Interacting Particle Systems with Applications in Finance

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Lecture 2: Mean-Field Analysis

The Ising model

Let Λ be a finite set and let $S = \{-1,+1\}^\Lambda$ be the set of all functions $x : \Lambda \to \{-1,+1\}$.

Recall that the *Ising model* with (continuous-time) Glauber dynamics is the Markov process $X = (X_t)_{t \geq 0}$ with state space S, where each coordinate $X_t(i)$ chooses a new state at times of a Poisson process with rate 1.

If $\mathcal{N}_i$ contains $M_i(\pm)$ sites of type ± 1, then site i chooses the new state

$$+1 \quad \text{with probability} \quad \frac{e^{\beta J M_i(+)}}{e^{\beta J M_i(+)} + e^{\beta J M_i(-)}},$$

and -1 with the remaining probability.

The Ising model

Possible choices for the neighborhood $\mathcal{N}_i \subset \Lambda$ of person $i \in \Lambda$:

1. the whole set Λ,
2. points $j \in \mathbb{Z}^d$ with $|i - j| = 1$,
3. neighbors in any locally finite graph.

Choice 1., made by Brock & Durlauf (2001), corresponds to a *mean-field* model. We can imagine Λ as a *complete graph* where every person is connected to everyone else.

We are interested in the limit $|\Lambda| \to \infty$. It is convenient to choose the coupling constant $J := 1/|\mathcal{N}_i| = 1/|\Lambda|$. Then site i chooses the new state

$$+1 \quad \text{with probability} \quad \frac{e^{\beta \overline{X}_t/2}}{e^{\beta \overline{X}_t/2} + e^{-\beta \overline{X}_t/2}},$$

where

$$\overline{X}_t := \frac{1}{|\Lambda|} \sum_{i \in \Lambda} X_t(i) \quad \text{is the mean choice of everybody.}$$

The Ising model

We are interested in the evolution of the mean choice

$$\overline{X}_t = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} X_t(i).$$

Set $\varepsilon := |\Lambda|^{-1}$. Since there are $\frac{1}{2}\varepsilon^{-1}(1 - \overline{X}_t)$ sites of type -1, the process $\overline{X}$ jumps

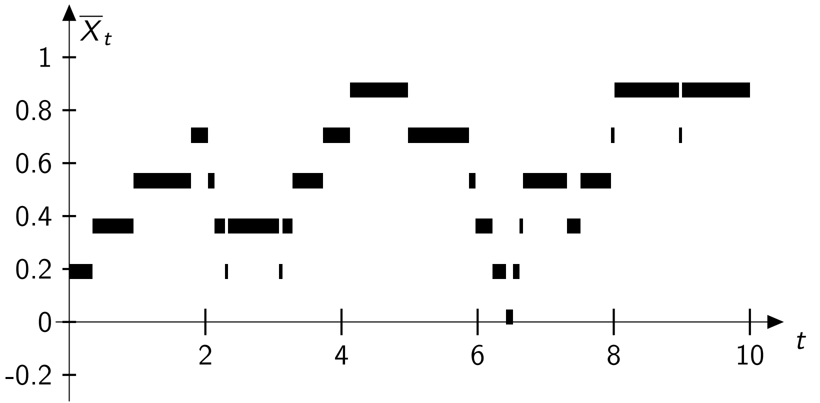
$$\overline{x} \mapsto \overline{x} + 2\varepsilon \quad \text{with rate} \quad r_+(x) = \tfrac{1}{2}\varepsilon^{-1}(1 - \overline{x}) \frac{e^{\beta \overline{x}/2}}{e^{\beta \overline{x}/2} + e^{-\beta \overline{x}/2}},$$

and similarly

$$\overline{x} \mapsto \overline{x} - 2\varepsilon \quad \text{with rate} \quad r_-(x) = \tfrac{1}{2}\varepsilon^{-1}(1 + \overline{x}) \frac{e^{-\beta \overline{x}/2}}{e^{\beta \overline{x}/2} + e^{-\beta \overline{x}/2}}.$$

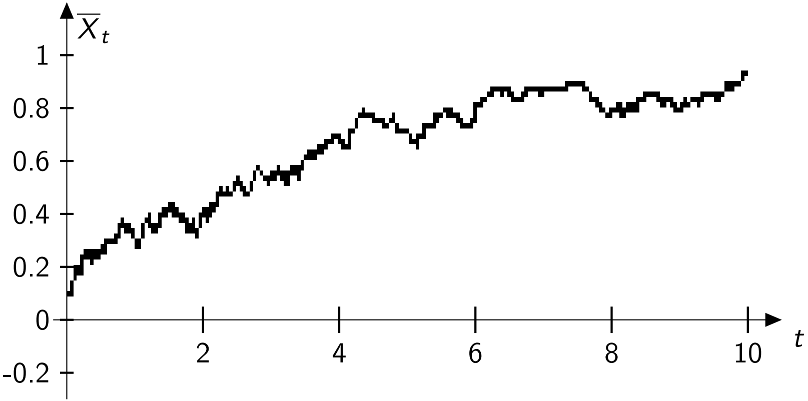
Since $r_\pm(x)$ are functions of $\overline{x}$ only, the process $(\overline{X}_t)_{t \geq 0}$ is itself a Markov process.

