Interacting Particle Systems with Applications in Finance

Jan M. Swart

Lecture 4: Monotonicity

Monotone Maps

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- The coalescing random walk map $\mathtt{rw}_{i,j}$ and the cooperative branching map $\mathtt{coop}_{i,i',j}$.

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- Branching annihilating random walks.
- Systems with branching and killing.

Stochastic Order

By definition, $f : S \to \mathbb{R}$ is monotone if $x \leq y$ implies $f(x) \leq f(y)$. We adopt the notation $\mu f := \int f \, \mathrm{d}\mu$.

Stochastic Order *Let μ, ν be probability measures on S. Then the following statements are equivalent:*

(i) *$\mu f \leq \nu f$ for every continuous monotone $f : S \to \mathbb{R}$.*

(ii) *$\mu f \leq \nu f$ for every bounded measurable monotone $f : S \to \mathbb{R}$.*

(iii) *Random variables X, Y with laws μ, ν can be coupled such that $X \leq Y$ a.s.*

If μ, ν satisfy the equivalent conditions (i) and (ii), then we say that they are *stochastically ordered* and write $\mu \leq \nu$.

Proof (iii)⇒(ii) is trivial, since

$$\nu f - \mu f = \mathbb{E}[f(Y) - f(X)] \geq 0,$$

and (ii)⇒(i) is even more trivial. For the much deeper and somewhat surprising converse, see Liggett (1985). See also Preston (1974) for a nice proof for finite spaces. ∎

Stochastic Order

Let $X = (X_t)_{t\geq 0}$ be an interacting particle system with transition probabilities $(P_t)_{t\geq 0}$ and generator of the form

$$Gf(x) = \sum_{m\in\mathcal{M}} r_m\big\{f\big(m(x)\big) - f\big(x\big)\big\},$$

where all maps in $\mathcal{M}$ are monotone. Construct X using a Poisson point process ω and random maps $(\mathbf{X}_{s,t})_{s\leq t}$. Then $\mathbf{X}_{s,t}$, being a composition of monotone maps, is a.s. monotone for each $s \leq t$. It follows that for every monotone measurable $f : S \to \mathbb{R}$ and $x \leq y$,

$$P_t f(x) = \mathbb{E}\big[f\big(\mathbf{X}_{0,t}(x)\big)\big] \leq \mathbb{E}\big[f\big(\mathbf{X}_{0,t}(y)\big)\big] = P_t f(y),$$

so P_t maps monotone functions into monotone functions.

Stochastic Order

As a consequence, we see that

$$\nu f \leq \mu f \quad \forall f : S \to \mathbb{R} \text{ monotone}$$

implies that

$$\nu P_t f \leq \mu P_t f \quad \forall f : S \to \mathbb{R} \text{ monotone} \quad (t \geq 0),$$

i.e., the time evolution preserves the stochastic order.

A more direct way to see this is as follows: couple random variables X_0, X'_0 with laws μ, ν in such a way that $X_0 \leq X'_0$, and let ω be independent of (X_0, X'_0). Then

$$X_t := \mathbf{X}_{0,t}(X_0) \leq \mathbf{X}_{0,t}(X'_0) =: X'_t \quad (t \geq 0).$$

Warning The fact that P_t maps monotone functions into monotone functions does not imply that G can be represented in monotone maps, see Fill & Machida (2001).

The lower and upper invariant laws

Below, we let $0, 1 \in S$ denote the configurations that are constantly zero or one, respectively.

> **Theorem 1** *Assume that G has a representation in monotone maps. Then there exist invariant laws $\underline{\nu}$ and $\overline{\nu}$ such that*
>
> $$\delta_0 P_t \underset{t\to\infty}{\Longrightarrow} \underline{\nu} \quad \textit{and} \quad \delta_1 P_t \underset{t\to\infty}{\Longrightarrow} \overline{\nu}.$$
>
> *If ν is any other invariant law, then $\underline{\nu} \leq \nu \leq \overline{\nu}$.*

We call $\underline{\nu}$ and $\overline{\nu}$ the *lower* and *upper invariant law*, respectively.

