# Recursive tree processes and the mean-field limit of stochastic flows

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Friday, October 9th, 2020

# Cooperative branching

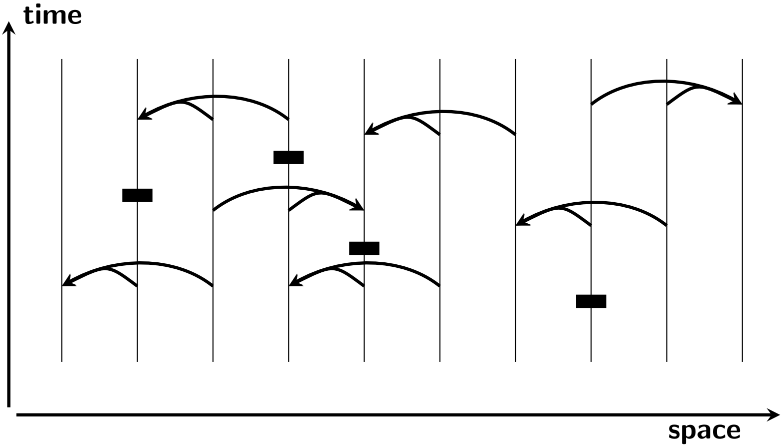
Let $S := \{0, 1\}$. Consider the maps:

$$\texttt{cob} : S^3 \to S \quad \text{with} \quad \texttt{cob}(x_1, x_2, x_3) := x_1 \vee (x_2 \wedge x_3),$$
$$\texttt{dth} : S^0 \to S \quad \text{with} \quad \texttt{dth}(\varnothing) := 0.$$

Let $G = (V, E)$ be a graph.
Let $X = (X_t)_{t \geq 0}$ with $X_t = (X_t(i))_{i \in V}$ be a Markov process with state space $S^V$ that evolves as follows:

- (*Cooperative branching*) For each $i \in V$, with Poisson rate $\alpha$, we pick $i \sim j \sim k$, all different, at random and replace $X_t(i)$ by $\texttt{cob}(X_t(i), X_t(j), X_t(k))$.
- (*death*) For each $i \in V$, with Poisson rate one, we replace $X_t(i)$ by $\texttt{dth}(\varnothing) = 0$.

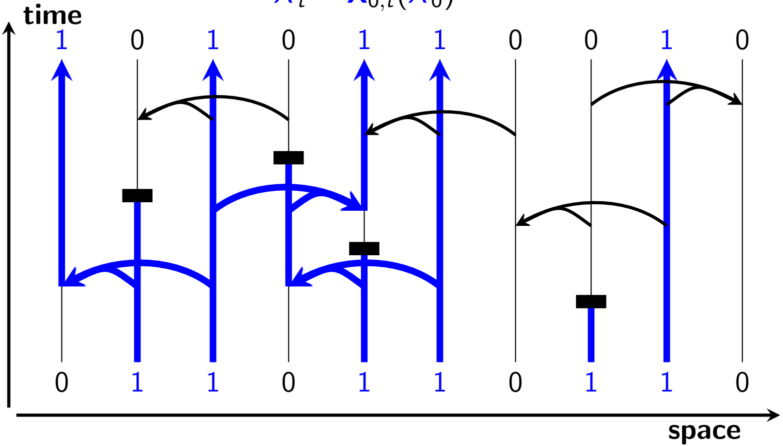
# A graphical representation




We denote `cob` and `dth` by suitable symbols.

# A graphical representation

$$X_t = \mathbf{X}_{0,t}(X_0)$$




$X_0$

The Poisson events define a random map $x \mapsto \mathbf{X}_{0,t}(x)$.

# A stochastic flow

The random maps $(\mathbf{X}_{s,t})_{s \leq t}$ form a *stochastic flow*

$$\mathbf{X}_{s,s} = 1 \quad \text{and} \quad \mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$$

with *independent increments*, in the sense that

$$\mathbf{X}_{t_0,t_1}, \ldots, \mathbf{X}_{t_{n-1},t_n}$$

are independent for each $t_0 < \cdots < t_n$.

If $X_0$ is independent of $(\mathbf{X}_{s,t})_{s \leq t}$, then setting

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t \geq 0}$ with the right jump rates.