The Ising model



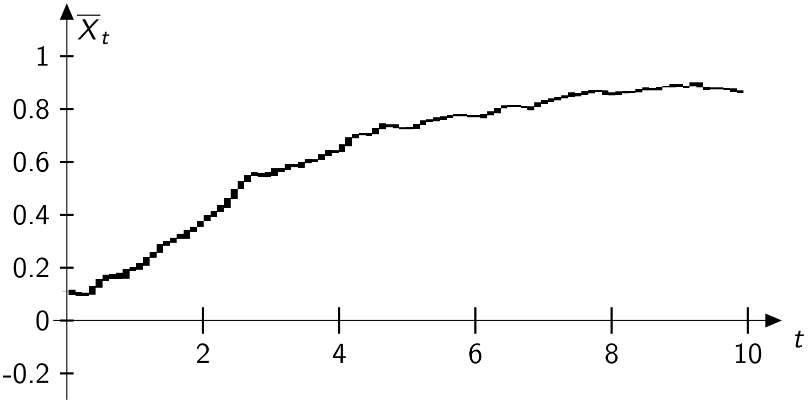
The Markov process $\overline{X}$ with $|\Lambda| = 10$, $\overline{X}_0 = 0.2$, and $\beta = 3$.

The Ising model



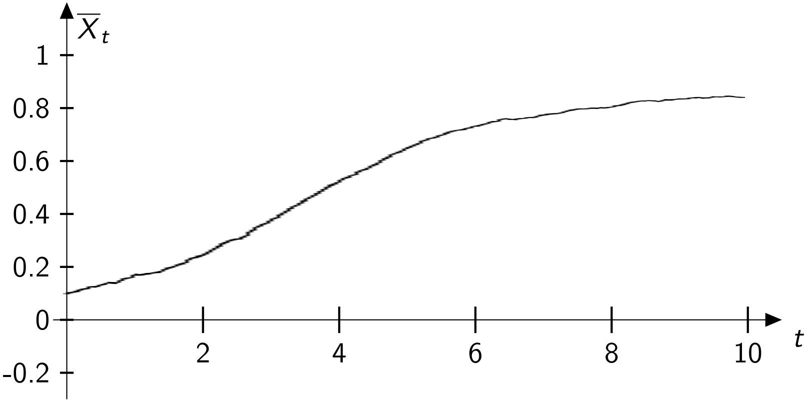
The Markov process $\overline{X}$ with $|\Lambda| = 100$, $\overline{X}_0 = 0.1$, and $\beta = 3$.

The Ising model



The Markov process $\overline{X}$ with $|\Lambda| = 1000$, $\overline{X}_0 = 0.1$, and $\beta = 3$.

The Ising model



The Markov process $\overline{X}$ with $|\Lambda| = 10,000$, $\overline{X}_0 = 0.1$, and $\beta = 3$.

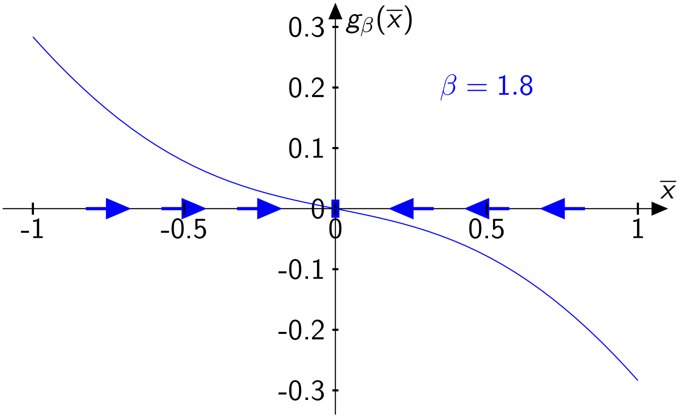
For small ε, by some law of large numbers, we expect that only the average displacement per unit time matters, i.e., the *local drift*

$$r_+(x) \cdot 2\varepsilon + r_-(x) \cdot (-2\varepsilon)$$
$$= \frac{(1-\overline{x})e^{\beta\overline{x}/2} - (1+\overline{x})e^{-\beta\overline{x}/2}}{e^{\beta\overline{x}/2} + e^{-\beta\overline{x}/2}} =: g_\beta(\overline{x}).$$

In the limit $\varepsilon \to 0$, we expect $\overline{X}_t$ to follow the deterministic differential equation