Proof By symmetry, it suffices to prove the statement for $\underline{\nu}$. We observe that

$$\delta_0 P_t = \mathbb{P}[\mathbf{X}_{0,t}(0) \in \cdot\,] = \mathbb{P}[\mathbf{X}_{-t,0}(0) \in \cdot\,] \quad (t \geq 0).$$

Now

$$\mathbf{X}_{-u,0}(0) = \mathbf{X}_{-t,0} \circ \mathbf{X}_{-u,-t}(0) \geq \mathbf{X}_{-t,0}(0) \quad \text{a.s.},$$

where we have used the monotonicity of $\mathbf{X}_{-t,0}$ and the fact that since 0 is the lowest element of S, we have $\mathbf{X}_{-u,-t}(0) \geq 0$. It follows that $t \mapsto X_{-t,0}(0)(i)$ is a.s. increasing, so the limit

$$\mathbf{X}_{-\infty,0}(0) := \lim_{t\to\infty} \mathbf{X}_{-t,0}(0)$$

exists a.s.

The lower and upper invariant laws

Set

$$\underline{X}_t := \mathbf{X}_{-\infty,t}(0) = \lim_{T \to -\infty} \mathbf{X}_{T,t}(0) \qquad (t \in \mathbb{R}),$$

and note that $\underline{X}_t$ is independent of $\mathbf{X}_{t,u}$ for each $t \leq u$, since these random variables are defined using disjoint parts of the Poisson point set ω.

Using this, we see that $(\underline{X}_t)_{t\in\mathbb{R}}$ is a stationary Markov process with transition probabilities $(P_t)_{t\geq 0}$, so

$$\underline{\nu} := \mathbb{P}[\underline{X}_t \in \cdot\,],$$

which does not depend on $t \in \mathbb{R}$, is an invariant law.

Since

$$\mathbb{P}^0[X_t \in \cdot\,] = \mathbb{P}[\mathbf{X}_{-t,0}(0) \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \underline{\nu},$$

this is the limit law of the process started in 0.

The lower and upper invariant laws

If ν is any other invariant law, then

$$\delta_0 f \leq \nu f \quad \forall \text{ continuous monotone } f : S \to \mathbb{R}.$$

Since P_t is Feller and monotone, it maps continuous monotone functions into continuous monotone functions, so

$$\delta_0 P_t f \leq \nu P_t f = \nu f \qquad (t \geq 0).$$

Taking the limit $t \to \infty$ and using the definition of weak convergence, we see that $\underline{\nu} f \leq \nu f$ for all continuous monotone $f : S \to \mathbb{R}$. ∎

The lower and upper invariant laws

Lemma 2 *Let* X, Y *be* S*-valued random variables whose laws are stochastically ordered as* $\mathbb{P}[X \in \cdot\,] \leq \mathbb{P}[Y \in \cdot\,]$*. Then* $\mathbb{E}[X(i)] \leq \mathbb{E}[Y(i)]$ *for all* $i \in \Lambda$*. If moreover* $\mathbb{E}[X(i)] = \mathbb{E}[Y(i)]$ *for all* $i \in \Lambda$*, then* X *and* Y *are equal in law.*

Proof By assumption, we can couple such that $X \leq Y$. For this coupling $Y(i) - X(i)$ is a nonnegative random variable, so

$$\mathbb{E}[Y(i)] - \mathbb{E}[X(i)] = \mathbb{E}[Y(i) - X(i)] \geq 0.$$

If this is zero for all i, then our coupling satisfies $X = Y$ a.s., proving that X and Y must be equal in law. ∎

The lower and upper invariant laws

Lemma 3 *Let X be a monotone interacting particle system with lower and upper invariant laws $\underline{\nu}$ and $\overline{\nu}$. If $\underline{\nu} = \overline{\nu}$, then X has a unique invariant law $\nu := \underline{\nu} = \overline{\nu}$ and is ergodic in the sense that starting from any initial law,*

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu.$$

Proof Let μ be any probability measure on S. Then

$$\delta_0 f \leq \mu f \quad \forall \text{ continuous monotone } f : S \to \mathbb{R}.$$

Since P_t is Feller and monotone, it maps continuous monotone functions into continuous monotone functions, so

$$\delta_0 P_t f \leq \mu P_t f \quad \forall \text{ continuous monotone } f : S \to \mathbb{R}, \; t \geq 0.$$

