors}, \\ 1 \mapsto 0 &\qquad \text{with rate } 1\cdot\text{fraction of type 0 neighbors}, \end{aligned}$$

where $s > 0$ gives type 1 a (small) advantage.

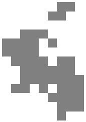
Contrary to the voter model, even if we start with just a single person of type 1, there is a positive probability that type 1 never dies out.

Models spread of *new idea* or *technology*, or *advantageous mutation* in biology.

Biased voter model with $s = 0.2$. Time $t = 0$.

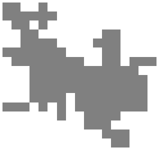


Biased voter model with $s = 0.2$. Time $t = 10$.



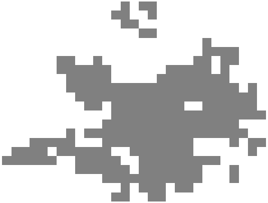
Biased voter model with $s = 0.2$. Time $t = 20$.

The biased voter model



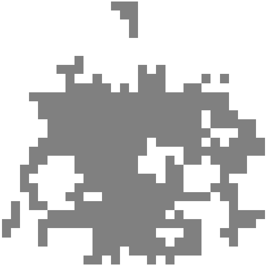
Biased voter model with $s = 0.2$. Time $t = 30$.

The biased voter model



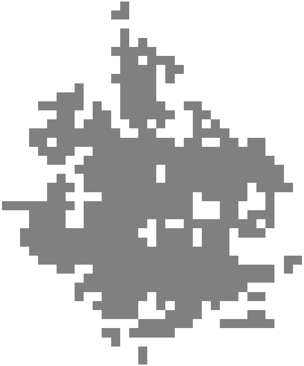
Biased voter model with $s = 0.2$. Time $t = 40$.

The biased voter model

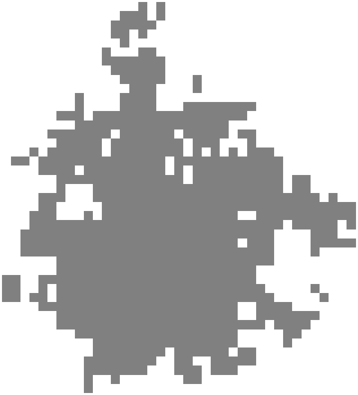


Biased voter model with $s = 0.2$. Time $t = 50$.

The biased voter model

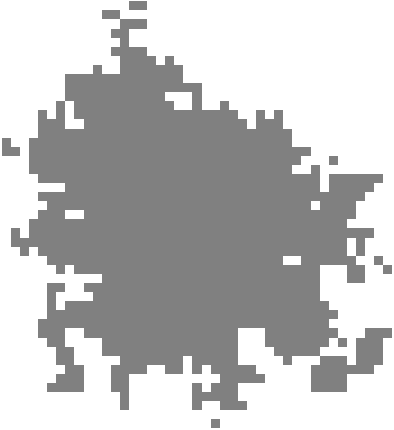


Biased voter model with $s = 0.2$. Time $t = 60$.



Biased voter model with $s = 0.2$. Time $t = 70$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 80$.

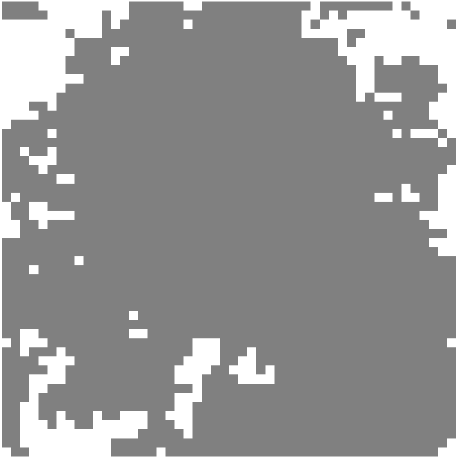
The biased voter model

Biased voter model with $s = 0.2$. Time $t = 90$.