# The mean-field limit

We are interested in the process on the *complete graph* with $N$ vertices. Let $\mathcal{P}(S) :=$ the space of probability measures on $S$.

For any deterministic map $g : S^k \to S$, define $\mathbf{T}_g : \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$\mathbf{T}_g(\mu) := \text{ the law of } g(X_1, \ldots, X_k),$$

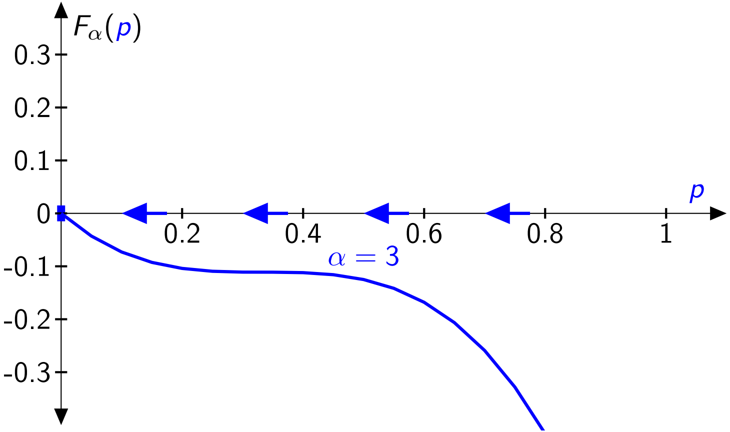
where $(X_i)_{i \geq 1}$ are i.i.d. with law $\mu$.
In the limit $N \to \infty$, the empirical measure $\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_t(i)}$ solves

$$\frac{\partial}{\partial t} \mu_t = \alpha \big\{ \mathbf{T}_{\mathtt{cob}}(\mu_t) - \mu_t \big\} + \big\{ \mathbf{T}_{\mathtt{dth}}(\mu_t) - \mu_t \big\}.$$

Rewriting this in terms of $p_t := \mu_t(\{1\})$ yields

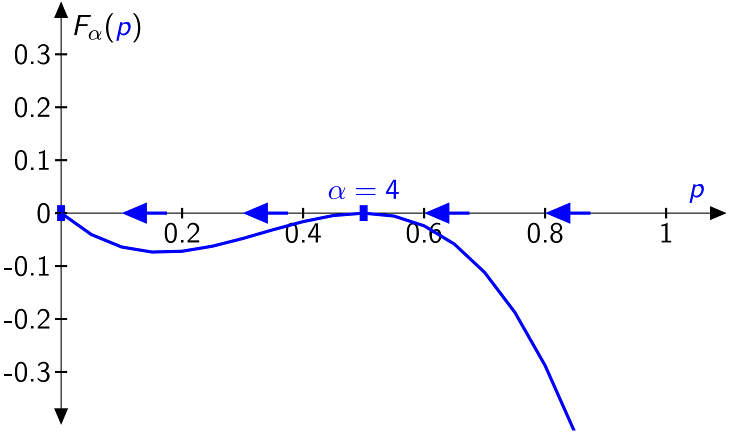
$$\frac{\partial}{\partial t} p_t = \alpha p_t^2 (1 - p_t) - p_t =: F_\alpha(p_t) \qquad (t \geq 0).$$

# Cooperative branching



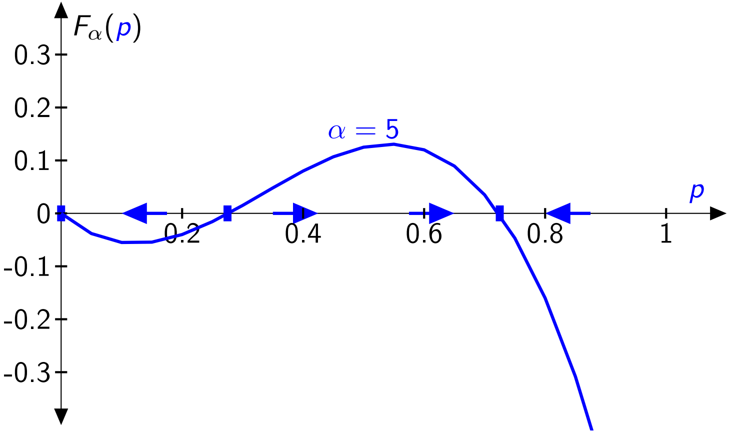

For $\alpha < 4$, the equation $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ has a single, stable fixed point $p = 0$.