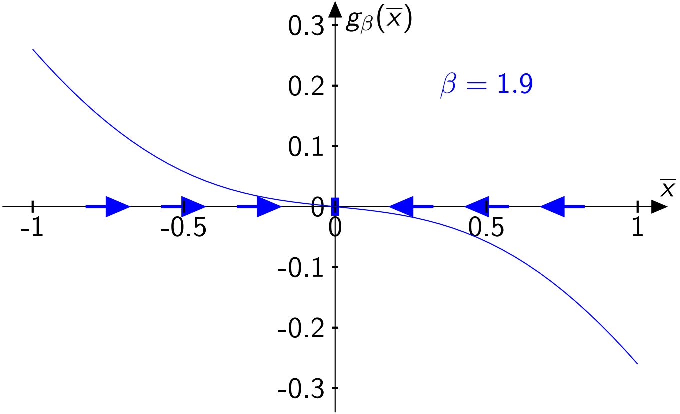
$$\tfrac{\partial}{\partial t}\overline{X}_t = g_\beta(\overline{X}_t) \qquad (t \geq 0).$$

The Ising model



For $\beta \leq 2$, the equation $\frac{\partial}{\partial t}\overline{X}_t = g_\beta(\overline{X}_t)$ has a single, stable fixed point $\overline{x} = 0$.

The Ising model



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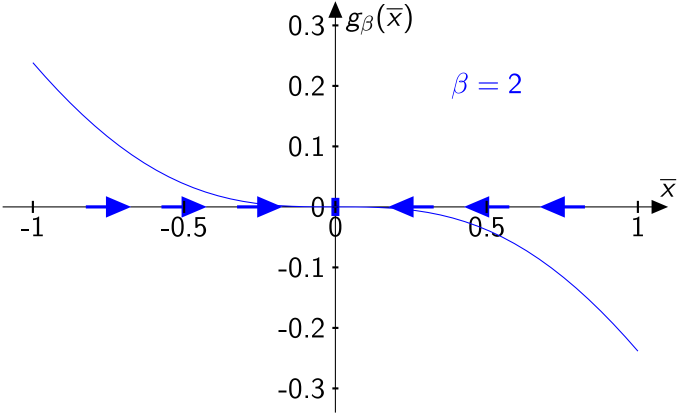

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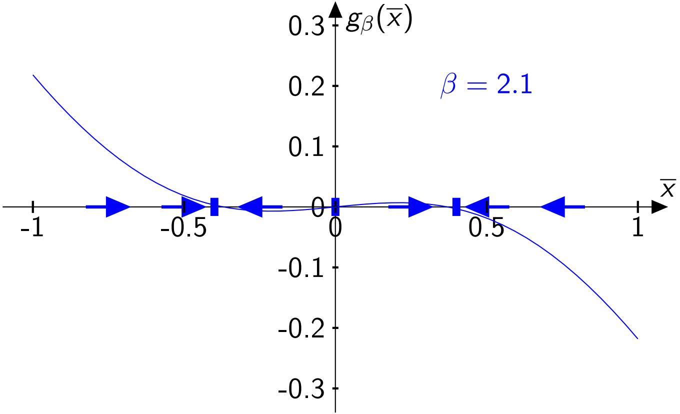
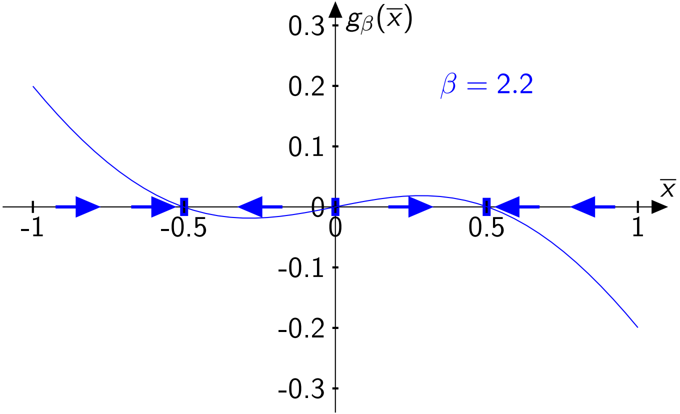


For $\beta > 2$, the fixed point $\overline{x} = 0$ becomes unstable and two new stable fixed points appear.