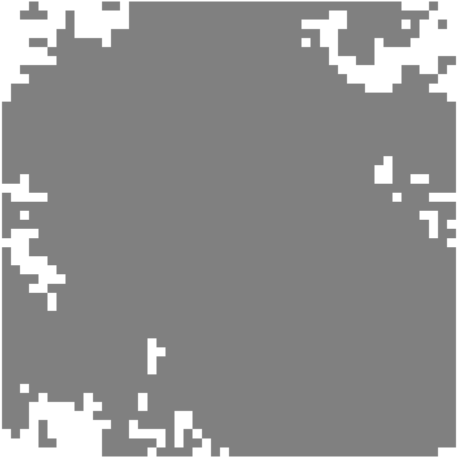
The biased voter model

Biased voter model with $s = 0.2$. Time $t = 100$.

The biased voter model

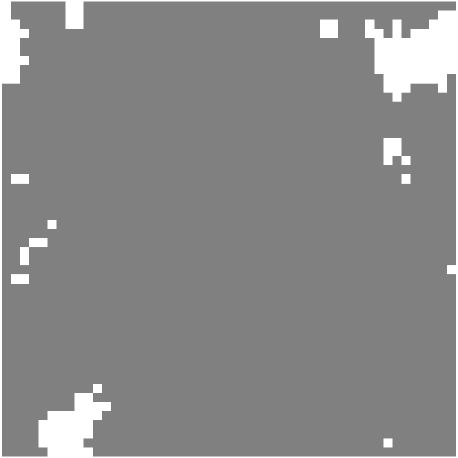


Biased voter model with $s = 0.2$. Time $t = 110$.



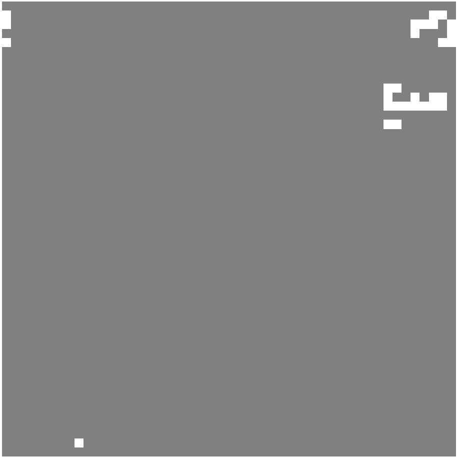
Biased voter model with $s = 0.2$. Time $t = 120$.

The biased voter model



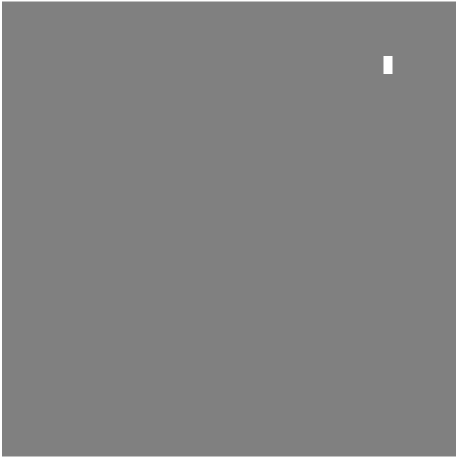
Biased voter model with $s = 0.2$. Time $t = 130$.

The biased voter model



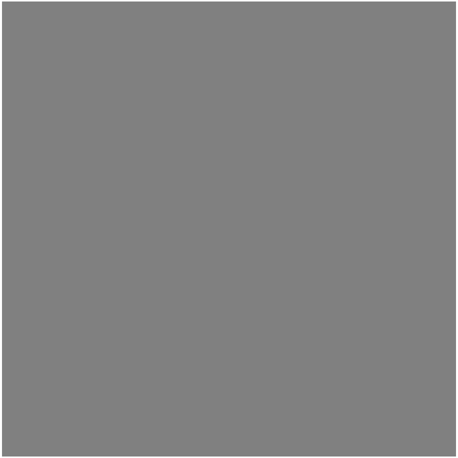
Biased voter model with $s = 0.2$. Time $t = 140$.

