# Interacting Particle Systems with Applications in Finance

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## Lecture 3: Construction of Infinite Systems and Uniqueness of the Invariant Law

# Construction of Particle Systems

Let $\Lambda$ be a countable set (the *lattice*, e.g., $\Lambda = \mathbb{Z}^d$), let $Q$ be a finite set (the *local state space*, e.g., $Q = \{1, \ldots, q\}$ or $\{0, 1\}$ or $\{-1, +1\}$), and let $S := Q^\Lambda$ be the space of all function $x : \Lambda \to Q$.

Let $\mathcal{M}$ be a countable collection of maps $m : S \to S$ and let $(r_m)_{m \in \mathcal{M}}$ be nonnegative rates. We wish to construct the Markov process $X = (X_t)_{t \geq 0}$ with formal generator

$$Gf(x) = \sum_{m \in \mathcal{M}} r_m \{ f\big(m(x)\big) - f\big(x\big) \}.$$

By Tychonoff, $S = Q^\Lambda$, equipped with the product topology, is a compact space. If $\Lambda$ is infinite and $|Q| > 1$, then $S$ is *uncountable*.

# Feller Processes

Let $S$ be a compact, metrizable space.
We let $\mathcal{C}(S)$ denote the Banach space of continuous real functions on $S$, equipped with the supremumnorm $\|f\| := \sup_{x\in S}|f(x)|$.
We let $\mathcal{M}_1(E)$ denote the space of probability measures on $E$, equipped with the topology of weak convergence. We note that $\mathcal{M}_1(E)$ is compact and metrizable.

By definition, a *continuous transition probability* on $S$ is a collection $(P_t(x, \mathrm{d}y))_{t\geq 0}$ of probability kernels on $S$ such that

(i) $(x,t) \mapsto P_t(x,\,\cdot\,)$ is continuous from $S \times [0,\infty)$ to $\mathcal{M}_1(S)$,

(ii) $\displaystyle\int_S P_s(x,\mathrm{d}y)P_t(y,\mathrm{d}z) = P_{s+t}(x,\mathrm{d}z) \quad \text{and} \quad P_0(x,\cdot\,) = \delta_x.$

# Feller Processes

Each continuous transition probability defines linear operators $P_t : \mathcal{C}(S) \to \mathcal{C}(S)$ by

$$P_t f(x) := \int_S P_t(x, \mathrm{d}y) f(y).$$

These satisfy

$$\begin{array}{rl}
\text{(i)} & \lim_{t\to 0} \|P_t f - f\| = 0 \qquad (f \in \mathcal{C}(S)),\\
\text{(ii)} & P_s P_t f = P_{s+t} f \quad \text{and} \quad P_0 f = f,\\
\text{(iii)} & f \geq 0 \text{ implies } P_t f \geq 0,\\
\text{(iv)} & P_t 1 = 1.
\end{array}$$

Conversely, each $(P_t)_{t\geq 0}$ with these properties corresponds to a continuous transition probability.
We call $(P_t)_{t\geq 0}$ a *Feller semigroup*.

By definition, the *generator* of a Feller semigroup is the operator

$$Gf := \lim_{t\to 0} t^{-1}\big(P_t f - f\big),$$

with *domain*

$$\mathcal{D}(G) := \big\{f \in \mathcal{C}(S) : \text{the limit } \lim_{t\to 0} t^{-1}\big(P_t f - f\big) \text{ exists}\big\}.$$

Here the limit should exist w.r.t. the topology on $\mathcal{C}(S)$, i.e., w.r.t. the supremumnorm $\|\cdot\|$.

The domain of a linear operator is an essential part of its definition!

# Feller Processes

An operator $A$ on a Banach space $\mathcal{C}$ is *closed* if its *graph* $\{(f, Af) : f \in \mathcal{D}(A)\}$ is a closed subset of $\mathcal{C} \times \mathcal{C}$.

We say that $A$ is *closeable* if there exists an operator $\overline{A}$ (the *closure* of $A$) with domain $\mathcal{D}(\overline{A})$, such that $\{(f, \overline{A}f) : f \in \mathcal{D}(\overline{A})\}$ is the closure in $\mathcal{C} \times \mathcal{C}$ of $\{(f, Af) : f \in \mathcal{D}(A)\}$.

We say that an operator $A$ on $\mathcal{C}(S)$ with domain $\mathcal{D}(A)$ satisfies the *maximum principle* if, whenever a function $f \in \mathcal{D}(A)$ assumes its maximum over $S$ in a point $x \in S$, we have $Af(x) \leq 0$.

# Feller Processes

**Feller semigroups** A linear operator $G$ on $\mathcal{C}(S)$ is the generator of a Feller semigroup $(P_t)_{t\geq 0}$ if and only if

(i) $1 \in \mathcal{D}(G)$ and $G1 = 0$.

(ii) $G$ satisfies the maximum principle.

(iii) $\mathcal{D}(G)$ is dense in $\mathcal{C}(S)$.

(iv) For every $f \in \mathcal{D}(G)$ there exists a continuously differentiable function $t \mapsto u_t$ from $[0, \infty)$ into $\mathcal{C}(S)$ such that $u_0 = f$, and $u_t \in \mathcal{D}(G)$, $\frac{\partial}{\partial t} u_t = G u_t$ for each $t \geq 0$.

(v) $G$ is closed.

Here, in point (iv), continuity and differentiability are defined w.r.t. the supremumnorm.

A Feller semigroup is uniquely determined by its generator. For $f \in \mathcal{D}(G)$, the function $u$ in (iv) is given by $u_t = P_t f$. More generally, $P_t$ is the closure of $\{(f, P_t f) : f \in \mathcal{D}(G)\}$.

# Feller Processes

If the domain $\mathcal{D}(A)$ of a linear operator $A$ is the whole Banach space $\mathcal{C}$, then $A$ is closed if and only if $A$ is *bounded* i.e., there exists a constant $C < \infty$ such that $\|Af\| \leq C\|f\|$.

As a consequence, the generator $G$ of a Feller semigroup is bounded if and only if it is everywhere defined, i.e., $\mathcal{D}(G) = \mathcal{C}(S)$.

