On rebellious voter models

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joint with Anja Sturm and Karel Vrbenský

The Neuhauser-Pacala model

1	0	1	1	1	0
1	0	0	0	1	1
1	0	0	0	1	1
0	1	0	1	1	0
1	0	1	0	1	0

0's and 1's represent two closely related species.

The Neuhauser-Pacala model

1	0	1	1	1	0
1	1	0	0	1	1
1	0	0	0	1	1
0	1	0	1	1	0
1	0	1	0	1	0

The 0's and 1's evolve in a Markovian way.

The Neuhauser-Pacala model

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1	1	0	0	1	1
1	0	0	0	1	1
0	0	0	1	1	0
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The 0's and 1's evolve in a Markovian way.

The Neuhauser-Pacala model

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1	0	0	0	1	1
0	0	0	1	1	0
1	0	1	0	1	0

The 0's and 1's evolve in a Markovian way.

The Neuhauser-Pacala model

1	0	1	1	0	0
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The 0's and 1's evolve in a Markovian way.

The Neuhauser-Pacala model

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The 0's and 1's evolve in a Markovian way.

The Neuhauser-Pacala model

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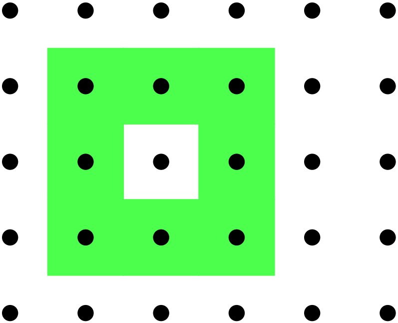
0	0	1	1	1	0
1	0	1	1	1	0

The 0's and 1's evolve in a Markovian way.

The Neuhauser-Pacala model



$\mathcal{N}_i := \{j \in \mathbb{Z}^d : 0 < \|i - j\|_\infty \leq R\}$ neighborhood of a site.
(Here $R = 1$, $d = 2$).

The Neuhauser-Pacala model

1	0	1	1	0	0
1	1	0	1	1	1
1	1	0	0	1	1
0	0	1	1	1	0
1	0	1	1	1	0

$f_0 = 3/8$, $f_1 = 5/8$ local frequencies of types $0, 1$.

The Neuhauser-Pacala model

1	0	1	1	0	0
1	1	0	1	1	1
1	1	†	0	1	1
0	0	1	1	1	0
1	0	1	1	1	0

With rate with rate $f_0 + \alpha_{01} f_1$ an organism of type 0 dies...

The Neuhauser-Pacala model

1	0	1	1	0	0
1	1	0	1	1	1
1	1	1	0	1	1
0	0	1	1	1	0
1	0	1	1	1	0

. . . and is replaced by a random type from the neighborhood.

The Neuhauser-Pacala model

Neuhauser & Pacala (1999): Markov process in the space $\{0,1\}^{\mathbb{Z}^d}$ of spin configurations $x = (x(i))_{i\in\mathbb{Z}^d}$, where spin $x(i)$ flips:

$$
\begin{aligned}
0 \mapsto 1 &\text{ with rate } f_1(f_0 + \alpha_{01} f_1),\\
1 \mapsto 0 &\text{ with rate } f_0(f_1 + \alpha_{10} f_0),
\end{aligned}
$$

with

$$
f_\tau(i) := \frac{\#\{j \in \mathcal{N}_i : x(j) = \tau\}}{\#\mathcal{N}_i} \quad \mathcal{N}_i := \{j : 0 < \|i - j\|_\infty \le R\}.
$$

the local frequency of type $\tau = 0, 1$.

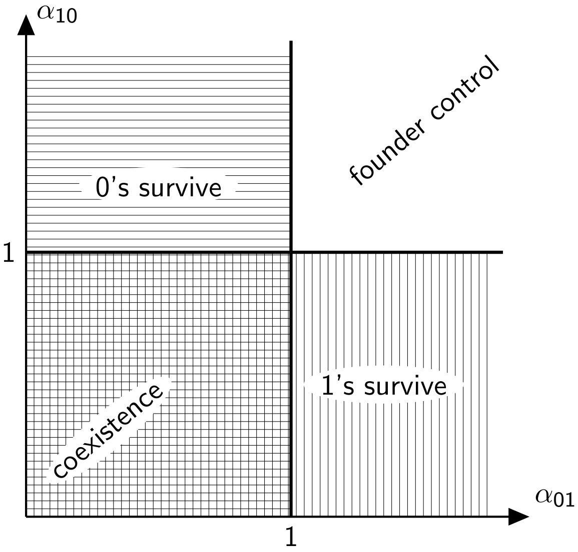
Interpretation: *Interspecific competition rates* α_{01}, α_{10}. Organism of type 0 dies with rate $f_0 + \alpha_{01} f_1$ and is replaced by type sampled at random from distance $\le R$.