The biased voter model



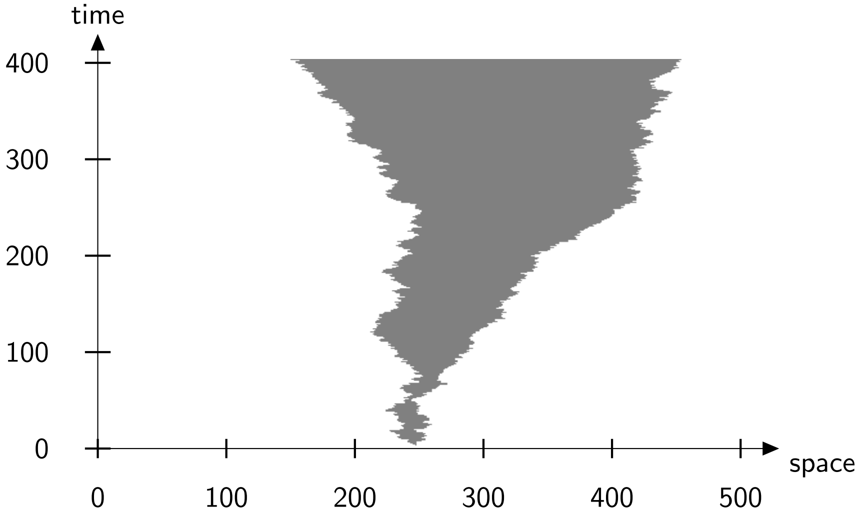
Biased voter model with $s = 0.2$. Time $t = 150$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 160$.

The biased voter model



A one-dimensional biased voter model with bias $s = 0.2$.

The biased voter model

We can extend the biased voter model by also allowing spontaneous jumps from 1 to 0.

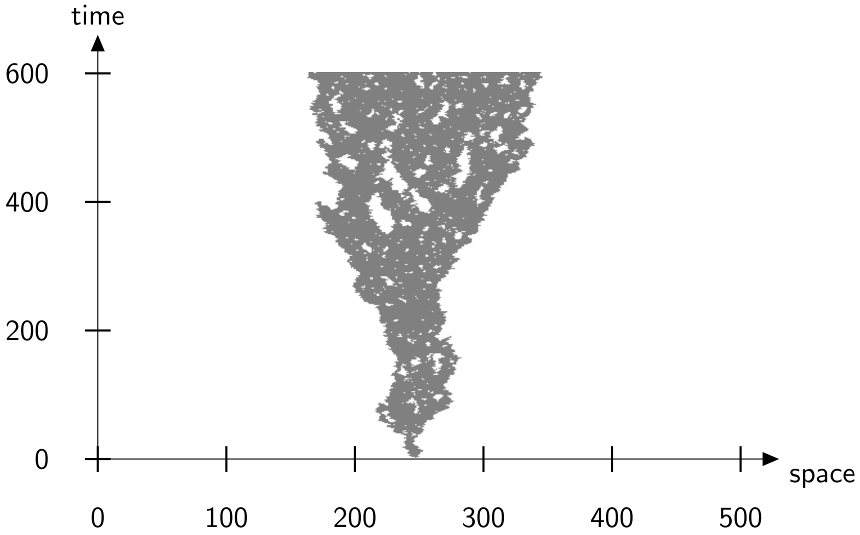
$$\begin{aligned} 0 \mapsto 1 \qquad & \text{with rate } (1+s)\cdot \text{fraction of type 1 neighbors},\\ 1 \mapsto 0 \qquad & \text{with rate } 1\cdot \text{fraction of type 0 neighbors}\\ & \qquad\qquad + d, \end{aligned}$$

where $s > 0$ gives type 1 an advantage and $d \geq 0$ is a *death rate*.

This models the fact that complicated new ideas may be forgotten or organisms may die.

Whether 1's have a positive probability to survive now depends in a nontrivial way on s and d.

The biased voter model



Process with bias $s = 0.5$, death rate $d = 0.02$.

The contact process

Simplifying, we can also look at a process that jumps as

$$\begin{aligned} 0 &\mapsto 1 \qquad \text{with rate } \lambda \cdot \text{number of type 1 neighbors,} \\ 1 &\mapsto 0 \qquad \text{with rate } d. \end{aligned}$$

This is the *contact process* with *infection rate* $\lambda > 0$ and *death rate* $d > 0$.

Again, this can be used to model the spread of ideas or biological populations.

By changing the time scale, we can set the intensity of one Poisson process to one, i.e., without loss of generality $d = 1$ (the usual convention) or also $\lambda = 1$ (if we wish).

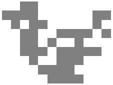
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.

Time $t = 0$.

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 1$.

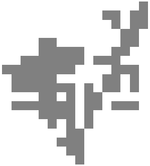
The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 2$.



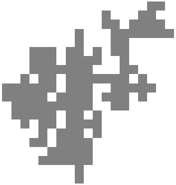
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 3$.