The Ising model



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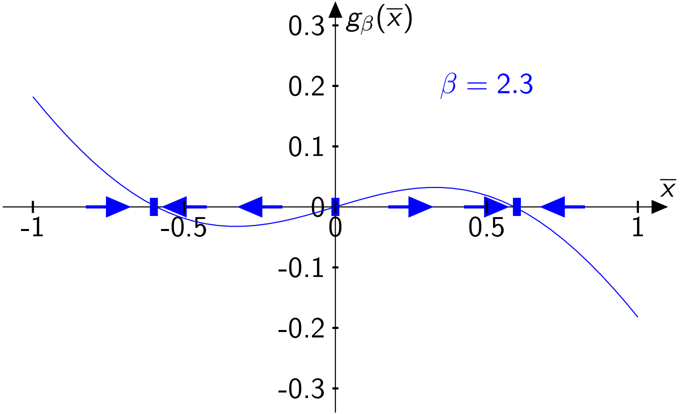
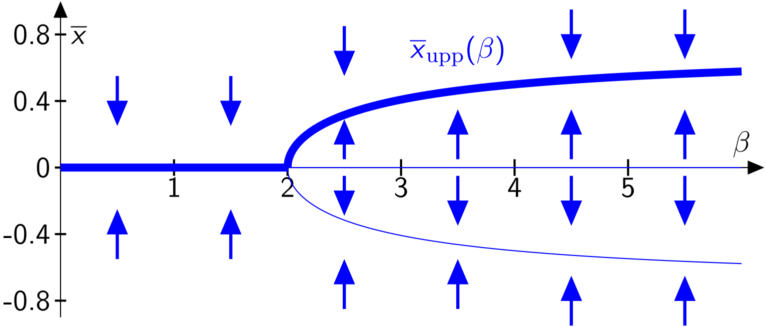
The Ising model



For $\beta > 2$, the fixed point $\overline{x} = 0$ becomes unstable and two new stable fixed points appear.

Starting with everybody in the state $+1$, the mean $\overline{X}_t$ converges as $t \to \infty$ to the upper fixed point $\overline{x}_{\mathrm{upp}}(\beta)$.



We observe a second-order phase transition.

The Ising model

Small β means large noise or equivalently a small tendency to choose the same state as your neighbors.

Large β means small noise or equivalently a strong tendency to choose the same state as your neighbors.

We observe a *phase transition* in β:

- For small β, the individuals behave essentially as if they were independent.
- For large β, two stable fixed points appear where a majority of people make the same choice.

The magnetization of the Ising model

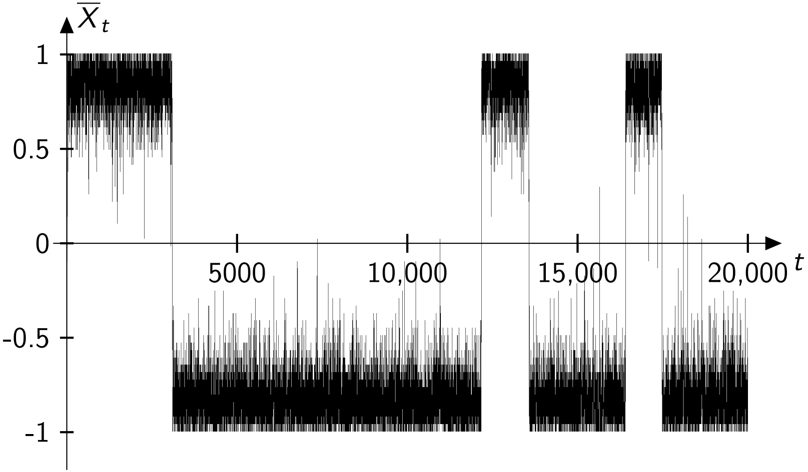
As long as the lattice Λ is *finite*, the (stochastic) Ising model is an irreducible continuous-time Markov chain with finite state space, and hence ergodic.

For the mean-field model, the limits $|\Lambda| \to \infty$ and $t \to \infty$ *cannot be interchanged*. Starting with $\overline{X}_0 > 0$, first letting $|\Lambda| \to \infty$ and then $t \to \infty$, our mean-field analysis showed that the magnetization has a positive limit $\overline{x}_{\rm upp}(\beta)$ for $\beta > 2$.

But ergodicity tells us that for any *fixed* $|\Lambda|$, letting $t \to \infty$, the process $\overline{X}_t$ eventually spends equally much time near $\overline{x}_{\rm upp}(\beta)$ and $-\overline{x}_{\rm upp}(\beta)$.

If Λ is large, then most of the time $\overline{X}_t \approx \pm\overline{x}_{\rm upp}(\beta)$, with rare transitions between the two values.

Metastability for the Ising model



The mean-field Ising model $\overline{X}$ with $|\Lambda| = 50$, $\overline{X}_0 = 0.1$, and $\beta = 3$.

Cooperative branching

Recall the *cooperative branching* map

$$\texttt{coop}_{i,i',j}x(k) := \begin{cases} \big(x(i) \wedge x(i')\big) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

and define a *death* map

$$\texttt{death}_i x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Consider the process $X = (X_t)_{t \geq 0}$ with generator

$$\begin{aligned} Gf(x) = b|\Lambda|^{-2} \sum_{i,i',j} \big\{ f\big(\texttt{coop}_{i,i',j}x\big) - f\big(x\big) \big\} \\ + \sum_i \big\{ f\big(\texttt{death}_i x\big) - f\big(x\big) \big\}. \end{aligned}$$

Factor $|\Lambda|^{-2}$ to make each site take part in a cooperative branching event with rate of order one.