By the compactness of $\mathcal{M}_1(S)$, every sequence $t_n \to \infty$ contains a subsequence $t_{n(m)}$ such that $\mu P_{t_{n(m)}}$ converges weakly to some limit μ_*. It follows that

$$\underline{\nu} \leq \mu_*,$$

and by the same argument also $\mu_* \leq \overline{\nu}$, which by the fact that $\nu = \underline{\nu} = \overline{\nu}$ implies that $\mu_* = \nu$. Since this holds for every subsequence, we conclude that μP_t converges weakly to ν. ∎

The lower and upper invariant laws

Alternative proof We have seen before that the limits

$$\mathbf{X}_{-\infty,0}(0) := \lim_{t\to\infty} \mathbf{X}_{-t,0}(0) \quad \text{and} \quad \mathbf{X}_{-\infty,0}(1) := \lim_{t\to\infty} \mathbf{X}_{-t,0}(1)$$

exist a.s. and have laws $\underline{\nu}$ and $\overline{\nu}$, respectively. By assumption, $\underline{\nu} = \overline{\nu}$, so $\mathbb{E}[\mathbf{X}_{-\infty,0}(0)(i)] = \mathbb{E}[\mathbf{X}_{-\infty,0}(1)(i)]$ for all i. Since moreover $\mathbf{X}_{-\infty,0}(0) \leq \mathbf{X}_{-\infty,0}(1)$, it follows that

$$\mathbf{X}_{-\infty,0}(0) = \mathbf{X}_{-\infty,0}(1) \quad \text{a.s.}$$

If X_0 has law μ, then $\mathbf{X}_{-t,0}(X_0)$ has law μP_t. Since

$$\mathbf{X}_{-t,0}(0) \leq \mathbf{X}_{-\infty,0}(X_0) \leq \mathbf{X}_{-t,0}(1)$$

and the left- and right-hand side converge to a.s. the same limit, the expression in the middle must converge to this too. ∎

Lemma 4 *Let X be a monotone interacting particle system with lower and upper invariant laws $\underline{\nu}$ and $\overline{\nu}$. Then X is ergodic if*

$$\int \underline{\nu}(\mathrm{d}x)x(i) = \int \overline{\nu}(\mathrm{d}x)x(i) \qquad (i \in \Lambda),$$

and has at least two invariant laws if

$$\int \underline{\nu}(\mathrm{d}x)x(i) < \int \overline{\nu}(\mathrm{d}x)x(i) \quad \textit{for some } i \in \Lambda.$$

Proof Immediate from the previous two lemmas. ∎

The voter model

For the voter model, we see that $X_0(i) = 0$ for all $i \in \Lambda$ implies $X_t(i) = 0$ for all $i \in \Lambda$, so

$$\delta_0 P_t = \delta_0 \quad \text{and likewise} \quad \delta_1 P_t = \delta_1 \quad (t \geq 0).$$

It follows that $\underline{\nu} = \delta_0$ and $\overline{\nu} = \delta_1$, so the voter model is never ergodic.

The magnetization of the Ising model

Consider the Ising model on $\mathbb{Z}^d$ with $\mathcal{N}_i = \{j \in \mathbb{Z}^d : |i-j| = 1\}$ and $J = 1$. Let $\underline{\nu}_\beta$ and $\overline{\nu}_\beta$ denote the lower and upper invariant laws, which depend on β. By symmetry, there exists a function $m_* : \mathbb{R} \to [0, \infty)$ such that

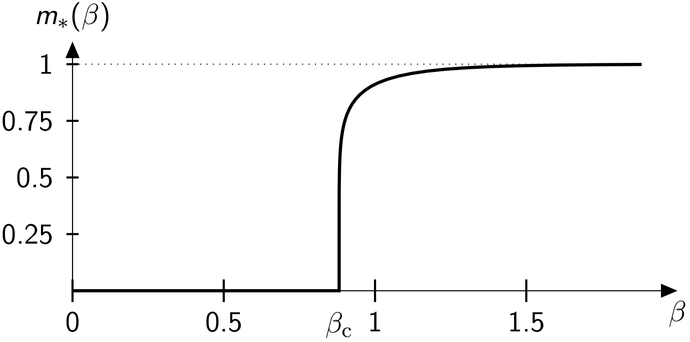
$$\int \underline{\nu}_\beta(\mathrm{d}x)x(i) = -m_*(\beta) \quad \text{and} \quad \int \overline{\nu}_\beta(\mathrm{d}x)x(i) = m_*(\beta).$$

Using terminology from physics, m_* is called the *spontaneous magnetization*.