The contact process



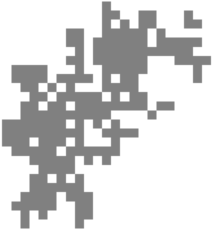
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 4$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 5$.

The contact process



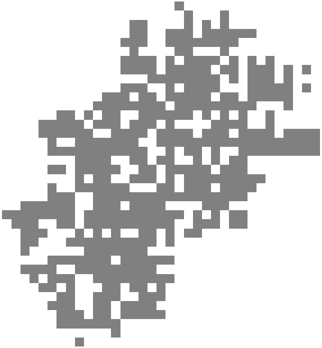
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 6$.

The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 7$.

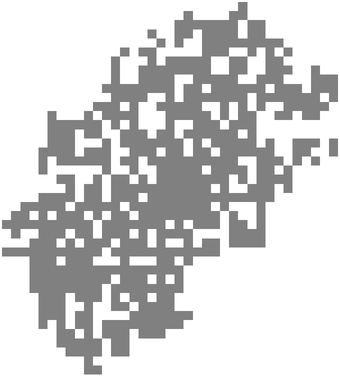
The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 8$.



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 9$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 10$.

The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 11$.

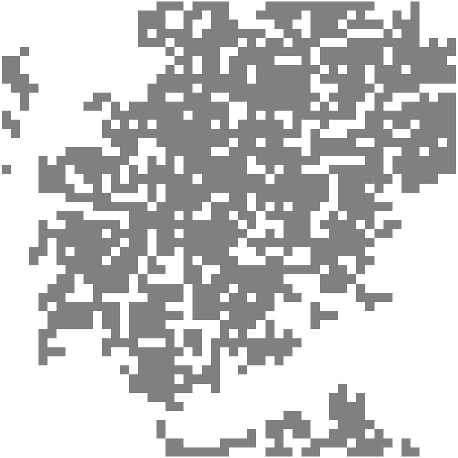
The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 12$.

The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 13$.

The contact process

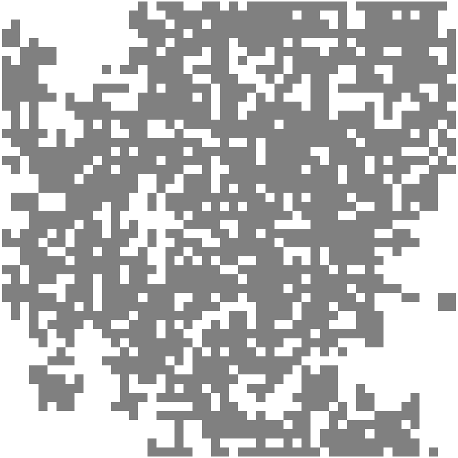


Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 14$.

The contact process

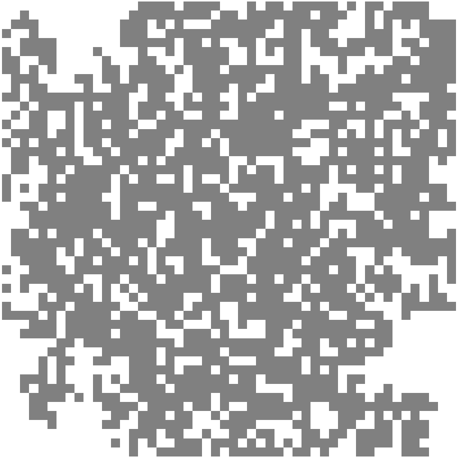
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 15$.

The contact process



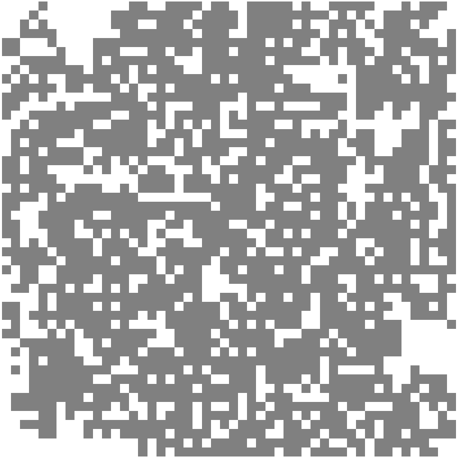
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 16$.