The Neuhauser-Pacala model

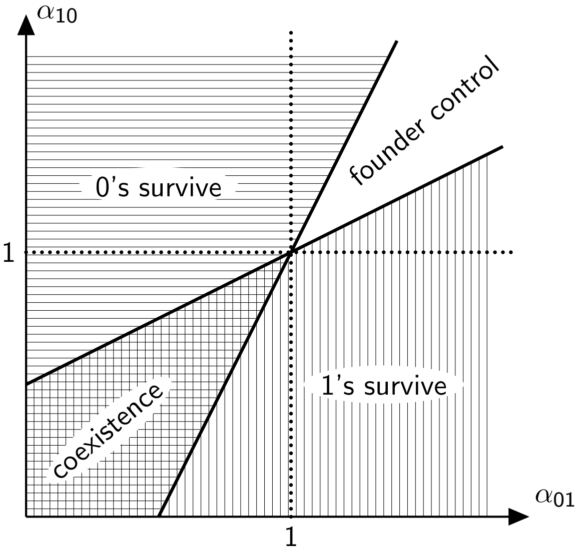
By definition, type 0 *survives* if starting from a single organism of type 0 and all other organisms of type 1, there is a positive probability that the organisms of type 0 never die out.

By definition, one has *coexistence* if there exists an invariant law concentrated on states where both types are present.