In this case, the Feller semigroup is given by

$$P_t = e^{Gt} := \sum_{n=0}^{\infty} \frac{1}{n!} G^n t^n \qquad (t \geq 0),$$

where the infinite sum converges absolutely in the operator norm, defined as $\|A\| := \sup\{\|Af\| : \|f\| \leq 1\}$.

In the general, unbounded case, it is usually not feasable to specify $\mathcal{D}(G)$ precisely.

**Hille-Yosida** The closure of a linear operator $A$ on $\mathcal{C}(S)$ is the generator of a Feller semigroup $(P_t)_{t\geq 0}$ if and only if

(i) $1 \in \mathcal{D}(\overline{A})$ and $\overline{A}1 = 0$.

(ii) $A$ satisfies the maximum principle.

(iii) $\mathcal{D}(A)$ is dense in $\mathcal{C}(S)$.

(iv) There exists an $r \in (0,\infty)$ and a dense subspace $\mathcal{D} \subset \mathcal{C}(S)$ with the property that for every $f \in \mathcal{D}$ there exists a $p_r \in \mathcal{D}(G)$ such that $(r - G)p_r = f$.

If (iv) holds for some $r \in (0,\infty)$, then it holds for every $r \in (0,\infty)$.

The function $p_r$ in (iv) is given by

$$p_r = \int_0^\infty e^{-rt} P_t f \,\mathrm{d}t.$$

If for some $f \in \mathcal{D}(A)$ one can solve the Cauchy problem

$$u_0 = f, \text{ and } u_t \in \mathcal{D}(A),\ \frac{\partial}{\partial t} u_t = G u_t \text{ for each } t \geq 0,$$

then $p_r := \int_0^\infty e^{-rt} u_t \,\mathrm{d}t$ solves $(r - G)p_r = f$.

# Feller Processes

Let $(P_t)_{t\geq 0}$ be a Feller semigroup on $\mathcal{C}(S)$.

Then, for each probability measure $\mu$ on $S$, there exists a process $X = (X_t)_{t\geq 0}$ with cadlag sample paths, unique in law, such that $\mathbb{P}[X_0 \in \cdot\,] = \mu$ and

$$\mathbb{E}\big[f(X_u)\,\big|\,(X_s)_{0\leq s\leq t}\big] = P_{u-t}f(X_t) \quad \text{a.s.} \qquad (0 \leq s \leq t).$$

The process $X$ is (strongly) Markov with transition probabilities $(P_t)_{t\geq 0}$.

See Ethier & Kurtz (1986).

# Poisson Construction of Particle Systems

We wish to adapt the Poisson construction of finite systems. Let $\omega$ be a Poisson point set on $\mathcal{M} \times \mathbb{R}$ with intensity

$$\mu\big(\{m\} \times A\big) = r_m \, \ell(A),$$

where $\ell$ denotes Lebesgue measure, and let

$$\omega_{s,t} := \omega \cap \mathcal{M} \times (s, t].$$

If $\omega_{0,t}$ is a.s. finite, then we can order its elements as

$$\omega_{s,t} = \big\{(m_1, t_1), \ldots, (m_n, t_n)\big\}$$

with $t_1 < \cdots < t_n$, and define as before

$$X_{t-s} = \mathbf{X}_{s,t}(X_0) := m_n \circ \cdots \circ m_1(X_0).$$

# Poisson Construction of Particle Systems

However, this is usually too restrictive. Recall the voter map

$$\mathtt{vot}_{ij}(x) := \begin{cases} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

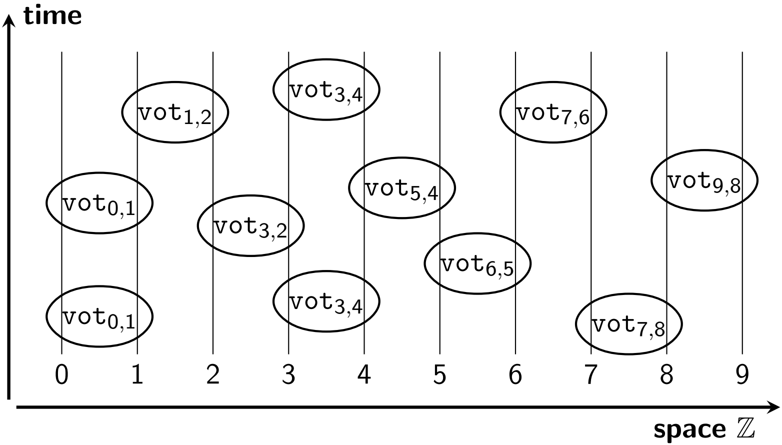
and consider the one-dimensional, nearest-neighbor voter model on $\mathbb{Z}$, with formal generator

$$\begin{aligned} Gf(x) = \tfrac{1}{2} \sum_{i\in\mathbb{Z}} \big\{ f\big(\mathtt{vot}_{i,i+1}x\big) - f\big(x\big) \big\} \\ + \tfrac{1}{2} \sum_{i\in\mathbb{Z}} \big\{ f\big(\mathtt{vot}_{i,i-1}x\big) - f\big(x\big) \big\}, \end{aligned}$$

which corresponds to $r_m = \frac{1}{2}$ for all $m \in \mathcal{M}$, and

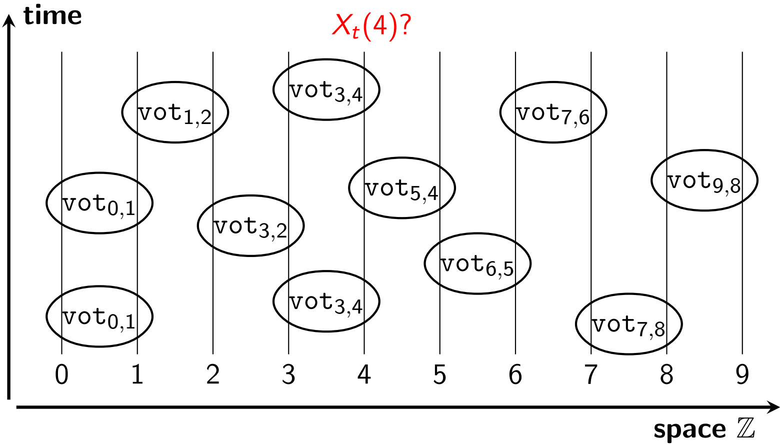
$$\mathcal{M} = \big\{ \mathtt{vot}_{i,i+1}, \mathtt{vot}_{i,i-1} : i \in \mathbb{Z} \big\}.$$