The contact process



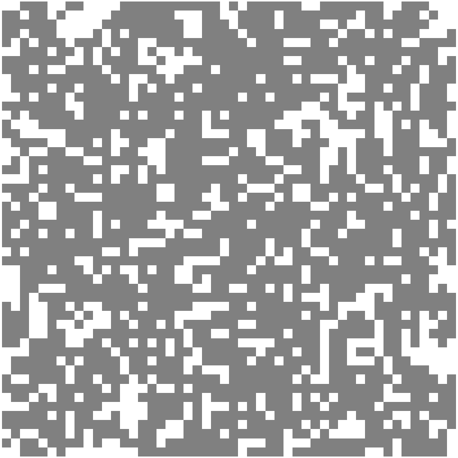
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 17$.

The contact process



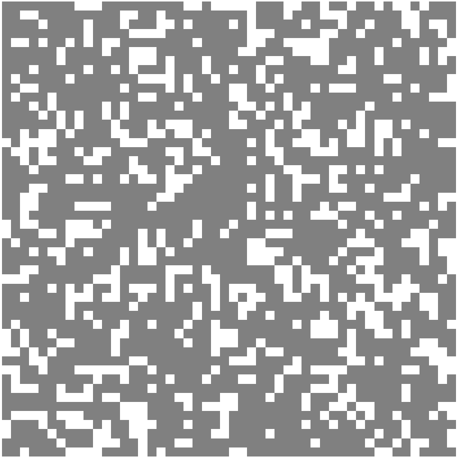
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 18$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 19$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 20$.

A rebellious voter model

Consider a model with two types $\{0,1\}$ and let

$$f_\tau := \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} 1_{\{x(j) = \tau\}}$$

be the frequency of type τ in the neighborhood $\mathcal{N}_i$.

A person of type τ chooses a new type with rate

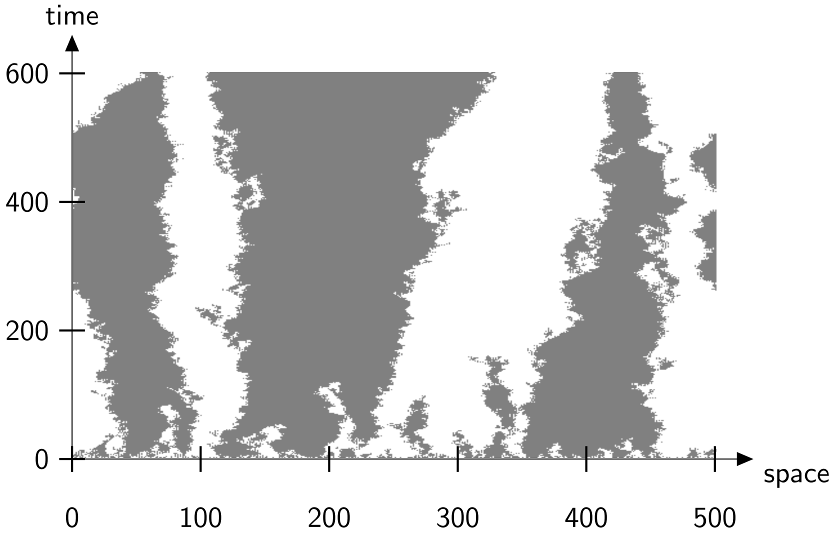
$$f_\tau + \alpha f_{1-\tau}.$$

For $\alpha < 1$, persons change their mind *less* often if they disagree with a lot of neighbors.

As in a normal voter model, the probability that the newly chosen type is τ' is $f_{\tau'}$.

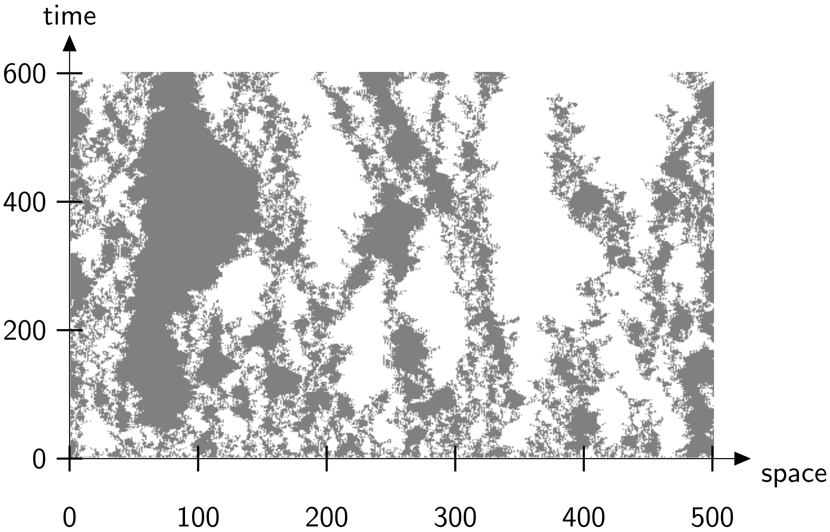
Used by Neuhauser & Pacala (1999) to model *balancing selection*.

