Interacting Particle Systems with Applications in Finance

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Lecture 6: The Contact Process

Introduction

Recall that the generator of the contact process has the random mapping representation

$$Gf(x) = \lambda \sum_{i \in \Lambda} \sum_{j \in \mathcal{N}_i} \big\{ f\big(\mathtt{bra}_{i,j} x\big) - f\big(x\big) \big\} + \sum_{i \in \Lambda} \big\{ f\big(\mathtt{death}_i x\big) - f\big(x\big) \big\}.$$

By duality, the intensity of the upper invariant measure

$$\theta(\lambda) := \int \overline{\nu}(\mathrm{d}x) x(i),$$

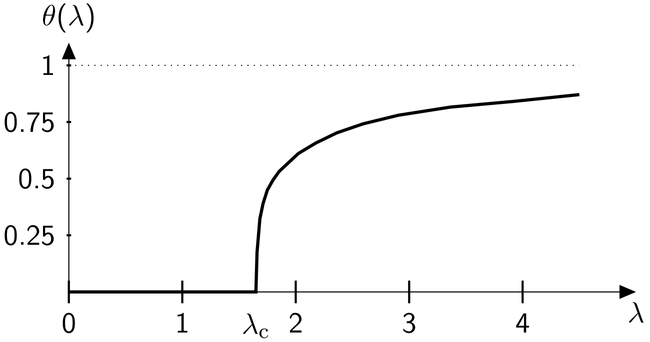
is equal to the survival probability

$$\mathbb{P}^{e_i}\big[X_t \neq 0 \ \forall t \geq 0\big] > 0.$$

We know that the process is ergodic (with unique invariant law δ_0) if and only if $\theta(\lambda) = 0$.

We have proved that θ is nondecreasing in λ and know how to prove that $\theta(\lambda) = 0$ for λ small enough.

Introduction



Our aim is to prove that $\theta(\lambda) > 0$ for λ sufficiently large, thereby establishing the existence of a phase transition.

Our method will be applicable to many other particle systems, including non-monotone ones.

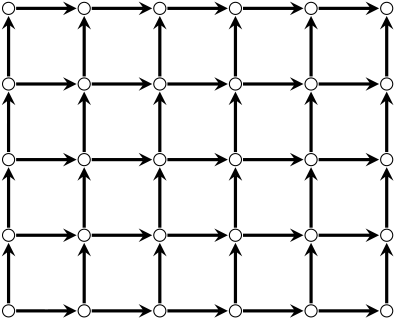
Oriented Percolation

We wish to show that the contact process survives with positive probability if the branching rate λ is large enough.

We will first prove this for a similar discrete-time process, and then use a comparison argument to transfer the result to the contact process.

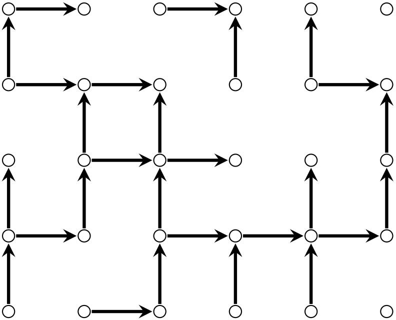
The discrete-time process that we will work with is *oriented percolation*.

The second step of the argument, comparison with oriented percolation, is a very common tool in the study of all kinds of interacting particle systems.



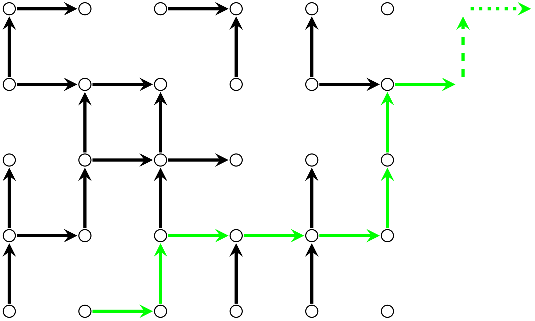
Equip $\mathbb{Z}^2$ with the structure of an oriented graph by drawing at each (i_1, i_2) two arrows, pointing to $(i_1 + 1, i_2)$ and $(i_1, i_2 + 1)$.