# Poisson Construction of Particle Systems

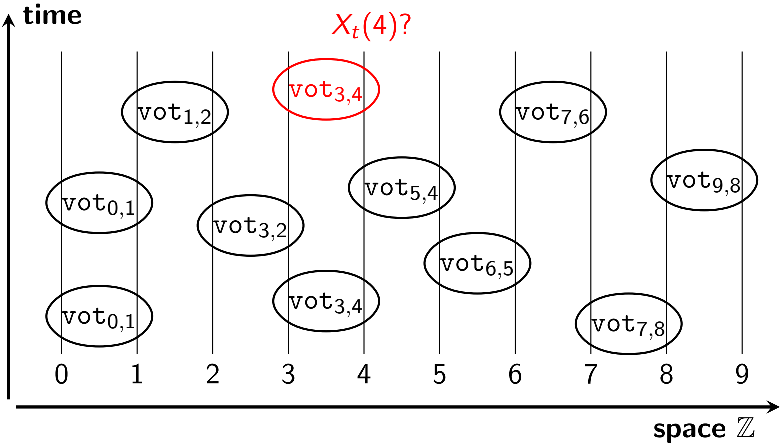



The Poisson set $\omega_{0,t}$ of local maps acting during the time interval $(0,t]$ is a.s. infinite for each $t > 0$!

# Poisson Construction of Particle Systems




However, we need to know only finitely many elements of $\omega_{0,t}$ to determine the local state of $X$ at a space-time point $(i, t)$.

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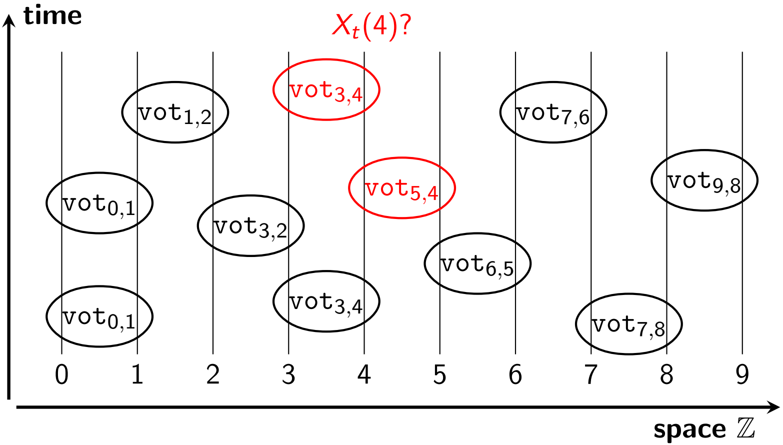


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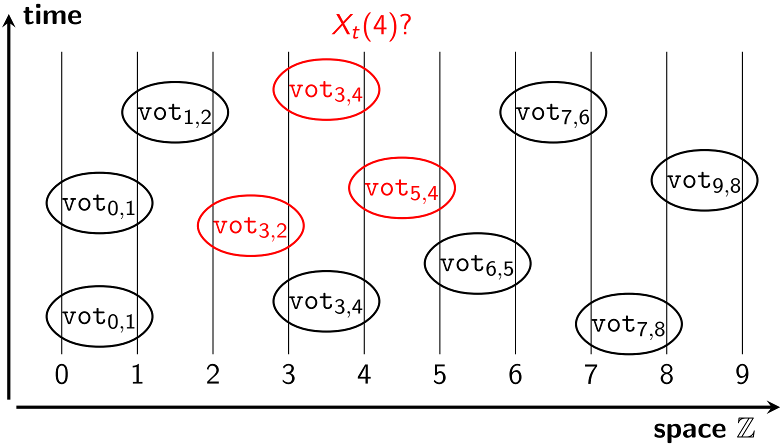


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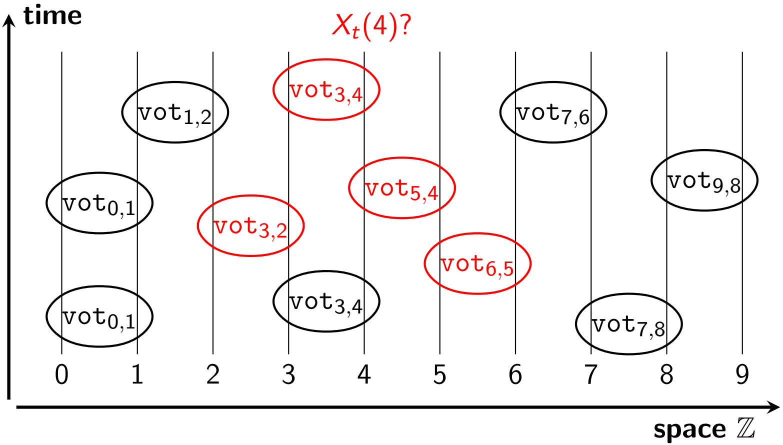


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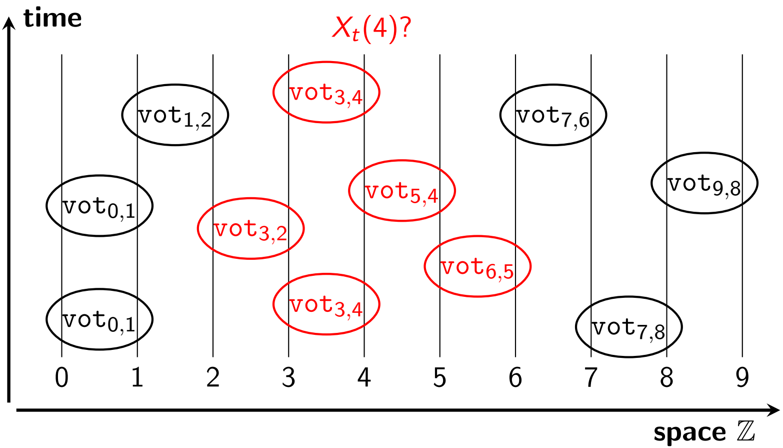



However, we need to know only finitely many elements of $\omega_{0,t}$ to determine the local state of $X$ at a space-time point $(i, t)$.