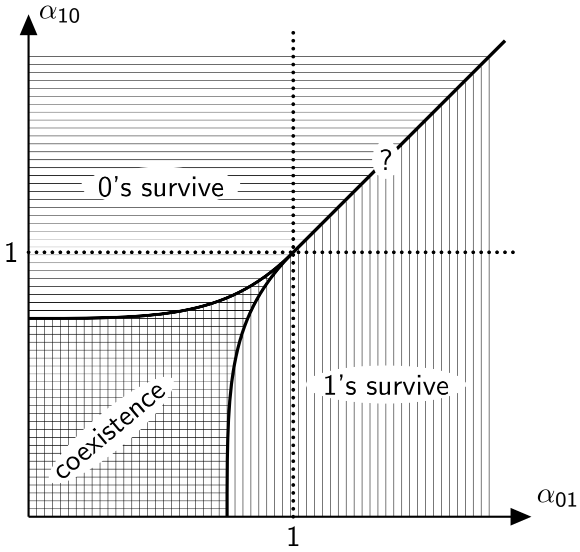
Mean field model



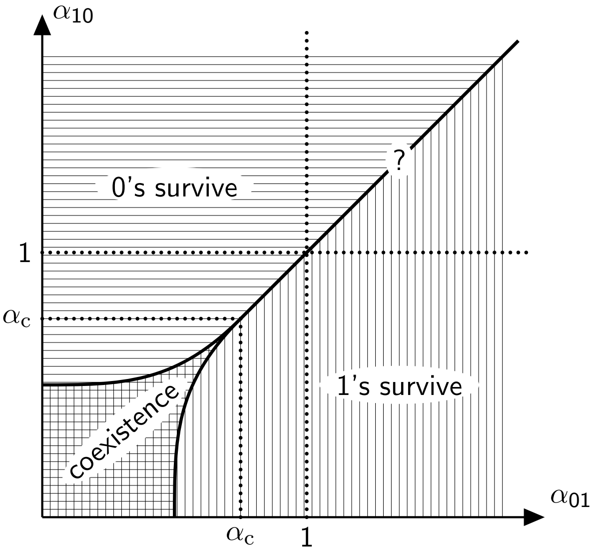
Dimension $d \geq 3$



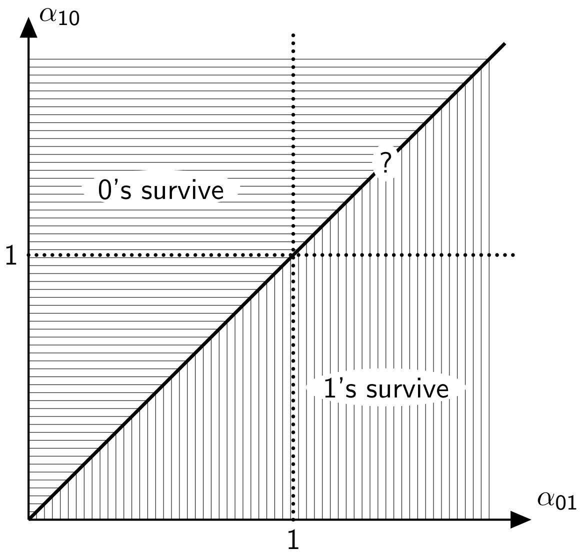
Dimension $d = 2$



Dimension $d = 1$, range $R \geq 2$



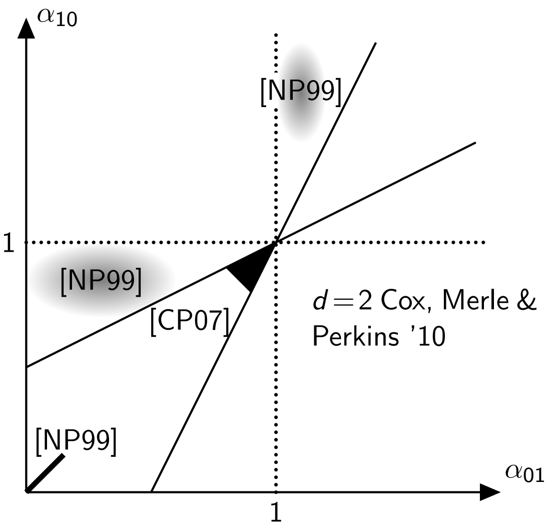
Dimension $d = 1$, range $R = 1$



Consequences for biodiversity

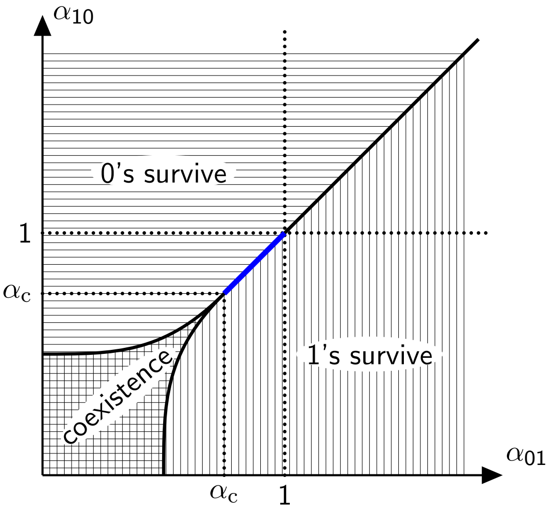
Conjecture There exists a critical dimension $d_{\rm c} \cong 4/3$ such that in dimensions $d < d_{\rm c}$, two species must be sufficiently different to be able to coexist, but in dimensions $d > d_{\rm c}$, any difference, no matter how small, suffices.

Rigorous results for $d \geq 3$



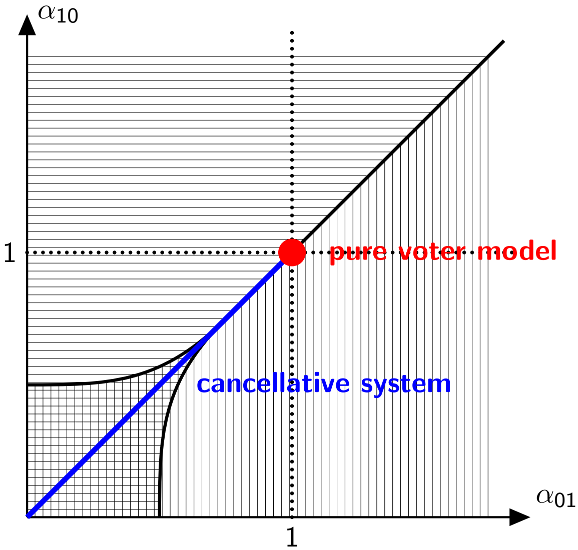
[NP99]=Neuhauser & Pacala '99, [CP07]=Cox & Perkins '07.

Open problem for $d = 1$



Open problem: noncoexistence.