Oriented Percolation

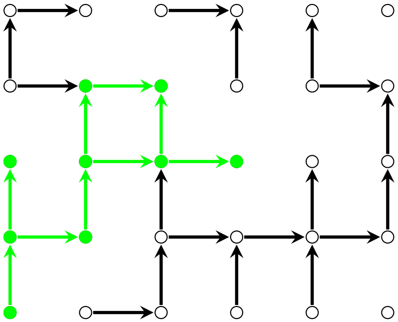


Thin the collection of arrows by independently keeping each arrow with probability p.

Oriented Percolation

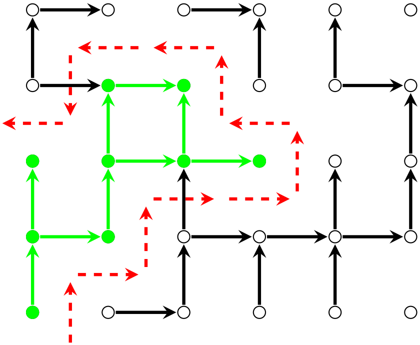


We want to prove that for p large enough *percolation* occurs, i.e., there are infinite oriented paths.



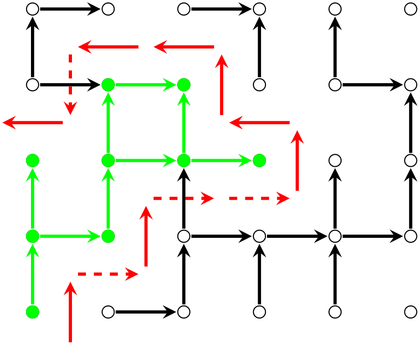
If the set C of points that can be reached by an open path starting at the origin is finite. . .

Oriented Percolation



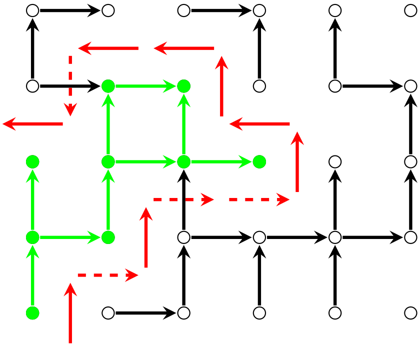
... then there is an oriented path separating this set from the infinite component of $\mathbb{N}^2 \backslash C$.

Oriented Percolation



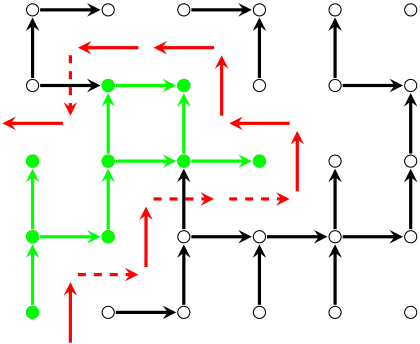
The *up* and *left* steps of this path cannot cross black arrows.

Oriented Percolation

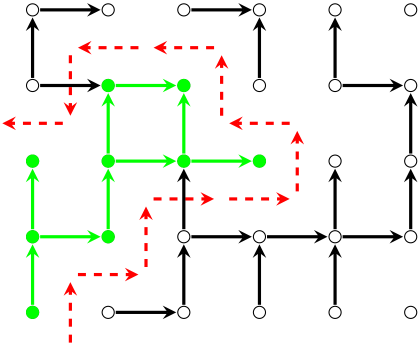


There are more up steps than down steps,
and more left steps than right steps.

Oriented Percolation

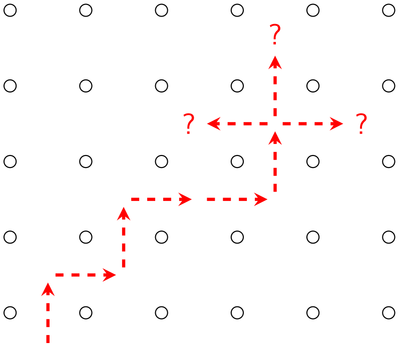


The probability that for a *given* path of L steps, no up or left steps cross a black arrow is $\leq (1-p)^{L/2}$.