For any map $m : S \to S$, let

$$\mathcal{D}(m) := \big\{ i \in \Lambda : \exists x \in S \text{ s.t. } m(x)(i) \neq x(i) \big\}$$

denote the set of lattice points whose values can possibly be changed by $m$. Let us say that a point $j \in \Lambda$ is *m-relevant* for some $i \in \Lambda$ if

$$\exists x, y \in S \text{ s.t. } m(x)(i) \neq m(y)(i) \text{ and } x(k) = y(k) \ \forall k \neq j,$$

and write

$$\mathcal{R}_i(m) := \big\{ j \in \Lambda : j \text{ is } m\text{-relevant for } i \big\}.$$

Note that $\mathcal{R}_i(m) = \emptyset$ for $i \notin \mathcal{D}(m)$. It may also happen that $\mathcal{R}_i(m) = \emptyset$ for some (or all!) $i \in \mathcal{D}(m)$.

If $\mathcal{D}(m)$ and $\mathcal{R}_i(m)$ with $i \in \mathcal{D}(m)$ are finite, then we say that $m$ is a *local map*.

**Example 1** The *voter map*

$$\mathtt{vot}_{ij}(x) := \begin{cases} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

says that site $i$ copies the type of site $j$. Therefore:

$\mathcal{D}(\mathtt{vot}_{ij}) = \{i\}$. *Only site $i$ can change.*

$\mathcal{R}_i(\mathtt{vot}_{ij}) = \{j\}$. *We only need to know $x(j)$ to decide what type site $i$ has after we apply $\mathtt{vot}_{ij}$.*

**Example 2** The *coalescing random walk map*

$$\mathtt{rw}_{i,j}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

says that if there is a particle at $i$, then this jumps to $j$, coalescing with any particle that may already be present there. Therefore:

$\mathcal{D}(\mathtt{rw}_{ij}) = \{i, j\}$. *Sites $i$ and $j$ can change.*

$\mathcal{R}_i(\mathtt{rw}_{ij}) = \emptyset$. *After we apply $\mathtt{rw}_{ij}$, the site $i$ is always empty. We do not need to known anything about $x$ to know that.*

$\mathcal{R}_j(\mathtt{rw}_{ij}) = \{i, j\}$. *In order to decide if after we apply $\mathtt{rw}_{ij}$, there is a particle at $j$, we need to know both $x(i)$ and $x(j)$.*

# Poisson Construction of Particle Systems

Assume that $\mathcal{M}$ contains only local maps.

By definition a *path of influence* from $(i,s)$ to $(j,u)$ is a cadlag function $\gamma:[s,u]\to\Lambda$ such that $\gamma_{s-}=i$, $\gamma_u=j$, and

(i) if $\gamma_{t-}\neq\gamma_t$ for some $t\in[s,u]$, then there exists some $m\in\mathcal{M}$ such that $(m,t)\in\omega$, $\gamma_t\in\mathcal{D}(m)$ and $\gamma_{t-}\in\mathcal{R}_{\gamma_t}(m)$,

(ii) for each $(m,t)\in\omega$ with $t\in[s,u]$ and $\gamma_t\in\mathcal{D}(m)$, one has $\gamma_{t-}\in\mathcal{R}_{\gamma_t}(m)$.

For any finite set $A\subset\Lambda$ and $s\leq u$, we set

$$\zeta_s^{A,u}:=\big\{i\in\Lambda:(i,s)\rightsquigarrow A\times\{u\}\big\},$$

where $(i,s)\rightsquigarrow A\times\{u\}$ denotes the presence of a path of influence from $(i,s)$ to some $(j,u)\in A\times\{u\}$.

**Proposition 1** *Assume that*

$$K_0 := \sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{M}\\ \mathcal{D}(m)\ni i}} r_m < \infty$$

$$K := \sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{M}\\ \mathcal{D}(m)\ni i}} r_m\big(|\mathcal{R}_i(m)| - 1\big) < \infty.$$

*Then, for each finite $A \subset \Lambda$, one has*

$$\mathbb{E}\big[|\zeta_s^{A,u}|\big] \leq |A| e^{K(u-s)} \qquad (s \leq u).$$

**Proof** Without loss of generality set $u=0$. Fix $A$ and write $\zeta_s := \zeta_s^{A,0}$. Let $\Lambda_n \subset \Lambda$ be finite sets such that $\Lambda_n \uparrow \Lambda$. For $n$ large enough such that $A \subset \Lambda_n$, write

$$\zeta_s^n := \big\{ i \in \Lambda_n : (i,s) \rightsquigarrow_n A \times \{0\} \big\},$$

where $(i,s) \rightsquigarrow_n A \times \{0\}$ denotes the presence of a path of influence from $(i,s)$ to $A \times \{0\}$ that stays in $\Lambda_n$. We observe that

$$\zeta_s^n \uparrow \zeta_s \qquad (s \leq 0).$$

It therefore suffices to prove

$$\mathbb{E}^A\big[|\zeta_s^n|\big] \leq |A| e^{K(-s)} \qquad (s \leq 0)$$

uniformly in $n$.

Let $\mathcal{M}_n := \{m \in \mathcal{M} : \mathcal{D}(m) \cap \Lambda_n \neq \emptyset\}$.

The process $(\zeta^n_{-t})_{t\geq 0}$ is a Markov process taking values in the (finite) space of all subsets of $\Lambda_n$, with generator

$$G_n f(B) := \sum_{m\in\mathcal{M}_n} r_m \big(f(B^{(m)}) - f(B)\big),$$

where

$$B^{(m)} := \big(B \backslash \mathcal{D}(m)\big) \cup \bigcup_{i\in B\cap\mathcal{D}(m)} \big(\mathcal{R}_i(m) \cap \Lambda_n\big).$$

By our assumption $K_0 < \infty$, we have $\sum_{m\in\mathcal{M}_n} r_m < \infty$, hence this Markov process is well-defined.