Cooperative branching

As before, set $\varepsilon := |\Lambda|^{-1}$ and $\overline{X}_t := \varepsilon \sum_{i \in \Lambda} X_t(i)$. Then

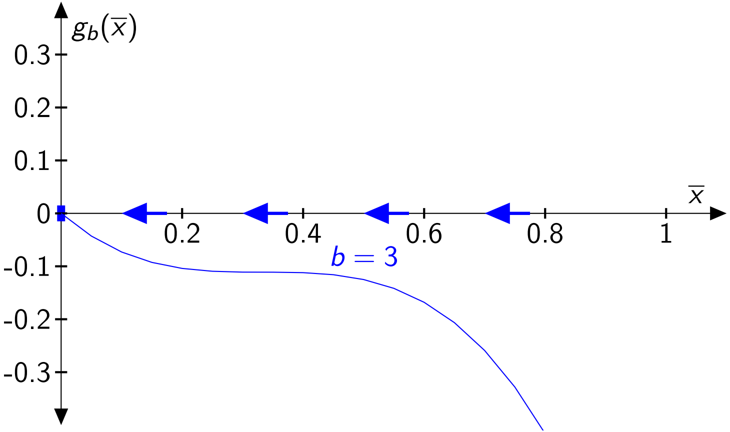
$$\begin{array}{lll} \overline{x} \mapsto \overline{x} + \varepsilon & \text{with rate} & r_+(x) = \varepsilon^{-1} b \overline{x}^2 (1 - \overline{x}), \\ \overline{x} \mapsto \overline{x} - \varepsilon & \text{with rate} & r_-(x) = \varepsilon^{-1} \overline{x}, \end{array}$$

from which we obtain in the limit $\varepsilon \to 0$ the mean-field equation

$$\frac{\partial}{\partial t} \overline{X}_t = b \overline{X}_t^2 (1 - \overline{X}_t) - \overline{X}_t =: g_b(\overline{X}_t) \qquad (t \geq 0).$$

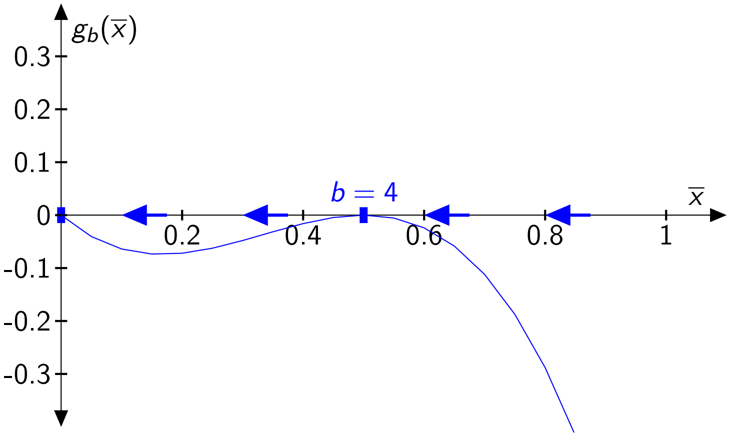
Interpretation: a sexually reproducing population.

Cooperative branching



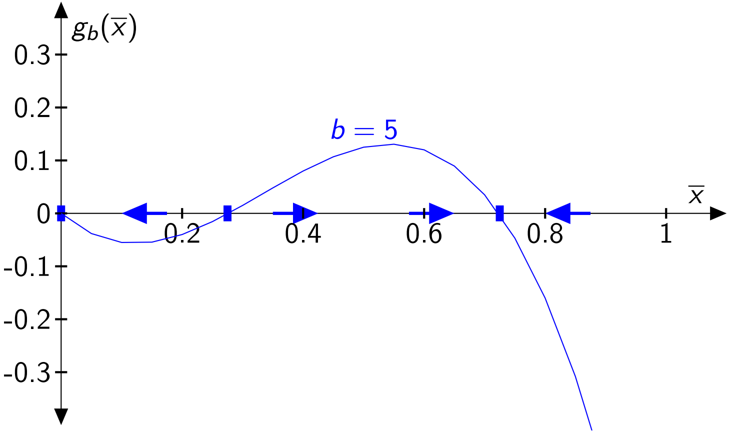
For $b < 4$, the equation $\frac{\partial}{\partial t}\overline{X}_t = g_b(\overline{X}_t)$ has a single, stable fixed point $\overline{x} = 0$.

Cooperative branching



For $b = 4$, a second fixed point appears at $\overline{x} = 0.5$.

