Interacting Particle Systems with Applications in Finance

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Lecture 5: Duality

The Contact Process

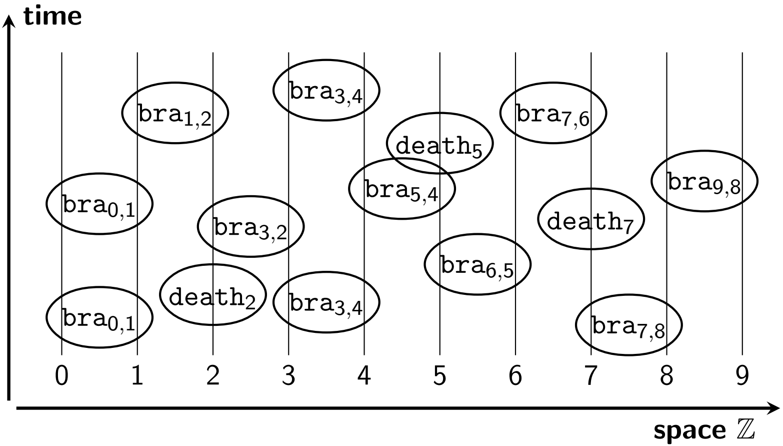
Recall that the *contact process* jumps as

$$
\begin{aligned}
0 \mapsto 1 & \qquad \text{with rate } \lambda \cdot \text{number of type 1 neighbors}, \\
1 \mapsto 0 & \qquad \text{with rate } 1,
\end{aligned}
$$

where $\lambda \geq 0$ is the *infection rate* and we have set the *death rate* equal to one. The generator of the contact process has the random mapping representation

$$Gf(x) = \lambda \sum_{i \in \Lambda} \sum_{j \in \mathcal{N}_i} \big\{ f\big(\mathtt{bra}_{i,j} x\big) - f\big(x\big) \big\} + \sum_{i \in \Lambda} \big\{ f\big(\mathtt{death}_i x\big) - f\big(x\big) \big\}.$$

The Graphical Representation



The contact process can be constructed by applying branching and death maps at Poissonian times.

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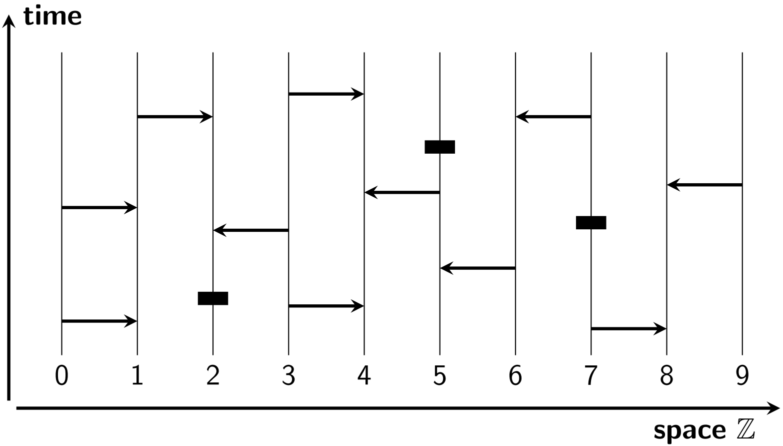
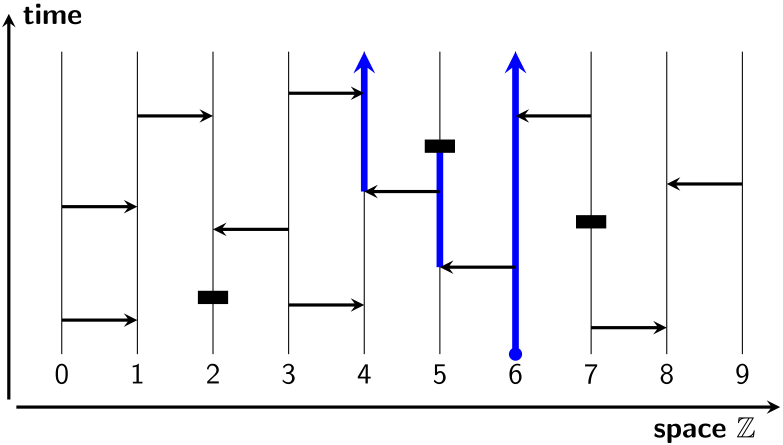