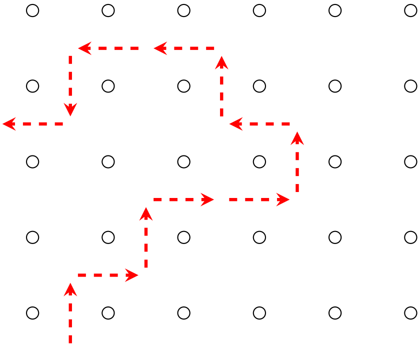
A path of length L must start somewhere between $(0,0)$ and $(L,0)$.

Oriented Percolation



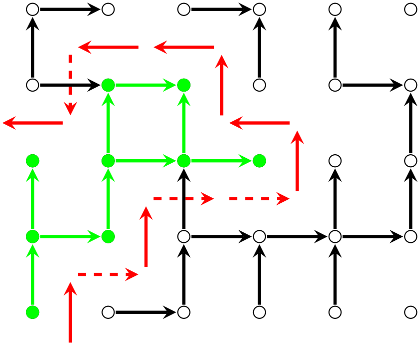
In each point, there are at most three directions in which the path can continue.

Oriented Percolation



It follows that the total number of red paths of length L is $\leq L3^L$.

Oriented Percolation



And the *expected number* of paths with the property that no up or left step crosses a black arrow is $\leq \sum_{L=2}^{\infty} L3^L(1-p)^{L/2}$.

Oriented Percolation

If $p > 8/9$, then

$$\begin{aligned}
&\mathbb{P}\big[\text{there is } \textit{no} \text{ infinite green path starting at } (0,0)\big] \\
&\quad = \mathbb{P}\big[\text{there is a red path blocking } (0,0)\big] \\
&\quad \leq \mathbb{E}\big[\#\text{ red paths blocking } (0,0)\big] \\
&\quad \leq \sum_{L=2}^{\infty} L 3^L (1-p)^{L/2} < \infty.
\end{aligned}$$

By choosing p very close to 1, we can make this sum as small as we wish. In particular, choosing p such that the sum is less than 1, we have proved that:

$$\mathbb{P}\big[\text{there is an infinite green path starting at } (0,0)\big] > 0.$$

This is a *Peierls argument*.

Oriented Percolation

We can actually do a little better, using a trick from the book of Durrett (1988). For any $L_0 \geq 0$,

$$\mathbb{P}\big[\text{there is } \textit{no} \text{ infinite green path starting at } (0,0),\ldots,(L_0,0)\big]$$
$$\leq \sum_{L=L_0}^{\infty} L3^L(1-p)^{L/2}.$$

As long as $p > 8/9$, we can make this sum as small as we wish by choosing L_0 large enough.
This proves that for any $p > 8/9$, there is an $L_0 \geq 0$ such that

$$\mathbb{P}\big[\text{there is an infinite green path starting at } (0,0),\ldots,(L_0,0)\big] > 0.$$

But then, of course, we must also have

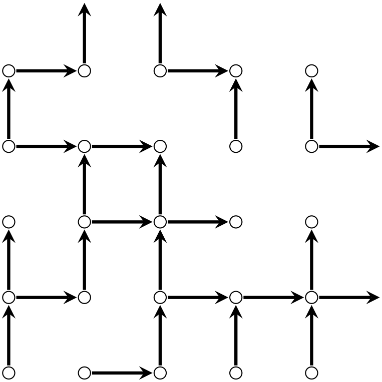
$$\mathbb{P}\big[\text{there is an infinite green path starting at } (0,0)\big] > 0.$$

Comparison with Oriented Percolation

We want to apply our knowledge about oriented percolation to prove that also in the contact process, with positive probability, there is an infinite open path starting at the origin provided that the infection rate is high enough.

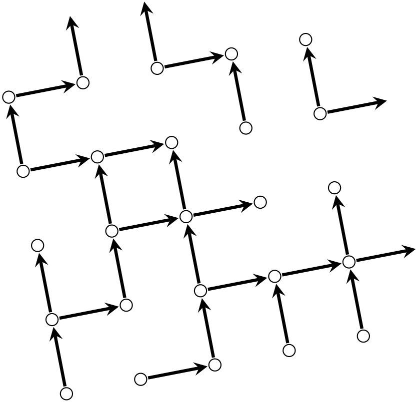
By Exercise 7 of Lecture 4, it suffices to prove the statement for the one-dimensional contact process.