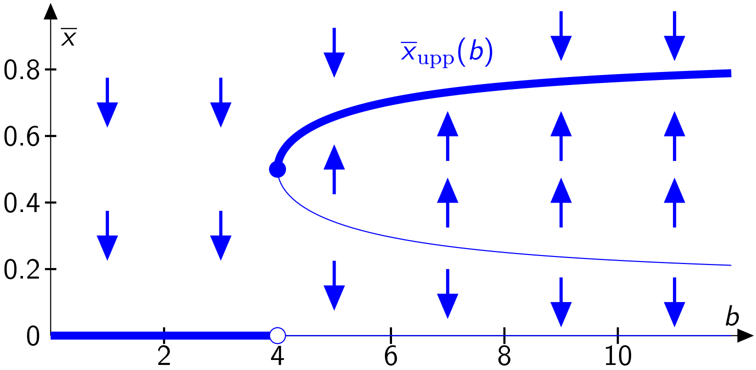
Cooperative branching



For $b > 4$, there are two stable fixed points and one unstable fixed point, which separates the domains of attraction of the other two.

Cooperative branching

Starting with all sites occupied by a 1, the mean $\overline{X}_t$ converges as $t \to \infty$ to the upper fixed point $\overline{x}_{\rm upp}(b)$.



We observe a first-order phase transition.

Cooperative branching

Again, the limits $|\Lambda| \to \infty$ and $t \to \infty$ *cannot be interchanged*. If $|\Lambda|$ is fixed but large, then $\overline{X}_t$ spends a long time near $\overline{x}_{\rm upp}(b)$, until eventually, the process dies out by chance.

This is *metastable* behavior.

The voter model

Recall the *voter map*

$$\mathtt{vot}_{ij}(x) := \begin{cases} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

We consider the mean-field model with generator

$$Gf(x) = |\Lambda|^{-1} \sum_{i,j \in \Lambda} \big\{ f\big(\mathtt{vot}_{i,j} x\big) - f\big(x\big) \big\}.$$

The factor $|\Lambda|^{-1}$ is chosen so that the rate at which a given site changes its type is of order one. This is the right time scale for the mean-field limit.

We can interpret G as saying each site chooses a new type at the times of a Poisson process with rate one, and the new type is chosen with equal probabilities from the population.

The voter model

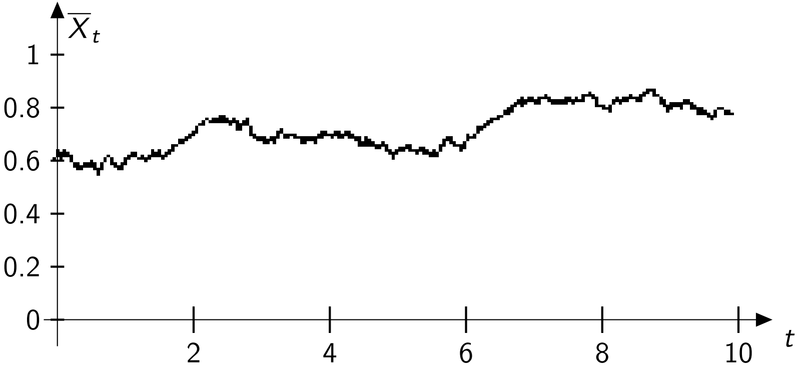
We consider the 2-type model with $X_t(i) \in \{0,1\}$ and as before, we set $\varepsilon := |\Lambda|^{-1}$ and $\overline{X}_t := \varepsilon \sum_{i\in\Lambda} X_t(i)$. Then $\overline{X}$ is a Markov process that jumps

$$\begin{aligned} \overline{x} &\mapsto \overline{x} + \varepsilon \qquad \text{with rate} \qquad r_+(x) = \varepsilon^{-1}\overline{x}(1-\overline{x}), \\ \overline{x} &\mapsto \overline{x} - \varepsilon \qquad \text{with rate} \qquad r_-(x) = \varepsilon^{-1}\overline{x}(1-\overline{x}), \end{aligned}$$

which in the mean-field limit yields the differential equation

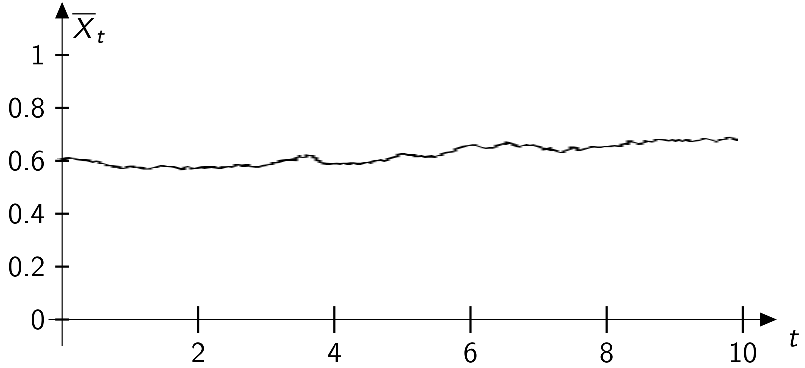
$$\frac{\partial}{\partial t}\overline{X}_t = 0.$$