Special models



Pure voter model

If started with finitely many organisms of type 1, then number of 1's is a martingale. Consequence: 1's (and likewise 0's) die out.

Dual to coalescing random walks. Consequence: coexistence in transient dimensions $d \geq 3$, clustering in $d = 1, 2$.

The symmetric case

The symmetric case $\alpha_{01} = \alpha_{10} = \alpha \leq 1$ is a *cancellative system*.

There is a *dual process* Y such that

$$\mathbb{P}\big[|X_t Y_0| \text{ is odd}\big] = \mathbb{P}\big[|X_0 Y_t| \text{ is odd}\big] \qquad (t \geq 0)$$

whenever X and Y are independent. Here

$$|x| := \textstyle\sum_i x(i) \quad \text{and} \quad xy(i) := x(i)y(i).$$

Cancellative systems

Equip $\{0,1\}$ with the usual product and with addition modulo 2, denoted as $\oplus$. Then $\{0,1\}$ is a *finite field*. We may view $\{0,1\}^{\mathbb{Z}^d}$ (equipped with $\oplus$) as a *linear space* over $\{0,1\}$.

A *cancellative system* $X = (X_t)_{t\geq 0}$ is a *linear system* w.r.t. to the finite field $\{0,1\}$, that evolves as

$$x \mapsto x \oplus Ax \quad \text{with rate } r(A) \geq 0,$$

where

$$Ax(i) := \bigoplus_{j\in\mathbb{Z}^d} A(i,j)x(j)$$

with $A(i,j) = 1$ for finitely many i,j and $A(i,j) = 0$ otherwise.

Example For $k \in \mathbb{Z}$, define:

$$A_k(k-1,k) := 1, \quad A_k(k,k) := 1, \quad A_k(i,j) := 0 \text{ otherwise},$$
$$A'_k(k-2,k) := 1, \quad A'_k(k-1,k) := 1, \quad A'_k(i,j) := 0 \text{ otherwise}.$$

Set $r(A_k) := \alpha$, $r(A'_k) := 1 = \alpha$, and $r(A) := 0$ for all other A.

This yields *one-sided rebellious voter model* where $x(k)$ flips

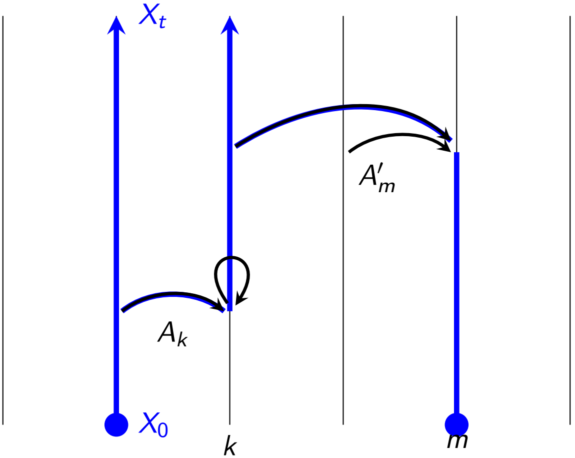
$$0 \leftrightarrow 1 \text{ with rate } \alpha 1_{\{x(k-1) \neq x(k)\}} + (1-\alpha) 1_{\{x(k-2) \neq x(k-1)\}}.$$

Graphical representation

Draw space horizontally, time vertically.

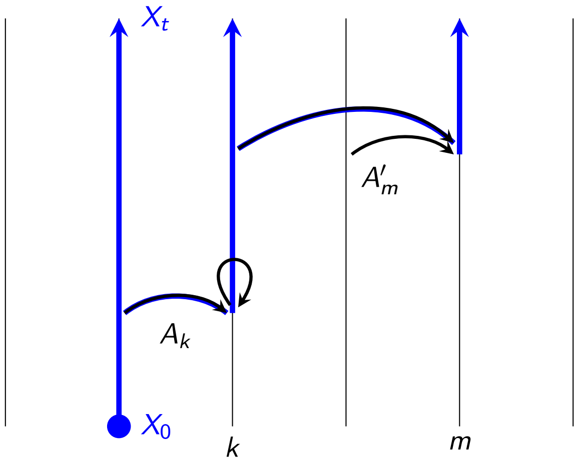
If the local map A applies at time t, draw an arrow from (i, t) to (j, t) whenever $A(i, j) = 1$.