Comparison with Oriented Percolation



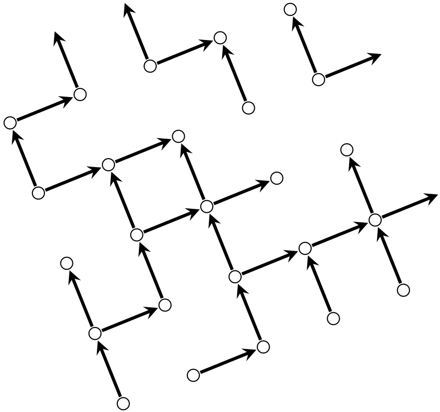
We take our percolation picture. . .

Comparison with Oriented Percolation



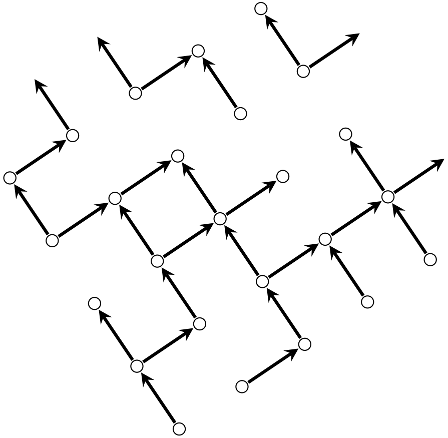
... and rotate it over 45°.

Comparison with Oriented Percolation



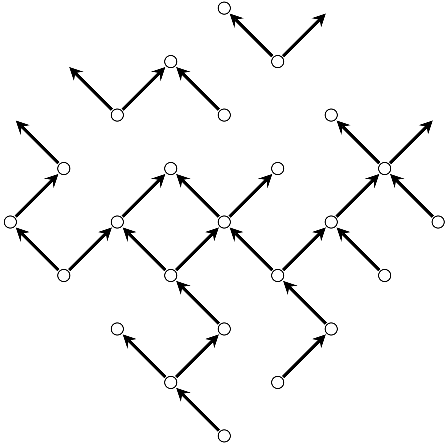
. . . and rotate it over 45°.

Comparison with Oriented Percolation



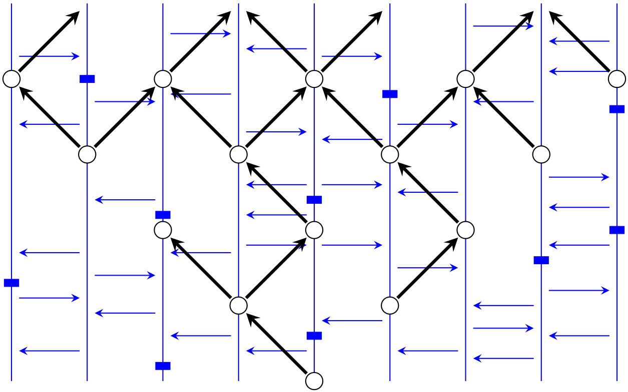
. . . and rotate it over 45°.

Comparison with Oriented Percolation

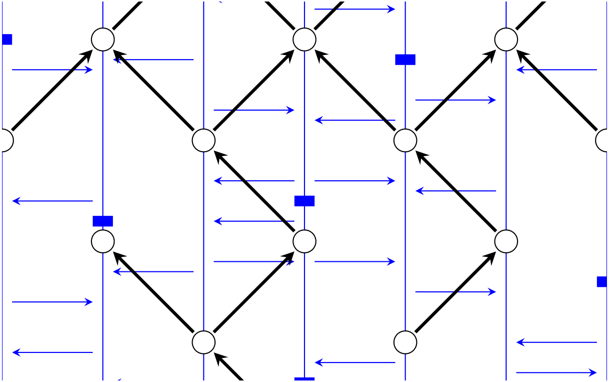


. . . and rotate it over 45°.