# Poisson Construction of Particle Systems

Let $(P_t^n)_{t\geq 0}$ be the associated semigroup and let $f$ denote the function $f(A) := |A|$. Then

$$\begin{aligned} G_n f(A) &= \sum_{m\in\mathcal{M}_n} r_m\big(f(A^m) - f(A)\big) \\ &\leq \sum_{m\in\mathcal{M}_n} r_m\Big(|A\backslash\mathcal{D}(m)| + \sum_{i\in A\cap\mathcal{D}(m)} |\mathcal{R}_i(m)| - |A|\Big) \\ &= \sum_{m\in\mathcal{M}_n} r_m\Big(\sum_{i\in A\cap\mathcal{D}(m)} \big(|\mathcal{R}_i(m)| - 1\big)\Big) \\ &= \sum_{i\in A} \sum_{\substack{m\in\mathcal{M}_n \\ \mathcal{D}(m)\ni i}} r_m\big(|\mathcal{R}_i(m)| - 1\big) \leq K|A|. \end{aligned}$$

# Poisson Construction of Particle Systems

We have just shown that the function $f(A) := |A|$ satisfies $G_n f(A) \leq K f(A)$. It follows that

$$\begin{aligned}\tfrac{\partial}{\partial t}\big(e^{-Kt}P^n_t f\big) &= -Ke^{-Kt}P^n_t f + e^{-Kt}P^n_t G_n f\\ &= e^{-Kt}P^n_t(G_n f - Kf) \leq 0\end{aligned}$$

and therefore $e^{-Kt}P^n_t f \leq e^{-K0}P^n_0 f = f$, which means that

$$\mathbb{E}^A\big[|\zeta^n_{-t}|\big] = P^n_t f(A) \leq |A|e^{Kt} \qquad (t \geq 0).$$

∎

# Poisson Construction of Particle Systems

Let

$$\omega_{s,(i,u)} := \big\{(m,t) \in \omega_{s,u} : \mathcal{D}(m) \times \{t\} \rightsquigarrow (i,u)\big\}$$

be the set of all Poisson events during $(s,u]$ that are relevant for the state of our process at $(i,u)$.

**Proposition 2** *Assume that*

$$K_1 := \sup_{i \in \Lambda} \sum_{\substack{m \in \mathcal{M} \\ \mathcal{D}(m) \ni i}} r_m |\mathcal{R}_i(m)| < \infty.$$

*Then $\omega_{s,(i,u)}$ is a.s. finite for each $s \leq u$ and $i \in \Lambda$.*

**Proof** This is the same proof as before, except that instead of

$$B^{(m)} := (B \backslash \mathcal{D}(m)) \cup \bigcup_{i \in B \cap \mathcal{D}(m)} \big(\mathcal{R}_i(m) \cap \Lambda_n\big),$$

we define

$$B^{(m)} := B \cup \bigcup_{i \in B \cap \mathcal{D}(m)} \big(\mathcal{R}_i(m) \cap \Lambda_n\big),$$

so that sites, once included in $\zeta^n_{-t}$, cannot be removed.
Our previous proof now shows that $\mathbb{E}^A[|\zeta_{-t}|] \leq |A| e^{K_1 t}$.
In particular, for fixed $s < u = 0$, the set $\zeta_s$ is a.s. finite and all sets $\zeta_{-t}$ with $s \leq -t \leq 0$ are contained in it. By the condition $K_0 < \infty$, the finite set $\zeta_s$ is intersected by only finitely many events $(m, t) \in \omega_{s,u}$. ∎

**Theorem 3 (Graphical Representation)** *Assume that*

$$\sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{M}\\ \mathcal{D}(m)\ni i}} r_m\big(|\mathcal{R}_i(m)|+1\big)<\infty.$$

*Then, a.s.,* $\omega_{s,(t,i)}$ *is finite for all* $s\leq t$ *and* $i\in\Lambda$*. For any finite* $\tilde{\omega}_{s,(t,i)}$ *with* $\omega_{s,(t,i)}\subset\tilde{\omega}_{s,(t,i)}\subset\omega_{s,t}$*, setting*

$$\tilde{\omega}_{s,(t,i)}:=\big\{(m_1,t_1),\ldots,(m_n,t_n)\big\}$$

*with* $t_1<\cdots<t_n$*, we can unambiguously define random maps* $\mathbf{X}_{s,t}:S\to S$ $(s\leq t)$ *by*

$$\mathbf{X}_{s,t}(x)(i):=m_n\circ\cdots\circ m_1(x)(i).$$

# Poisson Construction of Particle Systems

Let $(\mathbf{X}_{s,t})_{s\leq t}$ be the random maps defined in terms of the graphical representation (Poisson point set) $\omega$. Then

$$P_t(x,\,\cdot\,) := \mathbb{P}\big[\mathbf{X}_{0,t}(x) \in \,\cdot\,\big]$$

defines a collection of probability kernels $(P_t)_{t\geq 0}$.

**Theorem 4** *The probability kernels $(P_t)_{t\geq 0}$ form a continuous transition probability. Moreover, if $X_0$ is independent of $\omega$, then*

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

*defines a process with cadlag sample paths such that*

$$\mathbb{E}\big[f(X_u)\,\big|\,(X_s)_{0\leq s\leq t}\big] = P_{u-t}f(X_t) \quad \text{a.s.} \qquad (0 \leq s \leq t).$$

# Generator Construction of Particle Systems

For $f \in \mathcal{C}(S)$ and $i \in \Lambda$, define

$$\delta f(i) := \sup\big\{|f(x) - f(y)| : x, y \in S,\ x(j) = y(j)\ \forall j \neq i\big\}.$$

We call $\delta f$ the *variation* of $f$ and define

$$\mathcal{C}_{\rm sum}(S) := \big\{f \in \mathcal{C}(S) : \textstyle\sum_i \delta f(i) < \infty\big\}.$$

Recall that

$$K_0 := \sup_{i \in \Lambda} \sum_{\substack{m \in \mathcal{M} \\ \mathcal{D}(m) \ni i}} r_m < \infty.$$

For each $f \in \mathcal{C}_{\rm sum}(S)$,

$$\sum_{m \in \mathcal{M}} r_m \big| f(m(x)) - f(x) \big| \leq K_0 \sum_{i \in \Lambda} \delta f(i).$$

In particular, for such $f$, $Gf$ is well-defined by

$$Gf(x) = \sum_{m \in \mathcal{M}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}.$$

**Theorem 5** *Let $(P_t)_{t\geq 0}$ be the Feller semigroup arising from our Poisson construction. Then the generator of $(P_t)_{t\geq 0}$ is the closure of $G$ with domain $\mathcal{D}(G) := \mathcal{C}_{\rm sum}(S)$.*