A rebellious voter model



Process with $\alpha = 0.8$ behaves more or less as a voter model.

A rebellious voter model



In the process with $\alpha = 0.3$, cluster size remains bounded in time.

Another rich class of models are *reaction diffusion models*.

These are systems of particles that perform independent random walks and interact when they are near to each other.

Let $X_t(i) = 1$ (resp. 0) signify the presence (resp. absence) of a particle and consider the maps $\mathtt{rw}_{ij} : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$

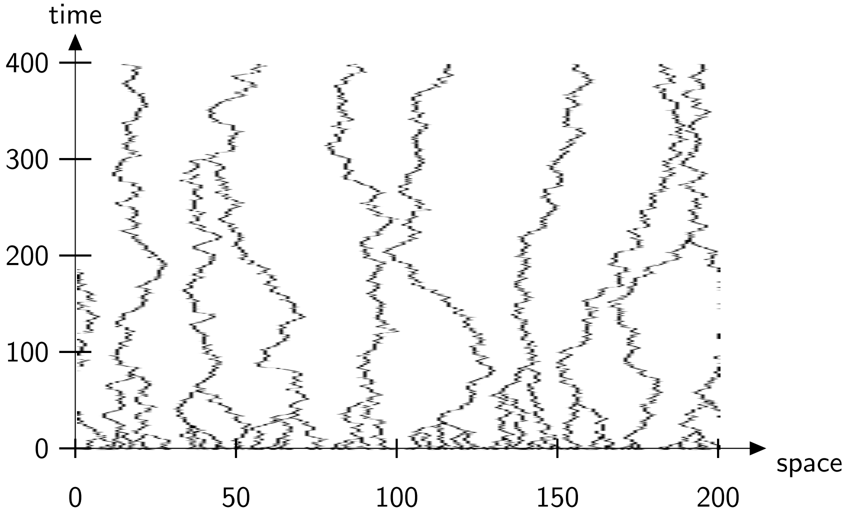
$$\mathtt{rw}_{i,j}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

The process with generator

$$G = \tfrac{1}{2} \sum_{i \in \mathbb{Z}} \big\{ f\big(\mathtt{rw}_{i,i+1}x\big) - f\big(x\big) \big\} + \tfrac{1}{2} \sum_{i \in \mathbb{Z}} \big\{ f\big(\mathtt{rw}_{i,i-1}x\big) - f\big(x\big) \big\}$$

describes *coalescing random walks*.

Coalescing random walks



Reaction diffusion models

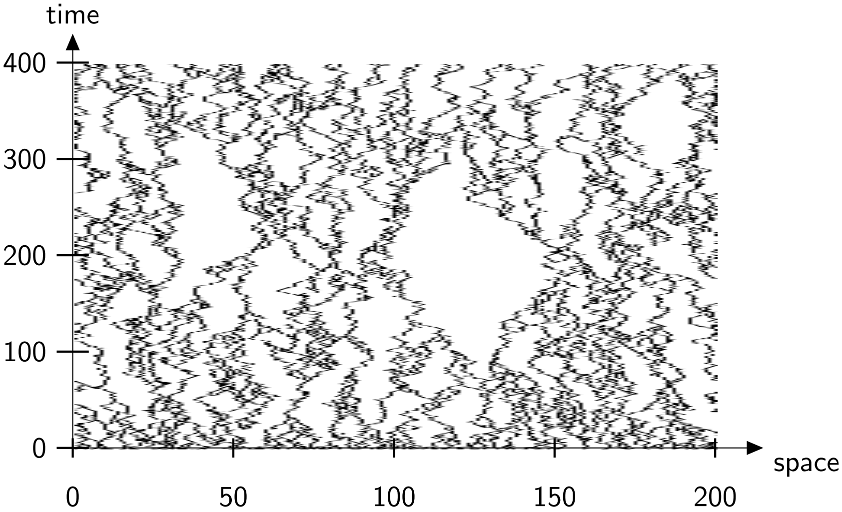
We can also add other maps to the dynamics, like the *branching map*