Comparison with Oriented Percolation



We overlay this with the graphical representation for the contact process.



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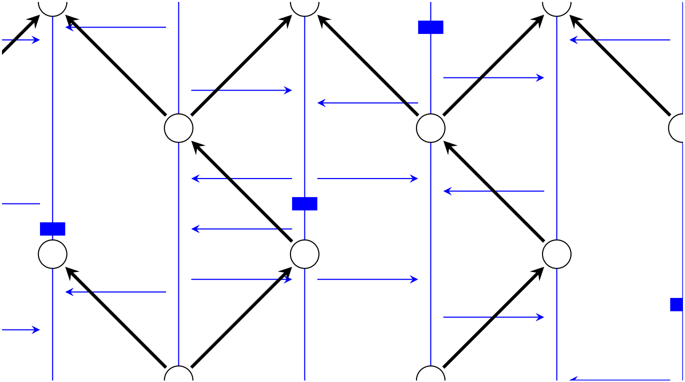
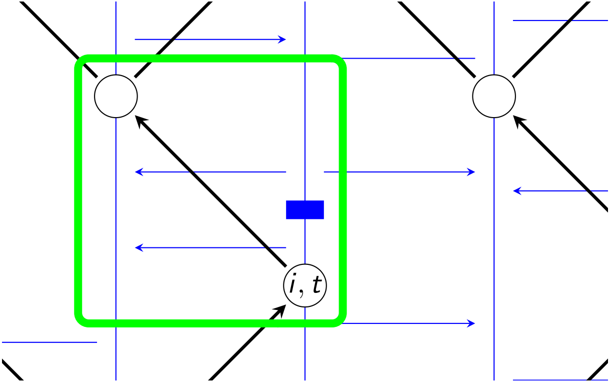
Comparison with Oriented Percolation



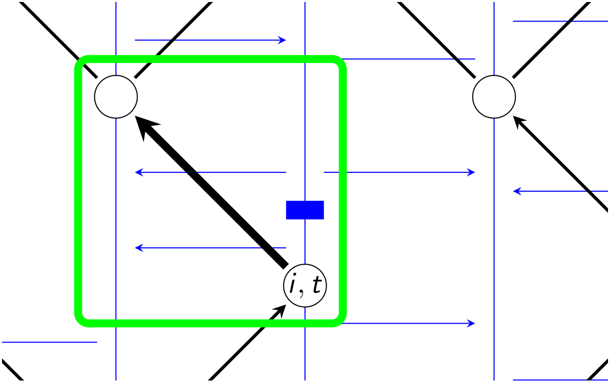
We overlay this with the graphical representation for the contact process.

Comparison with Oriented Percolation



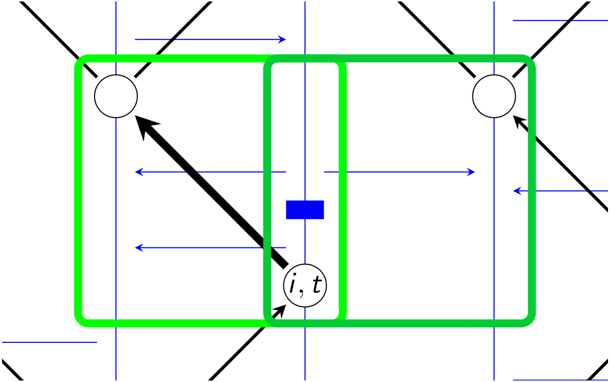
We draw a black arrow from (i, t) to $(i \pm 1, t + 1)$ if within the green square, there is an open path connecting these points.