We have already shown that for every dimension d there exists some $\beta' > 0$ such that the Ising model with $\beta < \beta'$ has a unique invariant law. It follows that $m_*(\beta) = 0$ for $\beta < \beta'$.

For the one-dimensional model, it is known that in fact $m_*(\beta) = 0$ for all $\beta \geq 0$.

The magnetization of the Ising model



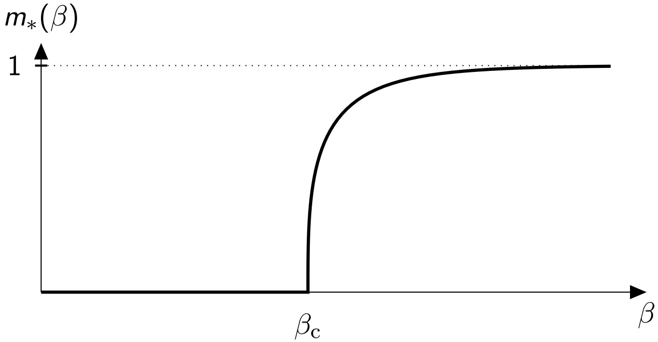
By contrast, Onsager (1944) proved that for the model on $\mathbb{Z}^2$,

$$m_*(\beta) = \begin{cases} \big(1 - \sinh(\beta)^{-4}\big)^{1/8} & \text{for } \beta \geq \beta_c := \log(1+\sqrt{2}), \\ 0 & \text{for } \beta \leq \beta_c. \end{cases}$$

The magnetization of the Ising model

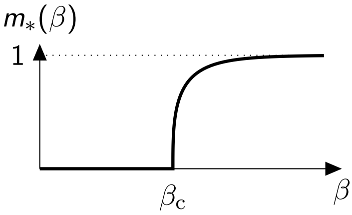
At β_{c}, the 2-dimensional Ising model has a second order *phase transition*.

The magnetization of the Ising model



For the model on $\mathbb{Z}^3$, it is known that m_* is continuous, nondecreasing in β, and there exists a $0 < \beta_c < \infty$ such that $m_*(\beta) = 0$ for $\beta \leq \beta_c$ while $m_*(\beta) > 0$ for $\beta > \beta_c$. Continuity at β_c proved by Aizenman, Duminil-Copin & Sidoravicius (2014).

The magnetization of the Ising model



Nonrigorous renormalization group theory explains that

$$m_*(\beta) \propto (\beta - \beta_c)^c \quad \text{as } \beta \downarrow \beta_c,$$

where the *critical exponent* c is given by

$c = 1/8$ in dim 2, $\quad c \approx 0.326$ in dim 3, $\quad$ and $\quad c = 1/2$ in dim ≥ 4.

Note that $c = 1/2$ is what we also found for the mean-field model.

The magnetization of the Ising model

Critical exponents are believed to be *universal*.

For example, in $d = 3$, whether we take $\mathcal{N}_i = \{j : |i - j| = 1\}$ or $\mathcal{N}_i = \{j : 0 < |i - j| \leq 2\}$ is believed to matter for the value of the *critical point* $\beta_{\rm c}$, but not for the *critical exponent* c.

The exponent $c \approx 0.33$ of the 3D Ising model can even be experimentally observed for certain physical systems.

For other (classes of) interacting particle systems, the situation is similar.
Little is known rigorously about critical exponents, except for certain 2-dimensional models, and for models in sufficiently high dimension where the *lace expansion* has been used to prove that critical exponents take on their mean-field values.

The Contact Process

Recall that the *contact process* jumps as

$$\begin{aligned} 0 &\mapsto 1 \qquad \text{with rate } \lambda \cdot \text{number of type 1 neighbors}, \\ 1 &\mapsto 0 \qquad \text{with rate } d, \end{aligned}$$

where $\lambda \geq 0$ is the *infection rate* and $d \geq 0$ the *death rate*. In what follows, for simplicity, we set $d = 1$.