**Proof (sketch)** One can check that

$$\sum_{i\in\Lambda}\delta P_t f(i) \leq e^{Kt}\sum_{i\in\Lambda}\delta f(i) \qquad \big(t\geq 0,\ f\in\mathcal{C}_{\rm sum}(S)\big)$$

In particular, for each $t\geq 0$, $P_t$ maps $\mathcal{C}_{\rm sum}(S)$ into itself. Moreover, $t\mapsto P_t f$ is continuously differentiable, $P_0 f = f$, and $\frac{\partial}{\partial t}P_t f = GP_t f$ for each $t\geq 0$. The claim now follows from Hille-Yosida. ∎

# Generator Construction of Particle Systems

Since a Feller process is uniquely characterized by its generator, we see that if

$$\begin{aligned} Gf(x) &= \sum_{m \in \mathcal{M}} r_m \{ f(m(x)) - f(x) \} \\ &= \sum_{m \in \tilde{\mathcal{M}}} \tilde{r}_m \{ f(m(x)) - f(x) \} \end{aligned}$$

are two *different* random mapping representations of the *same* generator, then the corresponding processes

$$X_t := \mathbf{X}_{0,t}(X_0) \quad \text{and} \quad \tilde{X}_t := \tilde{\mathbf{X}}_{0,t}(X_0)$$

are equal in law. In practise, one needs both the graphical representation (Poisson construction) and the Feller formalism.

# Generator Construction of Particle Systems

The first infinite interacting particle systems were constructed by Dobrushin (1971).

Liggett (1972) introduced the space $\mathcal{C}_{\rm sum}(S)$ and proved sufficient conditions directly in terms of the generator $G$ so that its closure generates a Feller semigroup (and hence corresponds to a well-defined Markov process). These conditions are more general and do not depend on finding a good random mapping representation.

Graphical representations of interacting particle systems have been used since Harris (1972), but I do not know a good general reference.

The fact that the Poisson and generator constructions yield the *same* process is usually considered self-evident.

# Uniqueness of the Invariant Law

Let $X$ be a Feller process in $S$ with semigroup $(P_t)_{t\geq 0}$.
For any probability measure $\mu$ on $S$, we define

$$\mu P_t(\mathrm{d}y) := \int_S \mu(\mathrm{d}x) P_t(x, \mathrm{d}y).$$

Then $\mu P_t$ is the law at time $t$ of the process with initial law $\mu$.
If $S$ is finite, this just says that

$$\mu P_t(y) = \sum_{x\in S} \mu(x) P_t(x, y),$$

where we simplify notation by writing $\mu(x) := \mu(\{x\})$ etc.

By definition, an *invariant law* of $X$ is a probability measure $\mu$ such that

$$\mu P_t = \mu \qquad (t \geq 0).$$

Equivalently, this says that the process $X$ with initial law $\mu$ is stationary.

# Uniqueness of the Invariant Law

Let $G$ have the random mapping representation

$$Gf(x) = \sum_{m \in \mathcal{M}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}.$$

Assume that

$$\sup_{i \in \Lambda} \sum_{\substack{m \in \mathcal{M} \\ \mathcal{D}(m) \ni i}} r_m \big( |\mathcal{R}_i(m)| + 1 \big) < \infty,$$

so that the corresponding particle systems is well-defined, and recall that

$$K := \sup_{i \in \Lambda} \sum_{\substack{m \in \mathcal{M} \\ \mathcal{D}(m) \ni i}} r_m \big( |\mathcal{R}_i(m)| - 1 \big).$$

Note that it is possible that $K < 0$.

Our proof that the random maps $(\mathbf{X}_{s,t})_{s\leq t}$ are well-defined by the graphical representation yields the following:

**Corollary 6** *For $A \subset \Lambda$ and $s \leq t$, let*

$$\zeta_s^{A,t} := \big\{ i \in \Lambda : (i,s) \rightsquigarrow A \times \{t\} \big\}$$

*be the set of lattice points at time $s$ whose value is relevant for the states in $A$ at time $t$. Then*

$$\mathbb{E}\big[|\zeta_s^{A,t}|\big] \leq |A| e^{K(t-s)}.$$

# Uniqueness of the Invariant Law

**Theorem 7** *Assume that* $K < 0$*. Then the interacting particle system* $X$ *with generator* $G$ *has a unique invariant law* $\nu$*. Moreover, the process started in any initial law satisfies*

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu.$$

Before we give the proof, it is good to see an example.

# Ergodicity of the Ising Model

Parameters $J, \beta > 0$. Assume $|\mathcal{N}_i| = N$ for all $i$. Utility function:

$$U_t^{\pm}(i,x) = \pm\tfrac{1}{2}J\sum_{j\in\mathcal{N}_i} x(j) \pm \tfrac{1}{2}W_t(i).$$

- $W_t(i)$ logistically distributed: $\mathbb{P}[W_t(i) \leq w] = (1 + e^{-\beta w})^{-1}$.
- The noise terms $W_t(i)$ are independent for each person and are redrawn at times of a Poisson process with intensity one.
- After the noise is redrawn, each person immediately chooses his or her new state according to the highest utility.

*If $|W_t(i)| > JN$, then the choice of a person does not depend on his/her neighborhood!*

# Ergodicity of the Ising Model

To apply Theorem 7, we need to device a random mapping representation. For $h \in \{-N-1, -N+1, \ldots, N+1\} =: H$, define

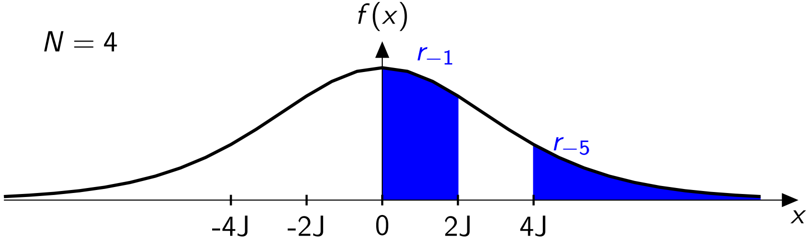
$$m_{i,h}(x)(k)(:= \begin{cases} +1 & \text{if } k = i \text{ and } \sum_{i\in\mathcal{N}_i} x(j) > h, \\ -1 & \text{if } k = i \text{ and } \sum_{i\in\mathcal{N}_i} x(j) < h, \\ x(k) & \text{if } k \neq i, \end{cases}$$

and define rates $r_h$ by

$$\begin{aligned} r_{-N-1} &:= \mathbb{P}[-W < JN], \\ r_h &:= \mathbb{P}[J(h-1) < -W < J(h+1)] \qquad (-N-1 < h < N+1), \\ r_{N+1} &:= \mathbb{P}[JN < -W], \end{aligned}$$

where $W$ is logistically distributed with parameter $\beta$.