Comparison with Oriented Percolation



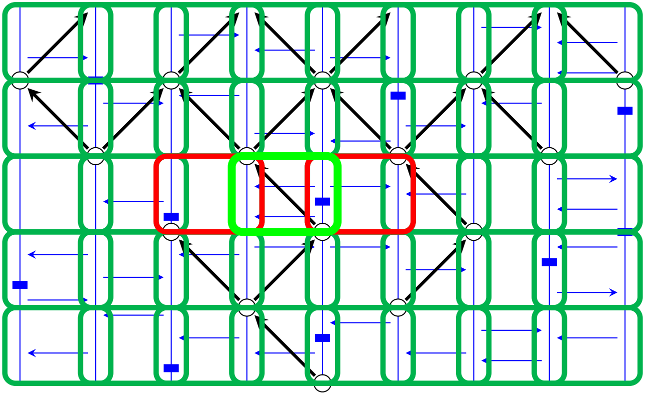
By choosing the infection rate λ large enough and the death rate d small enough, we can make the probability p of a black arrow as close to one as we wish.

Comparison with Oriented Percolation



The only problem is that, since the green squares overlap, these probabilities are not independent.

Comparison with Oriented Percolation



They are, however, *almost* independent. In fact, the bright green square is independent of all other squares, except the red ones.

K-dependence

The following result is due to Liggett, Schonmann, and Stacey (1997).

Theorem 1 *Let Λ be a countable set and let p, K be constants. Let $(\chi_i)_{i\in\Lambda}$ be Bernoulli random variables such that for each $i \in \Lambda$: 1° $P[\chi_i = 1] \geq p$, and 2° there exists $i \in \Delta_i \subset \Lambda$ with $|\Delta_i| \leq K$, such that*

$$\chi_i \text{ is independent of } (\chi_j)_{j\in\Lambda\setminus\Delta_i}.$$

Assume also that

$$\tilde{p} := \big(1 - (1-p)^{1/K}\big)^2 \geq 1/4.$$

Then it is possible to couple $(\chi_i)_{i\in\Lambda}$ to a collection of independent Bernoulli random variables $(\tilde{\chi}_i)_{i\in\Lambda}$ with $P[\tilde{\chi}_i = 1] = \tilde{p}$ in such a way that $\tilde{\chi}_i \leq \chi_i$ for all $i \in \Lambda$.

K-dependence

Warning: The property that there exists $i \in \Delta_i \subset \Lambda$ with $|\Delta_i| \leq K$, such that

$$\chi_i \text{ is independent of } (\chi_j)_{j \in \Lambda \setminus \Delta_i},$$

is *not* exactly what is traditionally called "k-dependence". Rather, in the literature, "k-dependence" is defined for random variables indexed by $\mathbb{Z}^d$ only and means that

$$\chi_i \text{ is independent of } \{\chi_j : j \in \mathbb{Z}^d,\ |j - i| > k\}.$$

This definition is a bit unfortunate since the structure of $\mathbb{Z}^d$ is in fact irrelevant for Theorem 1 and one often needs to apply the theorem to random variables that are not indexed by $\mathbb{Z}^d$.

K-dependence

Why is Theorem 1 good for us?

Since $\tilde{p}(p) \uparrow 1$ as $p \uparrow 1$, by choosing p close enough to 1, we can make $\tilde{p}$ as close to 1 as we wish.

Concretely, applying the theorem with $K = 3$ and choosing the infection and death rates such that

$$p > 1 - \Big(1 - \sqrt{\tfrac{8}{9}}\Big)^3 \approx 0.99981$$

we obtain $\tilde{p} > 8/9$ and can estimate from below by *independent* oriented percolation, for which we have proved that there are infinite open paths.

To prove the existence of a phase transition, all that remains to be done is to prove Theorem 1.

Lemma 2 *Let* $(\chi_n)_{n\geq 0}$ *be Bernoulli random variables such that*

$$P\big[\chi_n = 1 \,\big|\, \chi_0, \ldots, \chi_{n-1}\big] \geq q \qquad (n \geq 0). \tag{1}$$

Then we can couple to independent $(\tilde\chi_n)_{n\geq 0}$ *with* $\mathbb{P}[\tilde\chi_n = 1] = q$ *such that* $\tilde\chi_n \leq \chi_n \ \forall n \geq 0$.