The Contact Process

Using the *branching map*

$$\mathtt{bra}_{i,j}x(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

and *death* map

$$\mathtt{death}_i x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

we have the random mapping representation

$$Gf(x) = \lambda \sum_{i \in \Lambda} \sum_{j \in \mathcal{N}_i} \big\{ f\big(\mathtt{bra}_{i,j}x\big) - f\big(x\big) \big\} + \sum_{i \in \Lambda} \big\{ f\big(\mathtt{death}_i x\big) - f\big(x\big) \big\}.$$

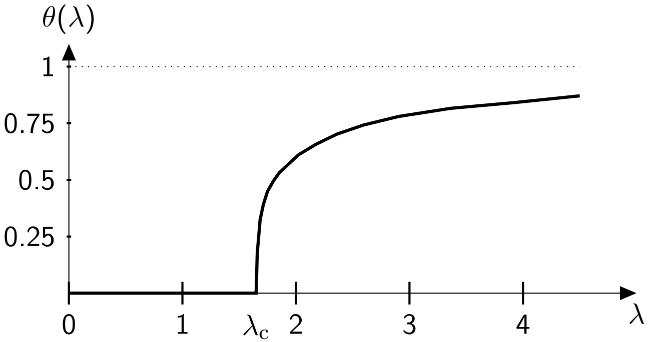
The process started in $X_0 = 0$ satisfies $X_t = 0$ for all $t \geq 0$, so clearly $\underline{\nu} = \delta_0$.

It follows that the process is ergodic if and only if $\overline{\nu} = \delta_0$. Let

$$\theta(\lambda) := \int \overline{\nu}(\mathrm{d}x)x(i),$$

which does not depend on i for the nearest-neighbor process on $\mathbb{Z}^d$.

Then the process is ergodic (with unique invariant law δ_0) if and only if $\theta(\lambda) = 0$.



For the one-dimensional nearest-neighbor model, one observes a second-order phase transition at $\lambda_c(1) \approx 1.649$ and

$$\theta(\lambda) \propto (\lambda - \lambda_c)^\beta \qquad \text{as } \lambda \downarrow \lambda_c,$$

with a critical exponent $\beta \approx 0.27648$.

Theorem 7 of Lecture 3, applied to the one-dimensional contact process, shows ergodicity and hence $\theta(\lambda) = 0$ for $\lambda < 0.5$.

The next proposition shows that there is a unique point $0 < \lambda_{\rm c} \leq \infty$ such that $\theta(\lambda) = 0$ for $\lambda < \lambda_{\rm c}$ and $\theta(\lambda) > 0$ for $\lambda > \lambda_{\rm c}$. We will later prove that $\lambda_{\rm c} < \infty$.

Proposition 5 *The function $\lambda \mapsto \theta(\lambda)$ is nondecreasing.*

Proof Let $0 \leq \lambda \leq \lambda'$ and let X, X' be contact processes with initial states $x \leq x'$ and branching rates λ, λ'. We will prove that X and X' can be coupled such that $X_t \leq X'_t$ for all $t \geq 0$. In particular, applying this to $x = x' = 1$ and letting $t \to \infty$, this then implies Proposition 5.

The Contact Process

We use the graphical representation. The set of local maps is

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 := \big\{\mathtt{bra}_{i,j} : i \in \Lambda,\ j \in \mathcal{N}_i\big\} \cup \big\{\mathtt{death}_i : i \in \Lambda\big\}.$$

Let ω be a Poisson point subset of $\mathcal{M} \times \mathbb{R}$ with local intensity $\mu(m, \mathrm{d}t)$ given by

$$\lambda 1_{\{m\in\mathcal{M}_1\}}\ell(\mathrm{d}t) + 1_{\{m\in\mathcal{M}_2\}}\ell(\mathrm{d}t),$$

and let ω'' be a Poisson point set with intensity

$$(\lambda' - \lambda)1_{\{m\in\mathcal{M}_1\}}\ell(\mathrm{d}t).$$

Then $\omega' := \omega + \omega''$ is a Poisson point set with intensity

$$\lambda' 1_{\{m\in\mathcal{M}_1\}}\ell(\mathrm{d}t) + 1_{\{m\in\mathcal{M}_2\}}\ell(\mathrm{d}t).$$

This means that we can couple the graphical representations of X and X' in such a way that they are equal, except that X' has more branching events.