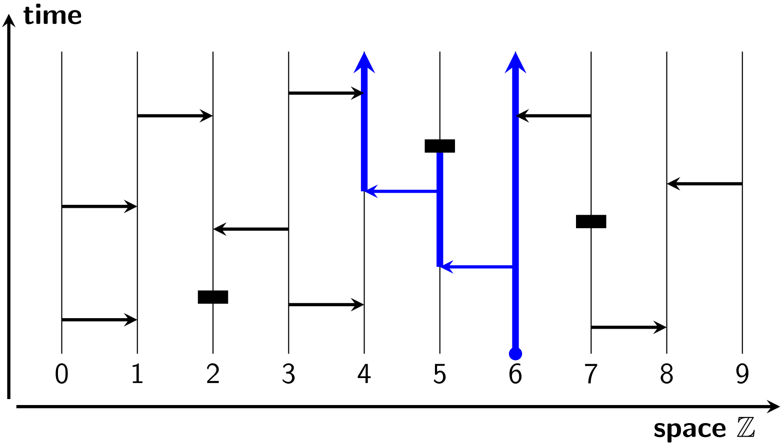
We will denote $\mathtt{bra}_{i,j}$ by an arrow from i to j
and $\mathtt{death}_i$ by a rectangle at i.

The Graphical Representation



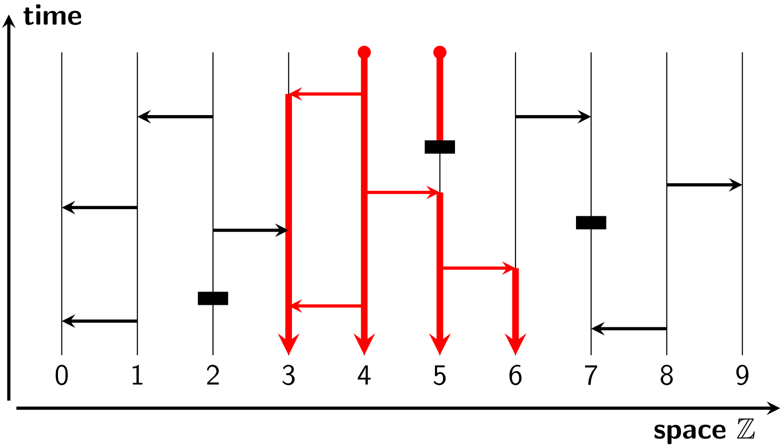
One has $X_t(i) = 1$ if and only if (i, t) can be reached through an *open path* from some point $(j, 0)$ with $X_0(j) = 1$.

The Graphical Representation



Open paths may use arrows but must stop at death symbols.

The Dual Contact Process



If we want to know the state of a space-time point (i, t), we reverse the arrows and look at paths backwards in time.

The Dual Contact Process

The backward paths define a *dual contact process* that runs backwards in time.

Let ω be the Poisson point process of arrows and death symbols, from which we define the random maps $(\mathbf{X}_{s,t})_{s\leq t}$ used to construct the contact process. Then

$$\mathbf{X}_{s,t}x(j) = 1 \qquad \begin{array}{l}\text{iff there is some } i \text{ such that } x(i) = 1 \\ \text{and an } \omega\text{-open path from } (i,s) \text{ to } (j,t).\end{array}$$

We define *dual* maps $(\mathbf{X}_{t,s})_{t\geq s}$

$$\hat{\mathbf{X}}_{t,s}x(i) = 1 \qquad \begin{array}{l}\text{iff there is some } j \text{ such that } x(j) = 1 \\ \text{and a } \textit{backward } \omega\text{-open path from } (j,t) \text{ to } (i,s),\end{array}$$

where we use the *same* Poisson point process ω.