$$\mathtt{bra}_{i,j}x(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

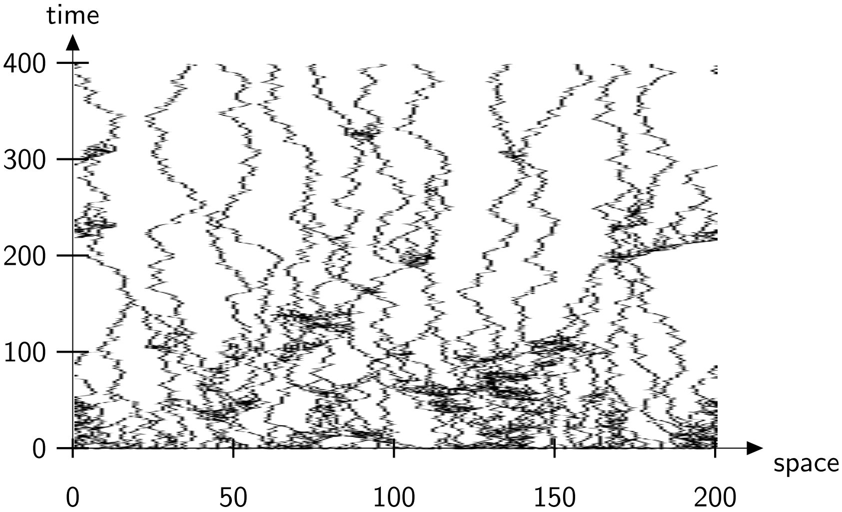
or even *cooperative branching*

$$\mathtt{coop}_{i,i',j}x(k) := \begin{cases} \big(x(i) \wedge x(i')\big) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

Branching and coalescing random walks

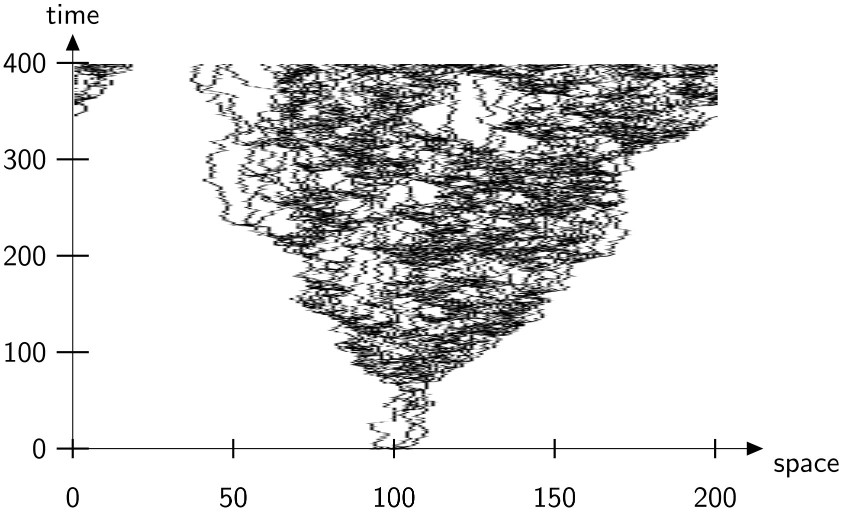


Cooperative branching and coalescence



Cooperative branching rate 2.2.

Cooperative branching



Cooperative branching rate 3.