More precisely, using the Poisson sets ω and ω' to construct random maps $(\mathbf{X}_{s,t})_{s\leq t}$ and $(\mathbf{X}'_{s,t})_{s\leq t}$, we observe that

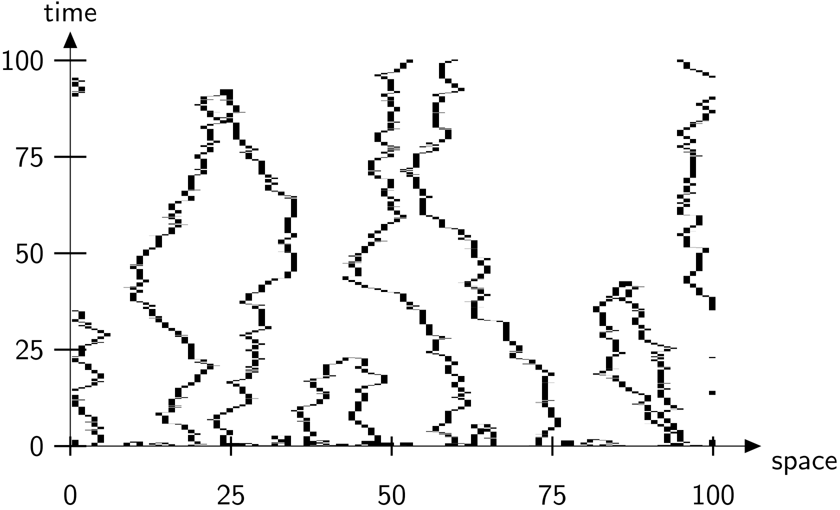
$$\mathbf{X}_{s,t}(x) \leq \mathbf{X}'_{s,t}(x') \quad \forall x \leq x',\ s \leq t.$$

Here we use that

$$\mathtt{bra}_{i,j}(x) \geq x \qquad (x \in S,\ i,j \in \Lambda),$$

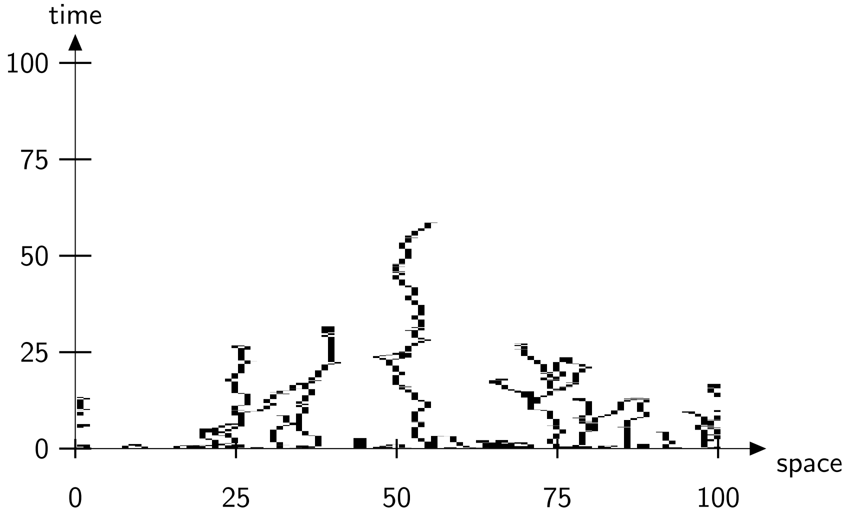
and that the maps $\mathtt{bra}_{\mathtt{i},\mathtt{j}}, \mathtt{death}_i$ are all monotone. ∎

A System that is Not Monotone



Annihilating random walks.

A System that is Not Monotone



Annihilating random walks with a bit of branching.

A System that is Not Monotone

In spite of this...

Numerical simulations of non-monotone particle systems often show phase transitions very similar to those of monotone systems.

Some even appear to have the same critical exponent as, e.g., the contact process.

In most cases, all one can prove is that the system is ergodic in some part of the parameter space and has at least two invariant measures in another part, without much information about how the transition occurs.

Exercise 1 Give an example of two probability measures μ, ν on a set of the form $\{0,1\}^\Lambda$ that satisfy

$$\int \mu(\mathrm{d}x)x(i) \leq \int \nu(\mathrm{d}x)x(i) \qquad (i \in \Lambda),$$

but that are *not* stochastically ordered as $\mu \leq \nu$.

Exercise 2 In Exercise 1 of Lecture 2, we calculated the fixed points of the mean-field contact process. Show that the largest fixed point $\overline{x}_{\rm upp}(\lambda)$ satisfies

$$\overline{x}_{\rm upp}(\lambda) \propto (\lambda - \lambda_{\rm c})^c$$

for some $c \in \mathbb{R}$, and determine this mean-field critical exponent.