The Dual Contact Process

We observe

$$
\begin{aligned}
&1_{\{\mathbf{X}_{s,t}(x) \wedge y \neq 0\}} \\
&\quad = 1_{\substack{\{\exists i,j \text{ with } x(i)=1 \text{ and } y(j)=1 \\ \text{and an } \omega\text{-open path from } (i,s) \text{ to } (j,t)\}}} \\
&\quad = 1_{\{x \wedge \hat{\mathbf{X}}_{t,s}(y) \neq 0\}}.
\end{aligned}
$$

Taking expectations, we derive the (self-) *duality* relation

$$\mathbb{P}^x\big[X_t \wedge y \neq 0\big] = \mathbb{P}^y\big[x \wedge \hat{X}_t \neq 0\big],$$

where X and $\hat{X}$ are contact processes started in x and y, respectively.

Lemma 1 *The upper invariant law $\overline{\nu}$ is nontrivial, i.e., $\overline{\nu} \neq \delta_0$, if and only if the contact process survives, i.e.,*

$$\mathbb{P}^{e_i}\big[X_t \neq 0 \ \forall t \geq 0\big] > 0.$$

Proof Let e_i denote the configuration with a single particle at i, i.e,, $e_i(j) := 1_{\{i=j\}}$. By duality, the intensity of the upper invariant law satisfies

$$\begin{aligned}\theta(\lambda) &:= \int \overline{\nu}(\mathrm{d}x)x(i) = \lim_{t\to\infty} \mathbb{E}^{1}[X_t(i)] \\ &= \lim_{t\to\infty} \mathbb{P}^{\Lambda}[X_t \wedge e_i \neq 0] = \lim_{t\to\infty} \mathbb{P}^{e_i}[1 \wedge \hat{X}_t \neq 0] \\ &= \lim_{t\to\infty} \mathbb{P}^{e_i}[\hat{X}_t \neq 0] = \mathbb{P}^{e_i}\big[\hat{X}_t \neq 0 \ \forall t \geq 0\big].\end{aligned}$$

∎

Additive Maps

By definition, a map $m : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ is *additive* if

- $m(0) = 0$
- $m(x \vee y) = m(x) \vee m(y)$.

  Lemma 2 *A map is additive if and only if it can be represented in arrows and blocking symbols. Each additive map m has a dual map* $\hat{m}$*, which is also additive.*

Additive maps are always monotone. Examples are:

- The branching map $\mathtt{bra}_{i,j}$

The map $\mathtt{coop}_{i,i',j}$ is monotone but not additive.

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- The voter map $\mathtt{vot}_{i,j}$.

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- The branching map $\texttt{bra}_{i,j}$
- The death map $\texttt{death}_i$
- The voter map $\texttt{vot}_{i,j}$.
- The coalescing random walk map $\texttt{rw}_{i,j}$

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The Voter Model

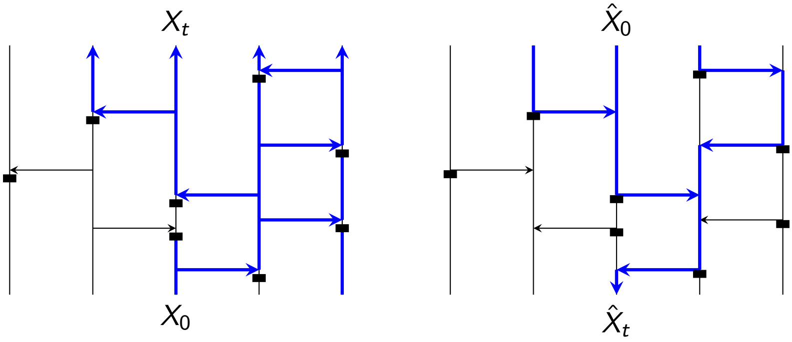
Recall that the *voter map*

$$\mathtt{vot}_{ij}(x)(k) := \begin{cases} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

says that the site i copies the type of j, while the *coalescing random walk map*

$$\mathtt{rw}_{i,j}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

says that a particle at i jumps to j, coalescing with any particle that may already be there.