Graphical representation



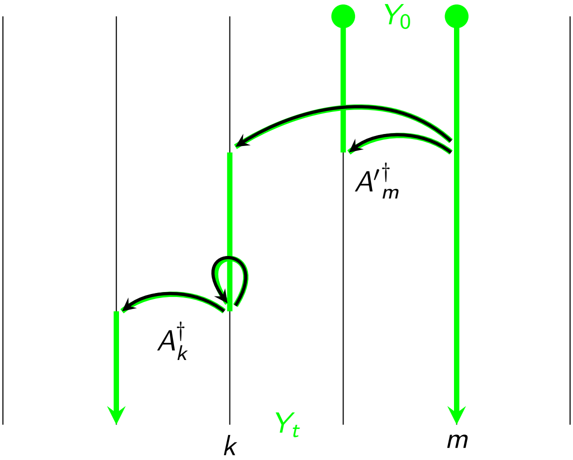
$X_t(i) = 1$ iff there is a odd number of paths from X_0 to (i, t).

Graphical representation



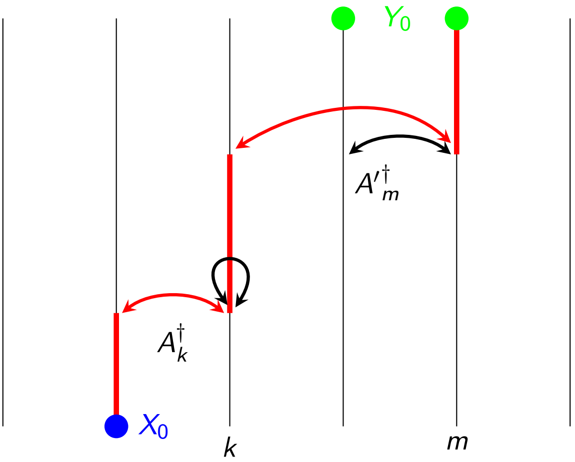
$X_t(i) = 1$ iff there is a odd number of paths from X_0 to (i, t).

Graphical representation of dual model



Time runs backwards and all arrows are reversed.

Duality



$|X_0 Y_t|$ is odd $\Leftrightarrow$ # paths from X_0 to Y_0 is odd $\Leftrightarrow$ $|X_t Y_0|$ is odd.

Cancellative system duality

Rates of the dual model:

$$r_Y(A^\dagger) = r_X(A),$$

where $A^\dagger(i,j) = A(j,i)$ denotes the *adjoint* of A.

Duality:

$$\mathbb{P}\big[|X_t Y_0| \text{ is odd}\big] = \mathbb{P}\big[|X_0 Y_t| \text{ is odd}\big] \qquad (t \geq 0)$$

whenever X and Y are independent, where

$$|x| := \sum_i x(i) \quad \text{and} \quad xy(i) := x(i)y(i).$$

Examples of duality

If X has *voter model* dynamics:

$$\big(x(k-1), x(k)\big) \mapsto \big(x(k-1), x(k-1)\big) \quad \text{with rate } \alpha,$$

then Y has *annihilating random walk* dynamics:

$$\big(y(k-1), y(k)\big) \mapsto \big(y(k-1) \oplus y(k), 0\big) \quad \text{with rate } \alpha 1_{\{y(k)=1\}},$$

i.e., a particle at k jumps to $k-1$;
if there is already a particle at $k-1$,
the two particles *annihilate*.

Examples of duality

If X has *rebellious* dynamics:

$$x(k) \mapsto 1 - x(k) \quad \text{with rate } (1-\alpha)1_{\{x(k-2)\neq x(k-1)\}},$$

then Y has *annihilating branching* dynamics:

$$y \mapsto y \oplus \delta_{k-2} \oplus \delta_{k-1}, \quad \text{with rate } (1-\alpha)1_{\{y(k)=1\}},$$

i.e., a particle at k produces *two* new particles
at positions $k-2$ and $k-1$
and these particles *annihilate* with any particles that may already be present.

Similarly, the Neuhauser-Pacala model is dual to a system of *branching-annihilating* particles, where:
particles jump with rate α to a new place at distance $\leq R$,
and each particle produces two new particles at distances $\leq R$ with rates proportional to $1 - \alpha$.

Since the number of particles of Y is always increased or decreased by 2, the process is *parity preserving*, i.e., $|Y_t|$ is odd $\Leftrightarrow$ $|Y_0|$ is odd.