The voter model

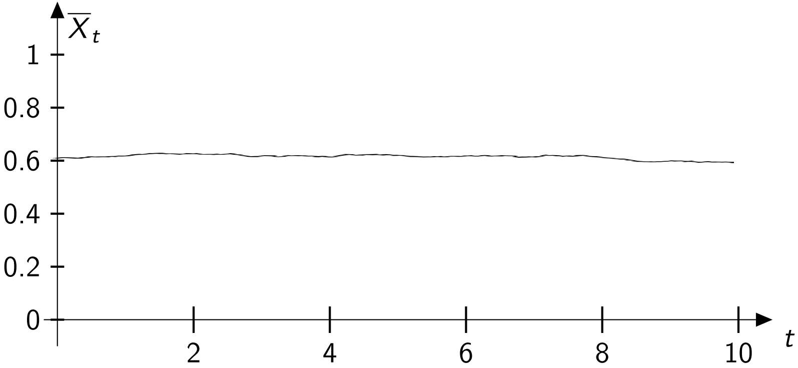


The Markov process $\overline{X}$ with $|\Lambda| = 100$ and $\overline{X}_0 = 0.6$.

The voter model



The Markov process $\overline{X}$ with $|\Lambda| = 1000$ and $\overline{X}_0 = 0.6$.



The Markov process $\overline{X}$ with $|\Lambda| = 10,000$ and $\overline{X}_0 = 0.6$.

The voter model

As before, we see random behavior on a larger time scale.

Let $\hat{X}_t := \overline{X}_{t/\varepsilon}$. Then

$$\mathbb{P}^{\overline{x}}\big[\hat{X}_t - \overline{x} = +\varepsilon\big] = \varepsilon^{-2}\overline{x}(1-\overline{x})\cdot t + O(t^2),$$
$$\mathbb{P}^{\overline{x}}\big[\hat{X}_t - \overline{x} = -\varepsilon\big] = \varepsilon^{-2}\overline{x}(1-\overline{x})\cdot t + O(t^2),$$

So

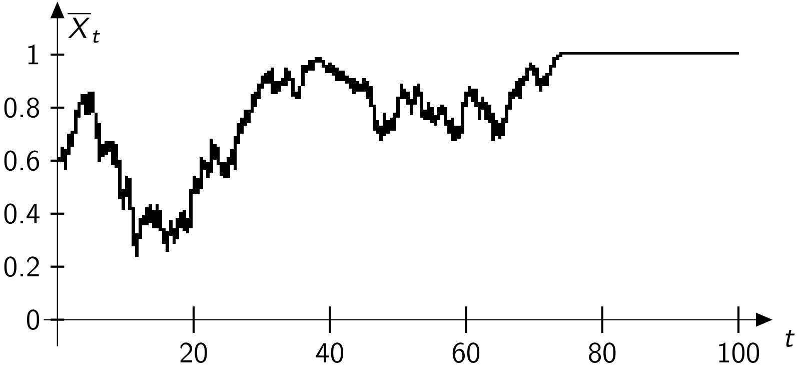
$$\mathbb{E}^{\overline{x}}\big[(\hat{X}_t - \overline{x})\big] = O(t^2),$$
$$\mathbb{E}^{\overline{x}}\big[(\hat{X}_t - \overline{x})^2\big] = \overline{x}(1-\overline{x})\cdot t + O(t^2).$$

One can use this to prove that the generator $\hat{G}_\varepsilon$ of $\hat{X}$ satisfies

$$\hat{G}_\varepsilon f(\overline{x}) \underset{\varepsilon\to 0}{\longrightarrow} \tfrac{1}{2}\overline{x}(1-\overline{x})\tfrac{\partial^2}{\partial\overline{x}^2} f(\overline{x})$$

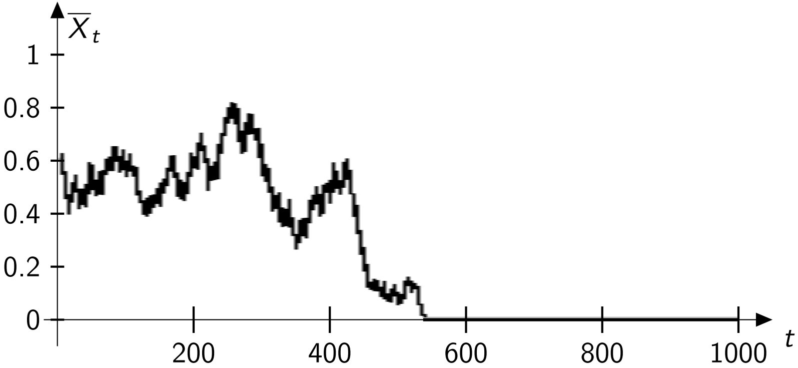
for suffiently smooth f.