# Ergodicity of the Ising Model




If $0 < W < 2J$, then $U^+ > U^-$ provided that $\sum_{j\in\mathcal{N}_i} x(i) > -1$, i.e., this quantity needs to be $0, 2$, or $4$ in order for $U^+$ to be larger that $U^-$.

If $4J < W$, then the person switches to $+1$ whatever the state of its neighbors, i.e., if $\sum_{j\in\mathcal{N}_i} x(i) > -5$.

Then we have the random mapping representation

$$Gf(x) = \sum_{i\in\Lambda}\sum_{h\in H} r_h\big\{f(m_{i,h}x) - f(x)\big\}.$$

We observe that

$$\mathcal{R}_i(m_{j,h}) = \emptyset \quad \text{if } i \neq j \text{ or } h = \pm(N+1),$$

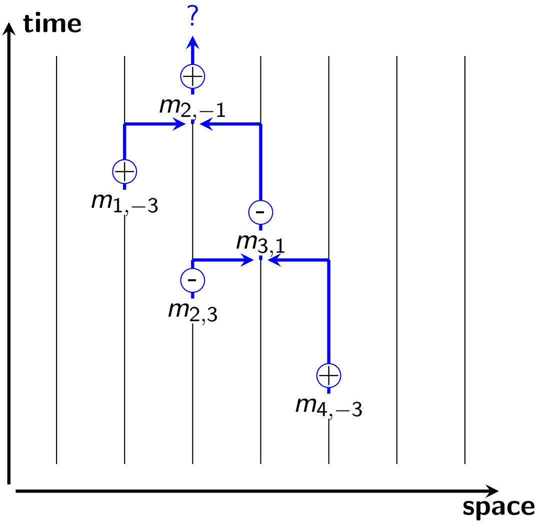
while

$$\mathcal{R}_i(m_{i,h}) = \mathcal{N}_i \quad \text{for } -N-1 < h < N+1.$$

It follows that

$$\begin{aligned} K &= \sup_i \sum_{h\in H} r_h\big(1_{\{-N-1<h<N+1\}}N - 1\big) \\ &= \mathbb{P}[-JN < W < JN]N - 1 = (N-1) - N\mathbb{P}\big[|W| > JN\big]. \end{aligned}$$

## Ergodicity of the Ising Model




If $\mathbb{P}\big[|W| > JN\big]$ is large enough, then $K < 0$, and we need to look back only finitely long in time to decide the state of any lattice point.

# Uniqueness of the Invariant Law

**Theorem 7** *Assume that $K < 0$. Then the interacting particle system $X$ with generator $G$ has a unique invariant law $\nu$. Moreover, the process started in any initial law satisfies*

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu.$$

**Proof** For $A \subset \Lambda$ and $s \leq t$, let

$$\zeta_s^{A,t} := \big\{ i \in \Lambda : (i,s) \rightsquigarrow A \times \{t\} \big\}$$

be the set of lattice points at time $s$ whose value is relevant for the states in $A$ at time $t$. In Corollary 6, we have seen that

$$\mathbb{E}\big[ |\zeta_s^{A,t}| \big] \leq |A| e^{K(t-s)}.$$

In particular, if $K < 0$, then a.s. there is some (random) $s_0 < t$ such that $\zeta_s^{A,t} = \emptyset$ for all $s \leq s_0$. This means that we need to look only finitely far back in time to decide what the state of $(i,t)$ is.

# Uniqueness of the Invariant Law

Recall that $\omega_{s,(i,t)}$ denotes the set of all Poisson events during $(s,t]$ that are relevant for the state of our process at $(i,t)$.
If $K < 0$, then $\omega_{-\infty,(i,u)}$ is finite a.s. Ordering its elements as

$$\omega_{-\infty,(t,i)} := \big\{(m_1,t_1),\ldots,(m_n,t_n)\big\}$$

with $t_1 < \cdots < t_n$, we can unambiguously define a stationary process $\overline{X} = (\overline{X}_t)_{t\in\mathbb{R}}$ by

$$\overline{X}_t(i) := \mathbf{X}_{-\infty,t}(x)(i) := m_n \circ \cdots \circ m_1(x)(i),$$

where the definition does not depend on the choice of $x \in S$.

Since $\overline{X}$ is stationary, $\nu := \mathbb{P}[\overline{X}_t \in \cdot\,]$ is an invariant law.

# Uniqueness of the Invariant Law

Let $Y$ be a random variable with values in $S$, independent of $\omega$, and let $(\overline{X}_t)_{t\in\mathbb{R}}$ be the stationary process constructed above. Then, for each finite $A \subset \Lambda$,

$$\begin{aligned}&\mathbb{P}\big[\overline{X}_0(i) \neq \mathbf{X}_{-t,0}(Y)(i) \text{ for some } i \in A\big]\\ &\quad \leq \mathbb{P}\big[\zeta^{A,0}_{-t} \neq \emptyset\big] \leq \mathbb{E}\big[|\zeta^{A,0}_{-t}|\big] \leq |A|e^{-Kt} \underset{t\to\infty}{\longrightarrow} 0.\end{aligned}$$

It follows that

$$\lim_{t\to\infty} \mathbf{X}_{-t,0}(Y)(i) = \overline{X}_0(i) \quad \text{a.s.} \quad (i \in \Lambda).$$

Let $X$ be the process started in the initial law $\mu = \mathbb{P}[Y \in \cdot\,]$. Then

$$\mathbb{P}\big[X_t \in \cdot\,\big] = \mathbb{P}\big[\mathbf{X}_{-t,0}(Y) \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu.$$

In particular, this implies also the uniqueness of $\nu$. ∎

# Ergodicity of the Ising Model

For our stochastic Ising model, Theorem 7 yields ergodicity provided that

$$K = (N-1) - N\mathbb{P}\big[|W| > JN\big] < 0,$$

which yields

$$\frac{2}{1+e^{J\beta N}} > \frac{N-1}{N} \quad \Leftrightarrow \quad J\beta < -N^{-1}\log\big((2N)/(N-1)-1\big).$$

where $N = |\mathcal{N}_i|$. Concretely, for the Ising model on $\mathbb{Z}^2$ this proves ergodicity provided

$$J\beta < 0.12771$$

which is quite far from the known critical point 0.88137. Our bound gets worse when $N$ becomes large.