Proof Define

$$q_n(\varepsilon_0,\ldots,\varepsilon_{n-1}) := \mathbb{P}\big[\chi_n = 1 \,\big|\, \chi_0 = \varepsilon_0,\ldots,\chi_{n-1} = \varepsilon_{n-1}\big].$$

Let $(U_n)_{n\geq 0}$ be independent, uniformly distributed $[0,1]$-valued random variables and define inductively

$$\chi'_n := 1_{\{U_n < q_n(\chi'_0,\ldots,\chi'_{n-1})\}} \qquad (n \geq 0). \tag{2}$$

Then the $(\chi'_n)_{n\geq 0}$ are equally distributed with $(\chi_n)_{n\geq 0}$. Moreover,

$$\tilde\chi_n := 1_{\{U_n < q\}} \qquad (n \geq 0)$$

are i.i.d. with intensity q and satisfy $\tilde\chi_n \leq \chi'_n$. ∎

Proof of Theorem 1 Since Λ is countable, without loss of generality we may assume $\Lambda = \mathbb{N}$.

Unfortunately, in general, the random variables $(\chi_i)_{i\geq 0}$ from Theorem 1 do not satisfy condition (1) of Lemma 2 for any $q > 0$.

To remedy this, we construct i.i.d. Bernoulli random variables $(\psi_i)_{i\geq 0}$ with $\mathbb{P}[\psi_i = 1] = r$ to be chosen later, independent of $(\chi_i)_{i\geq 0}$, and set

$$\chi'_i := \psi_i \chi_i.$$

We will show that the "thinned" random variables $(\chi'_i)_{i\geq 0}$ satisfy condition (1) with $q = \tilde{p}$.

We will prove by induction that for an appropriate choice of r,

$$\mathbb{P}[\chi_n = 0 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}] \leq 1 - r. \qquad (3)$$

Note that this is true for $n = 0$ provided that $r \leq p$. Let us put

$$E_0 := \{i \in \Delta_n : 0 \leq i \leq n-1,\ \varepsilon_i = 0\},$$
$$E_1 := \{i \in \Delta_n : 0 \leq i \leq n-1,\ \varepsilon_i = 1\},$$
$$F := \{i \notin \Delta_n : 0 \leq i \leq n-1\}.$$

Then. . .

K-dependence

$$
\begin{aligned}
&\mathbb{P}[\chi_n = 0 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}] \\
&\quad = \mathbb{P}\big[\chi_n = 0 \,\big|\, \chi'_i = 0 \ \forall i \in E_0,\ \chi_i = 1 = \psi_i \ \forall i \in E_1,\ \chi'_i = \varepsilon_i \ \forall i \in F\big] \\
&\quad = \mathbb{P}\big[\chi_n = 0 \,\big|\, \chi'_i = 0 \ \forall i \in E_0,\ \chi_i = 1 \ \forall i \in E_1,\ \chi'_i = \varepsilon_i \ \forall i \in F\big] \\
&\quad = \frac{\mathbb{P}\big[\chi_n = 0,\ \chi'_i = 0 \ \forall i \in E_0,\ \chi_i = 1 \ \forall i \in E_1,\ \chi'_i = \varepsilon_i \ \forall i \in F\big]}{\mathbb{P}\big[\chi'_i = 0 \ \forall i \in E_0,\ \chi_i = 1 \ \forall i \in E_1,\ \chi'_i = \varepsilon_i \ \forall i \in F\big]} \\
&\quad \leq \frac{\mathbb{P}\big[\chi_n = 0,\ \chi'_i = \varepsilon_i \ \forall i \in F\big]}{\mathbb{P}\big[\psi_i = 0 \ \forall i \in E_0,\ \chi_i = 1 \ \forall i \in E_1,\ \chi'_i = \varepsilon_i \ \forall i \in F\big]} \\
&\quad = \frac{\mathbb{P}\big[\chi_n = 0 \,\big|\, \chi'_i = \varepsilon_i \ \forall i \in F\big]}{\mathbb{P}\big[\psi_i = 0 \ \forall i \in E_0,\ \chi_i = 1 \ \forall i \in E_1 \,\big|\, \chi'_i = \varepsilon_i \ \forall i \in F\big]} \\
&\quad \leq \frac{1-p}{(1-r)^{|E_0|}\,\mathbb{P}\big[\chi_i = 1 \ \forall i \in E_1 \,\big|\, \chi'_i = \varepsilon_i \ \forall i \in F\big]} \leq \frac{1-p}{(1-r)^{|E_0|}\, r^{|E_1|}},
\end{aligned}
$$