Behavior of the dual model

By definition, Y *survives* if starting from an *even* number of particles, there is a positive probability that the particles never die out.

By definition, Y is *stable* if starting from *one* particle, the process 'modulo translations' is positively recurrent, i.e., the system returns to a state with only one particle infinitely often and spends a positive fraction of its time in such states.

By definition, Y is *persistent* if there exists an invariant law concentrated on nonempty configurations.

Simple consequences of duality

Lemma X has coexistence $\Leftrightarrow$ Y survives.

Proof Start X in product measure with intensity $1/2$ and let $Y_0 := \delta_i + \delta_j$. Then

$$\mathbb{P}\big[X_t(i) \neq X_t(j)\big] = \mathbb{P}\big[|X_t Y_0| \text{ is odd}\big] = \mathbb{P}\big[|X_0 Y_t| \text{ is odd}\big]$$
$$= \tfrac{1}{2}\mathbb{P}\big[Y_t \neq 0\big] \underset{t\to\infty}{\longrightarrow} \tfrac{1}{2}\mathbb{P}\big[Y_s \neq 0 \ \forall s \geq 0\big].$$

∎

Lemma In X both types survive $\Leftrightarrow$ Y is persistent.

Proofs of coexistence

For small α, coexistence and survival of both types for the Neuhauser-Pacala model proved by showing that dual model survives and is persistent.

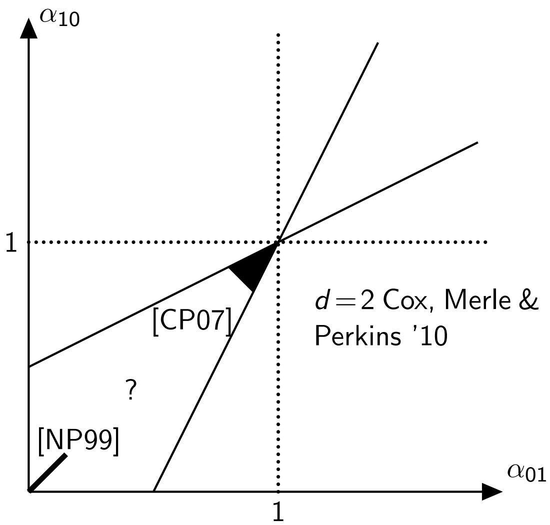
Main tool: comparison with oriented percolation.

In dimensions $d \geq 2$, Cox, Merle and Perkins prove that as $\alpha_{01}, \alpha_{10} \to 1$ and $\alpha_{01} \approx \alpha_{10}$, rescaled sparse models converge to supercritical *super Brownian motion*.

Using this, for α_{01}, α_{10} fixed but very close to *one*, they are able to compare with oriented percolation and prove coexistence.

Intermediate α still open since monotonicity not proved.

Coexistence results for $d \geq 3$



[NP99]=Neuhauser & Pacala '99, [CP07]=Cox & Perkins '07.

Interfaces

By definition, a cancellative spin-system X is *type-symmetric* if it treats the types symmetrically, i.e., $1 - X_t$ has the same dynamics as X_t.

Lemma X type-symmetric $\Leftrightarrow$ dual Y parity preserving.

For *one-dimensional*, type-symmetric X, setting

$$Y'_t(i) := 1_{\{X_t(i - \frac{1}{2}) \neq X_t(i + \frac{1}{2})\}} \qquad (i \in \mathbb{Z} + \tfrac{1}{2})$$

defines the *interface model* of X.

Lemma [Swa13] Y' is a also a parity preserving cancellative system and

$$\begin{tikzcd} X \arrow[r, "\text{interface}"] \arrow[d, leftrightarrow, "\text{dual}"'] & Y' \arrow[d, leftrightarrow, "\text{dual}"] \\ Y & X' \arrow[l, "\text{interface}"] \end{tikzcd}$$

The rebellious voter model

For the rebellious voter model, the dual and interface models coincide:

$$\begin{tikzcd} X \arrow[d, bend right, leftrightarrow, "\text{dual}"'] \arrow[d, bend left, "\text{interface}"] \\ Y \end{tikzcd}$$

Lemma [SS08] For the rebellious voter model both types survive $\Leftrightarrow$ the model exhibits coexistence.

Interface tightness

By definition, X exhibits *interface tightness* if its interface model Y' is stable, i.e., Y' modulo translations is positively recurrent.