The map $\mathtt{vot}_{ij}$ is dual to $\mathtt{rw}_{ij}$.
Two sites in X_t have the same type if the dual random walks starting there coalesce before time 0.

We start the voter model on $\mathbb{Z}^d$ by assigning i.i.d., uniformly distributed types to the sites.

Proposition 3 *For each $i \neq j$, the limit*

$$\lim_{t\to\infty} \mathbb{P}\big[X_t(i) \neq X_t(j)\big]$$

exists, and satisfies $= 0$ or > 0 depending on whether the dimension d is ≤ 2 or ≥ 3.

Proof This limit is just the probability that two random walks started in i and j eventually coalesce. Since the distance between two random walkers is itself a random walk and random walks are recurrent in dimensions $d \leq 2$ and transient in $d \geq 3$, the result follows. ∎

Duality

Let $(X_t)_{t\geq 0}$ be a Markov process with state space S, generator G, semigroup $(P_t)_{t\geq 0}$.
Let $(Y_t)_{t\geq 0}$ be a Markov process with state space R, generator H, semigroup $(Q_t)_{t\geq 0}$.
Let $\psi : S \times R \to \mathbb{R}$ be bounded measurable.

Def X and Y are *dual* with *duality function* ψ iff

$$\mathbb{E}^x\big[\psi(X_t, y)\big] = \mathbb{E}^y\big[\psi(x, Y_t)\big] \qquad (t \geq 0,\ x \in S,\ y \in R).$$

Equivalently

$$\int P_t(x, \mathrm{d}x')\psi(x', y) = \int Q_t(y, \mathrm{d}y')\psi(x, y').$$

In "good" situations, it can be proved that this is equivalent to

$$G\psi(\,\cdot\,, y)(x) = H\psi(x,\,\cdot\,)(y) \qquad (x \in S,\ y \in R).$$

Duality

Example 1 The duality function

$$\psi(x, y) = 1_{\{x \wedge y \neq 0\}}$$

leads to *additive systems duality*.
Example 2 The duality function

$$\psi(x, y) = 1_{\{\sum_i x(i)y(i) \text{ is odd}\}}$$

leads to *cancellative systems duality*. Here a map m is *cancellative* if and only if

- $m(0) = 0$
- $m\big(x + y \mod(2)\big) = m(x) \vee m(y) \mod(2)$.

Examples of cancellative maps are $\texttt{vot}_{ij}$ and the annihilating maps $\texttt{arw}_{i,j}$ and $\texttt{abra}_{i,j}$.

Duality

Lloyd and Sudbury (1995, 1997, 2000) have shown that many particle systems have a dual w.r.t. to the duality function

$$\psi(x,y) := q^{\sum_i x(i)y(i)}.$$

Example 1 $q = 0$ gives

$$0^{|x \cap y|} = 1_{\{x \wedge y = \emptyset\}} \qquad \text{additive duality.}$$

Example 2 $q = -1$ gives

$$(-1)^{|x \cap y|} = 1 - 21_{\{\sum_i x(i)y(i) \text{ is odd}\}} \qquad \text{cancellative duality.}$$

For $q \neq 0, 1$, there is in general no pathwise construction of these dualities.

Exercises

Exercise 1 Let

$$\psi_{\rm add}(x,y) = 1_{\{x \wedge y \neq 0\}},$$
$$\psi_{\rm add}(x,y) = 1_{\{\sum_i x(i)y(i) \text{ is odd}\}}$$

denote the duality functions for additive and cancellative systems, respectively. The voter map $\mathtt{vot}_{ij}$ is both additive and cancellative. Show that

$$\psi_{\rm add}\big(\mathtt{vot}_{ij}x, y\big) = \psi_{\rm add}\big(x, \mathtt{rw}_{ij}y\big).$$

Find a map m that is the cancellative dual of $\mathtt{vot}_{ij}$, in the sense that

$$\psi_{\rm can}\big(\mathtt{vot}_{ij}x, y\big) = \psi_{\rm can}\big(x, m(y)\big).$$