K-dependence

Here, in the last step, we have used K-dependence and the (nontrivial) fact that

$$\mathbb{P}\big[\chi_i = 1 \ \forall i \in E_1 \,\big|\, \chi'_i = \varepsilon_i \ \forall i \in F\big] \geq r^{|E_1|}. \qquad (4)$$

We claim that (4) is a consequence of the induction hypothesis (3). Indeed, we may assume that the induction hypothesis (3) holds regardless of the ordering of the first n elements, so without loss of generality we may assume that $E_1 = \{n-1, \ldots, m\}$ and $F = \{m-1, \ldots, 0\}$, for some m. Then the left-hand side of (4) may be written as

$$\begin{aligned}
&\prod_{k=m}^{n-1} \mathbb{P}\big[\chi_k = 1 \,\big|\, \chi_i = 1 \ \forall m \leq i < k,\ \chi'_i = \varepsilon_i \ \forall 0 \leq i < m\big] \\
&= \prod_{k=m}^{n-1} \mathbb{P}\big[\chi_k = 1 \,\big|\, \chi'_i = 1 \ \forall m \leq i < k,\ \chi'_i = \varepsilon_i \ \forall 0 \leq i < m\big] \geq r^{n-m}.
\end{aligned}$$

K-dependence

If we assume moreover that $r \geq \frac{1}{2}$, then $r^{|E_1|} \geq (1-r)^{|E_1|}$ and the r.h.s. of our previous estimate

$$\mathbb{P}[\chi_n = 0 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}] \leq \frac{1-p}{(1-r)^{|E_0|}\, r^{|E_1|}}$$

can be further estimated as

$$\frac{1-p}{(1-r)^{|E_0|}\, r^{|E_1|}} \leq \frac{1-p}{(1-r)^{|\Delta_n \cap \{0,\ldots,n-1\}|}} \leq \frac{1-p}{(1-r)^{K-1}}.$$

We see that in order for our proof to work, we need $\frac{1}{2} \leq r \leq p$ and

$$\frac{1-p}{(1-r)^{K-1}} \leq 1-r.$$

In particular, choosing $r = 1-(1-p)^{1/K}$ yields equality here.

K-dependence

Having proved (3), using moreover that

$$\begin{aligned}&\mathbb{P}[\chi'_n = 1 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}] \\ &\quad = r\mathbb{P}[\chi_n = 1 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}],\end{aligned}$$

we see that Theorem 1 holds provided that we put $\tilde{p} := r^2$. ∎

Excercises

Exercise 1 The one-dimensional contact process with double deaths has been introduced before Exercise 4 of Lecture 4. Use comparison with oriented percolation to prove that the one-dimensional contact process with double deaths survives with positive probability if its branching rate λ is large enough. When you apply Theorem 1, what value of K do you (at least) need to use?

Exercise 2 Use the previous exercise, Exercise 4 of Lecture 4, and Exercise 2 of Lecture 5 to conclude that for the cooperative branching process considered there, if λ is large enough, then: 1° the process survives with positive probability if initially there are at least two particles, and: 2° its upper invariant law is nontrivial.

Excercises

Exercise 3 Fill in the necessary details in our proof that the one-dimensional contact process survives for λ large enough to derive an explicit upper bound on $\lambda_{\rm c}$.

Exercise 4 For any $x \in \{0,1\}^{\mathbb{Z}}$, let us write $|x| := \sum_i x(i)$ and let $e_i \in \{0,1\}^{\mathbb{Z}}$ be defined as $e_i(j) := 1_{\{i=j\}}$. Assume that there exists some $t > 0$ such that the one-dimensional contact process satisfies

$$r := \mathbb{E}^{e_0}\big[|X_t|\big] < 1.$$

Show that this then implies that

$$\mathbb{E}^{e_0}\big[|X_{nt}|\big] \leq r^n \qquad (n \geq 0)$$

and the process started in any finite initial state dies out a.s. Use this to derive the bound $\frac{1}{2} \leq \lambda_{\rm c}$. Can you do even better?