Interface tightness means that starting from $\ldots 000000111111 \ldots$, the system spends a positive fraction of time in such states: the types *cannot invade* each other's territory.

Interface tightness for long-range voter models was proved by Cox and Durrett (1995) under a third moment condition on the infection rates. This was improved to a second moment condition, which is sharp, by Belhaouari, Mountford and Valle (2007). A simpler proof was given by S. & Sturm (2008).

Interface tightness for the Neuhauser-Pacala model with $R \geq 2$ or for the rebellious voter model is an *open problem*.

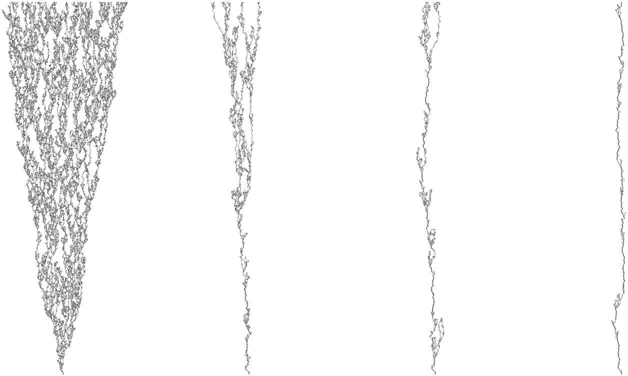
Strong interface tightness

By definition, X exhibits *strong interface tightness* if for the invariant law of the interface model modulo shifts $\mathbb{E}[|Y'_\infty|] < \infty$, i.e., the expected number of sites such that $X_t(i - \frac{1}{2}) \neq X_t(i + \frac{1}{2})$ is finite.

Strong interface tightness is known to hold for voter models and numerically observed for the rebellious voter model with $\alpha > \alpha_{\rm c}$. By contrast, the expected *length* of the interface (distance from left-most one to right-most zero) is known to be infinite for voter models.

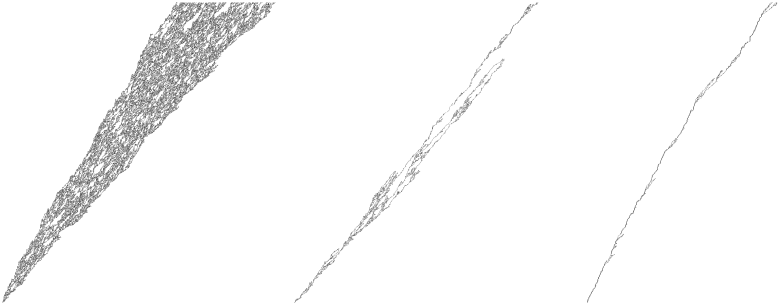
Theorem [Swa13] Strong interface tightness implies noncoexistence.

Numerical simulation



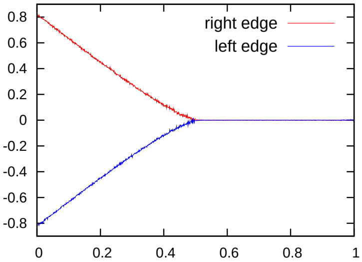
Interface process Y of the two-sided rebellious voter model for $\alpha = 0.4, 0.5, 0.51, 0.6$.

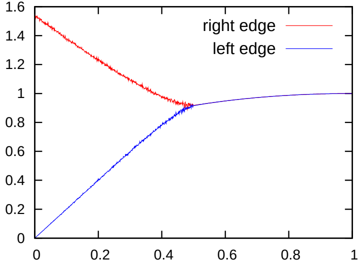
One-sided rebellious interface model



Interface process Y of the one-sided rebellious voter model for $\alpha = 0.3, 0.5, 0.6$.

Edge speeds





Edge speeds for the rebellious voter model (left) and its one-sided counterpart (right) [S. & Vrbenský '10].

Two functions of the process

Define the *survival probability*

$$\rho(\alpha) := \mathbb{P}^{\delta_0}[X_t \neq 0 \ \forall t \geq 0].$$

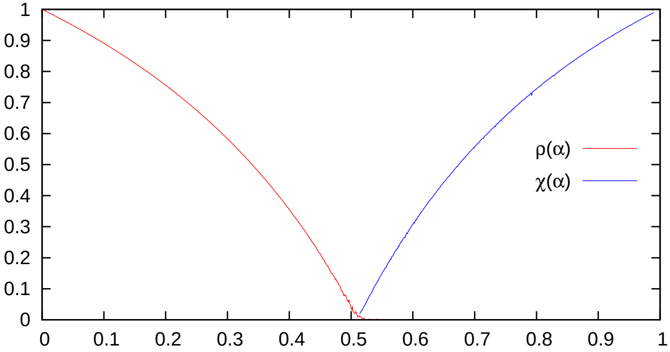
- coexistence $\Leftrightarrow$ $\rho(\alpha) > 0$.