Exercise 3 Prove that the function $\lambda \mapsto \theta(\lambda)$ from Proposition 5 is right-continuous. *Hint:* Use that the decreasing limit of continuous functions is upper semi-continuous.

Excercises

For the next exercise, let us define a *double death* map

$$\mathtt{death}_{i,j}x(k) := \begin{cases} 0 & \text{if } k \in \{i,j\}, \\ x(k) & \text{otherwise.} \end{cases}$$

Consider the cooperative branching process X with values in $\{0,1\}^{\mathbb{Z}}$ with generator

$$\begin{aligned} G_X f(x) = \lambda \sum_{i\in\mathbb{Z}} \sum_{\sigma\in\{-1,+1\}} & \big\{ f\big(\mathtt{coop}_{i,i+\sigma,i+2\sigma}x\big) - f\big(x\big)\big\} \\ & + \sum_{i\in\mathbb{Z}} \big\{ f\big(\mathtt{death}_i x\big) - f\big(x\big)\big\}, \end{aligned}$$

and the contact process with double deaths Y with generator

$$\begin{aligned} G_Y f(y) = \lambda \sum_{i\in\mathbb{Z}} \sum_{\sigma\in\{-1,+1\}} & \big\{ f\big(\mathtt{bra}_{i,i+\sigma}y\big) - f\big(y\big)\big\} \\ & + \sum_{i\in\mathbb{Z}} \big\{ f\big(\mathtt{death}_{i,i+1} y\big) - f\big(y\big)\big\}, \end{aligned}$$

Excercises

Exercise 4 Let X be the process with cooperative branching defined above and set

$$X^{(2)}_t(i) := 1_{\{X_t(i)=1=X_t(i+1)\}} \qquad (i \in \mathbb{Z},\ t \geq 0).$$

Show that X can be coupled to a contact process with double deaths Y (with the same parameter λ) in such a way that

$$Y_0 \leq X^{(2)}_0 \quad \text{implies} \quad Y_t \leq X^{(2)}_t \quad (t \geq 0).$$

Exercise 5 Show that a system $(X_t)_{t\geq 0}$ of annihilating random walks can be coupled to a system $(Y_t)_{t\geq 0}$ of coalescing random walks such that

$$X_0 \leq Y_0 \quad \text{implies} \quad X_t \leq Y_t \quad (t \geq 0).$$

Note that the annihilating random walks are not a monotone particle system.

Exercise 6 Let X be a system of branching and coalescing random walks with generator

$$G_X f(x) = \tfrac{1}{2} b \sum_{i \in \mathbb{Z}} \sum_{\sigma \in \{-1,+1\}} \big\{ f\big(\mathtt{bra}_{i,i+\sigma} x\big) - f\big(x\big) \big\} + \tfrac{1}{2} \sum_{i \in \mathbb{Z}} \big\{ f\big(\mathtt{rw}_{i,i+\sigma} x\big) - f\big(x\big) \big\},$$

and let Y be a system of coalescing random walks with positive drift, with generator

$$G_Y f(y) = \tfrac{1}{2}(1+b) \sum_{i \in \mathbb{Z}} \big\{ f\big(\mathtt{rw}_{i,i+1} y\big) - f\big(y\big) \big\} + \tfrac{1}{2} \sum_{i \in \mathbb{Z}} \big\{ f\big(\mathtt{rw}_{i,i-1} y\big) - f\big(y\big) \big\}.$$

Show that X and Y can be coupled such that

$$Y_0 \leq X_0 \quad \text{implies} \quad Y_t \leq X_t \quad (t \geq 0).$$

Exercise 7 Let $d < d'$ and identify $\mathbb{Z}^d$ with the subset of $\mathbb{Z}^{d'}$ consisting of all $(i_1, \ldots, i_{d'})$ with $(i_{d+1}, \ldots, i_{d'}) = (0, \ldots, 0)$. Let X and X' denote the contact processes on $\mathbb{Z}^d$ and $\mathbb{Z}^{d'}$, respectively, with the same infection rate λ. Show that X and X' can be coupled such that

$$X_0(i) \leq X'_0(i) \qquad (i \in \mathbb{Z}^d)$$

implies

$$X_t(i) \leq X'_t(i) \qquad (t \geq 0,\ i \in \mathbb{Z}^d).$$