A cancellative system

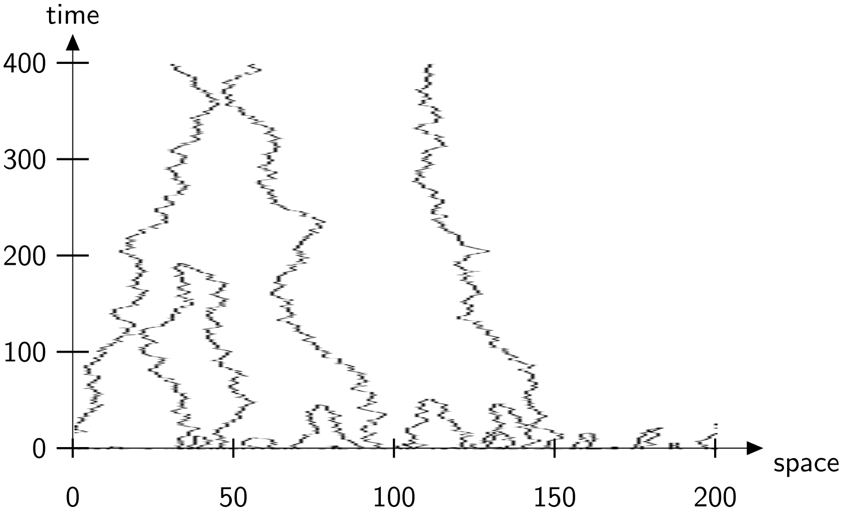
Two more maps of interest are the *annihilating random walk* map

$$\mathtt{arw}_{i,j}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) + x(j) \mod(2) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

and the *annihilating branching map*

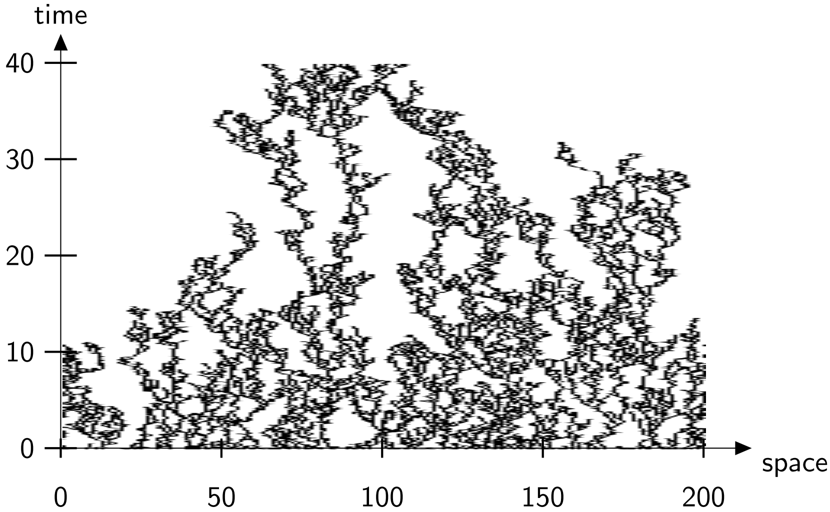
$$\mathtt{abra}_{i,j}x(k) := \begin{cases} x(i) + x(j) \mod(2) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

A cancellative system



Annihilating random walks.

A cancellative system



A system of branching annihilating random walks.

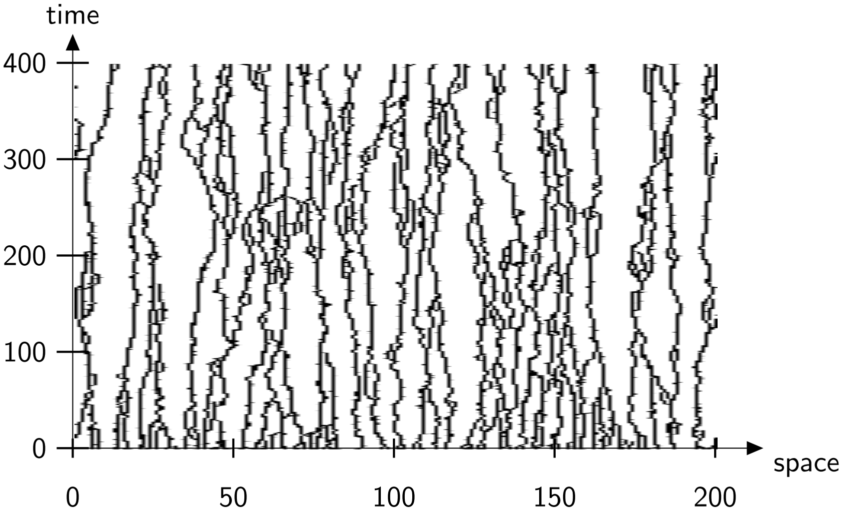
Killing

Define a killing map as

$$\texttt{kill}_{i,j}x(k) := \begin{cases} \big(1 - x(i)\big) \wedge x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

which says that the particle at i, if present, kills any particle at j.

Branching and killing



A system with branching and killing.