Define the *fraction of time spent with a single interface*

$$\chi(\alpha) := \mathbb{P}\big[|Y_\infty| = 1\big].$$

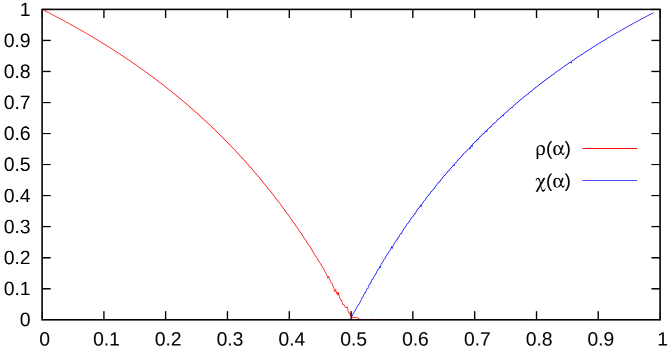
- interface tightness $\Leftrightarrow$ $\chi(\alpha) > 0$.

Numerical data



The functions ρ and χ for the two-sided rebelious voter model.

Numerical data



The functions ρ and χ for the one-sided rebelious voter model.

Explicit formulas

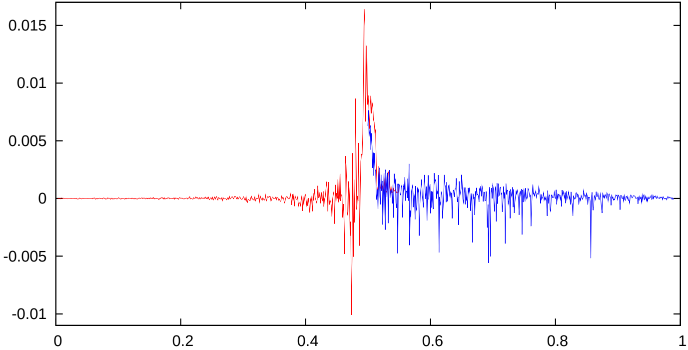
It seems that for the one-sided model, the functions ρ and χ are described by the explicit formulas:

$$\rho(\alpha) = 0 \vee \frac{1-2\alpha}{1-\alpha} \quad \text{and} \quad \chi(\alpha) = 0 \vee \big(2 - \frac{1}{\alpha}\big).$$

In particular, one has the symmetry $\rho(1-\alpha) = \chi(\alpha)$ and the critical parameter seems to be given by $\alpha_{\rm c} = 1/2$.

Explanation?

Numerical data



Differences of ρ and χ with presumed explicit formulas.

A critical exponent

Theoretical physicists believe that

$$\rho(\alpha) \sim (\alpha_c - \alpha)^\beta \quad \text{as} \quad \alpha \uparrow \alpha_c,$$

where β is a *critical exponent*.

It has been conjectured by I. Jensen (1994) that $\beta = 13/14$ and that $\beta = 1$ [Inui & Tretyakov '98]. More recent estimates are $\beta \approx 0.92$, $\beta \approx 0.95$ [Hinrichsen '00] [Ódor & Szolnoki '05]. Our formula would imply $\beta = 1$.

Interface tightness

Let ξ^1, ξ^2, ξ^3 be independent random walks started from $(\xi^1_0, \xi^2_0, \xi^3_0) = (-1, 0, 1)$.

Set $\tau_{ij} := \inf\{t \geq 0 : \xi^i_t = \xi^j_t\}$ and

$$\tau := \tau_{12} \wedge \tau_{23} \wedge \tau_{31}.$$

Then

$$\mathbb{P}[\tau > t] \sim t^{-3/2} \quad \text{as} t \to \infty,$$
$$\mathbb{E}[\tau] = 1 < \infty.$$

Observed: for small branching rate $1 - \alpha$, system Y spends fraction of time of order $(1 - \alpha)^m$ with $1 + 2m$ particles.