The voter model



The Markov process $\overline{X}$ with $|\Lambda| = 100$ and $\overline{X}_0 = 0.6$.

The voter model



The Markov process $\overline{X}$ with $|\Lambda| = 1000$ and $\overline{X}_0 = 0.6$.

In the limit $\varepsilon \downarrow 0$, the process $\hat{X}_t$ converges in law to a solution of the stochastic differential

$$\mathrm{d}\hat{X}_t = \sqrt{\hat{X}_t(1-\hat{X}_t)}\mathrm{d}B_t \qquad (t \geq 0).$$

This is the *Wright-Fisher diffusion* with generator

$$Gf(\overline{x}) = \tfrac{1}{2}\overline{x}(1-\overline{x})\tfrac{\partial^2}{\partial \overline{x}^2} f(\overline{x}).$$

These calculations can be made rigorous using methods from the theory of convergence of Markov processes; see, e.g., the book by Ethier & Kurtz (1986).

The same methods may be applied to give rigorous proofs of the mean-field limits, where now one finds a first-order differential operator in the limit and an ODE instead of an SDE.

Exercises

Exercise 1 Do a mean-field analysis of the contact process with generator

$$\begin{aligned} Gf(x) = \lambda|\Lambda|^{-1} \sum_{i,j} \big\{ f\big(\mathtt{bra}_{i,j}x\big) - f\big(x\big) \big\} \\ + \sum_{i} \big\{ f\big(\mathtt{death}_{i}x\big) - f\big(x\big) \big\}. \end{aligned}$$

Do you observe a phase transition? Is it first- or second order?

Exercise 2 Same as above for the model with generator

$$\begin{aligned} Gf(x) = b|\Lambda|^{-2} \sum_{i,i',j} \big\{ f\big(\mathtt{coop}_{i,i',j}x\big) - f\big(x\big) \big\} \\ + \sum_{i,j} |\Lambda|^{-1} \big\{ f\big(\mathtt{rw}_{i,j}x\big) - f\big(x\big) \big\}. \end{aligned}$$

Exercise 3 Derive an SDE in the limit $|\Lambda| \to \infty$ for the density of the mean-field voter model with small bias and death rates, with generator

$$
\begin{aligned}
Gf(x) = |\Lambda|^{-2} & \sum_{i,j \in \Lambda} \big\{ f\big(\mathtt{vot}_{i,j} x\big) - f\big(x\big) \big\} \\
& + s|\Lambda|^{-1} \sum_{i,j \in \Lambda} \big\{ f\big(\mathtt{bra}_{i,j} x\big) - f\big(x\big) \big\} \\
& + d \sum_{i \in \Lambda} \big\{ f\big(\mathtt{death}_i x\big) - f\big(x\big) \big\}.
\end{aligned}
$$

Hint: You should find expressions of the form

$$
\begin{aligned}
\mathbb{E}^{\overline{x}}\big[(\overline{X}_t - \overline{x})\big] &= b(\overline{x}) \cdot t + O(t^2), \\
\mathbb{E}^{\overline{x}}\big[(\overline{X}_t - \overline{x})^2\big] &= a(\overline{x}) \cdot t + O(t^2),
\end{aligned}
$$

which leads to a limiting generator of the form

$$
Gf(\overline{x}) = \tfrac{1}{2} a(\overline{x}) \tfrac{\partial^2}{\partial \overline{x}^2} f(\overline{x}) + b(\overline{x}) \tfrac{\partial}{\partial x} f(\overline{x}).
$$

Exercise 4 Do a mean-field analysis of the following extension of the voter model, considered by Neuhauser & Pacala (1999). In their model, the site i flips

$$
\begin{aligned}
0 \mapsto 1 &\quad \text{with rate}\big(f_0 + \alpha_{01} f_1\big) f_1,\\
1 \mapsto 0 &\quad \text{with rate}\big(f_1 + \alpha_{10} f_1\big) f_0,
\end{aligned}
$$

where $\alpha_{01}, \alpha_{10} > 0$ and $f_\tau = |\mathcal{N}_i|^{-1} \sum_{j \in \mathcal{N}_i} 1_{\{x(j)=\tau\}}$ is the relative frequency of type τ in the neigborhood of i.

Find all stable and unstable fixed points of the mean-field model in the regimes: I. $\alpha_{01}, \alpha_{10} < 1$, II. $\alpha_{01} < 1 < \alpha_{10}$, III. $\alpha_{10} < 1 < \alpha_{01}$, IV. $1 < \alpha_{01}, \alpha_{10}$.