# Ergodicity of the Ising Model

A sharper bound can be found in Theorem IV.3.1 of Liggett's (1985) book.

This theorem is based on a similar approach: it is shown that

$$\sum_{i\in\Lambda}\delta P_t f(i) \le e^{K' t}\sum_{i\in\Lambda}\delta f(i) \qquad \big(t\ge 0,\ f\in\mathcal{C}_{\rm sum}(S)\big),$$

where the constant $K'$ is negative for a suitable choice of the parameters.

If this happens, then $\sum_{i\in\Lambda}\delta P_t f(i)\to 0$ as $t\to\infty$ which implies that $P_t f$ converges to a constant $c_f$. Since this happens for *every* $f$, one can deduce that there is a unique invariant law $\nu$ and

$$P_t f \underset{t\to\infty}{\longrightarrow} \int f\,\mathrm{d}\nu = c_f \qquad \big(f\in\mathcal{C}_{\rm sum}(S)\big).$$

The usefulness of Theorem 7 depends on finding a good random mapping representation.

**Exercise 1** The generator of the contact process has the random mapping representation

$$
\begin{aligned}
Gf(x) = \lambda \sum_{i\in\Lambda} \sum_{j\in\mathcal{N}_i} \big\{ f\big(\mathtt{bra}_{i,j}x\big) - f\big(x\big)\big\} \\
+ \sum_{i\in\Lambda} \big\{ f\big(\mathtt{death}_i x\big) - f\big(x\big)\big\}.
\end{aligned}
$$

Letting $0 \in S$ denote the state that is identically zero, we observe that the contact process started in $X_0 = 0$ satisfies $X_t = 0$ for all $t \geq 0$. It follows that $\delta_0$ is an invariant law for the contact process. Assuming that $|\mathcal{N}_i| = N$ does not depend on $i \in \Lambda$, apply Theorem 7 to give suffient conditions in terms of $\lambda$ and $N$ for $\delta_0$ to be the only invariant law and for the contact process to be ergodic.

# Exercises

**Exercise 2** The *threshold voter model* is a particle system with state space $S = \{0,1\}^\Lambda$, where in the state $x$, the site $i$ flips with the following rates:

$$\begin{array}{lll} 0 \mapsto 1 & \text{with rate} & 1_{\{x(j) = 1 \quad \text{for some } j \in \mathcal{N}_i\}}, \\ 1 \mapsto 0 & \text{with rate} & 1_{\{x(j) = 0 \quad \text{for some } j \in \mathcal{N}_i\}}. \end{array}$$

Assume that $i \notin \mathcal{N}_i$ and set $\overline{\mathcal{N}}_i := \mathcal{N}_i \cup \{i\}$. For each $\Delta \subset \Lambda$, define a local map $m_{i,\Delta}$ by

$$m_{i,\Delta}(x)(k) := \begin{cases} \sum_{j \in \Delta} x(j) \mod(2) & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Show that the threshold voter model has the random mapping representation

$$Gf(x) = 2^{1-|\mathcal{N}_i|} \sum_{i \in \Lambda} \sum_{\substack{\Delta \subset \overline{\mathcal{N}}_i \\ |\Delta| \text{ is even}}} \big\{ f\big(m_{i,\Delta} x\big) - f\big(x\big) \big\}.$$

**Exercise 3** A commonly studied stochastic Ising model, somewhat different from the one we considered before, is the process with state space $\{-1,+1\}^\Lambda$ in which the site $i$ flips with the following rates:

$$\begin{aligned} -1 \mapsto +1 \quad &\text{with rate} \quad e^{-\beta J M_i(-)}, \\ +1 \mapsto -1 \quad &\text{with rate} \quad e^{-\beta J M_i(+)}, \end{aligned}$$

where $M_i(\pm) := \sum_{j\in\mathcal{N}_i} 1_{\{x(j)=\pm\}}$. For each $i \in \Lambda$ and finite $\Delta \subset \Lambda$, define a local map $m_{i,\Delta}$ by

$$m_{i,\Delta}(x)(k) := \begin{cases} 1-x(i) & \text{if } k=i \quad \text{and} \quad x(i)\neq x(j) \ \ \forall j\in\Delta, \\ x(k) & \text{otherwise.} \end{cases}$$

Assume that $i \notin \mathcal{N}_i$ and set $p := 1-e^{-\beta J}$. Show that our model has the random mapping representation

$$Gf(x) = \sum_{i\in\Lambda}\sum_{\Delta\subset\mathcal{N}_i} p^{|\Delta|}(1-p)^{|\mathcal{N}_i\backslash\Delta|}\big\{f\big(m_{i,\Delta}x\big)-f\big(x\big)\big\}.$$

# Exercises

**Exercise 4** Let $m_i^{\pm}$ be the local maps defined as

$$m_i^{\pm}(x)(k) := \begin{cases} \pm 1 & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Then an alternative random mapping representation for the stochastic Ising model from Exercise 3 is

$$\begin{aligned} Gf(x) = & \sum_{i\in\Lambda} \sum_{\sigma\in\{-,+\}} (1-p)^{|\mathcal{N}_i|} \{ f(m_i^{\sigma} x) - f(x) \} \\ & + \sum_{i\in\Lambda} \sum_{\substack{\Delta\subset\mathcal{N}_i \\ \Delta\neq\emptyset}} p^{|\Delta|}(1-p)^{|\mathcal{N}_i\setminus\Delta|} \{ f(m_{i,\Delta} x) - f(x) \}. \end{aligned}$$

Assuming that $|\mathcal{N}_i| = N$ does not depend on $i \in \Lambda$, apply Theorem 7 to give suffient conditions in terms of $\beta$ and $N$ for this stochastic Ising model to be ergodic.