# Cooperative branching



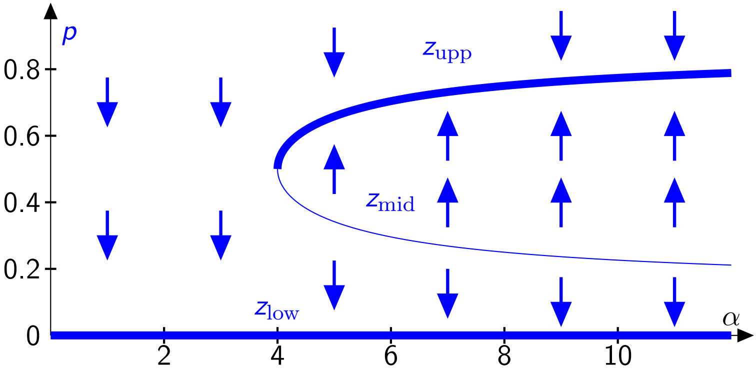

For $\alpha = 4$, a second fixed point appears at $p = 0.5$.

# Cooperative branching




For $\alpha > 4$, there are two stable fixed points and one unstable fixed point, which separates the domains of attraction of the other two.

# Cooperative branching




Fixed points of $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ for different values of $\alpha$.

# The general set-up

(i) Polish space $S$ *local state space.*

(ii) $(\Omega, \mathcal{B}, \mathbf{r})$ Polish space with Borel $\sigma$-field and finite measure: *source of external randomness.*

(iii) $\kappa : \Omega \to \mathbb{N}$ measurable function.

(iv) For each $\omega \in \Omega$, a measurable function $\gamma[\omega] : S^{\kappa(\omega)} \to S$.

Then the *mean-field equation* takes the form

$$\frac{\partial}{\partial t}\mu_t = \int_\Omega \mathbf{r}(\mathrm{d}\omega)\big\{\mathbf{T}_{\gamma[\omega]}(\mu_t) - \mu_t\big\} \qquad (t \geq 0). \tag{1}$$

In our example $S = \{0, 1\}$, $\Omega = \{1, 2\}$,

$$\begin{aligned}
\gamma[1] &= \texttt{cob} : S^3 \to S, & \kappa(1) &= 3, & \mathbf{r}(\{1\}) &= \alpha,\\
\gamma[2] &= \texttt{dth} : S^0 \to S, & \kappa(2) &= 0, & \mathbf{r}(\{2\}) &= 1.
\end{aligned}$$

# The mean-field equation

**Theorem [Mach, Sturm, S. '20]** Assume that

$$\int_\Omega \mathbf{r}(\mathrm{d}\omega)\,\kappa(\omega) < \infty \tag{2}$$

Then for each initial state, the mean-field equation (1) has a unique solution.

Define a (nonlinear) semigroup $(\mathbf{T}_t)_{t\geq 0}$ of operators acting on probability measures by

$$\mathbf{T}_t(\mu) := \mu_t \quad \text{where } (\mu_t)_{t\geq 0} \text{ solves (1) with } \mu_0 = \mu.$$

**Proposition [Mach, Sturm, S. '20]** Assume that $\forall k,\ x \in S^k$

$$\mathbf{r}\big(\{\omega : \kappa(\omega) = k,\ \gamma[\omega] \text{ is discontinuous at x}\}\big) = 0. \tag{3}$$

Then the operators $\mathbf{T}_t$ are continuous w.r.t. weak convergence.

# The mean-field equation

Let $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t(i)}$ denote the empirical measure.
Let $d$ be any metric that generates the topology of weak convergence and let $\|\cdot\|$ denote the total variation norm.

**Theorem [Mach, Sturm, S. '20]** Assume (2) and at least one of the following conditions:

(i) $\mathbb{P}\big[d(\mu_0^N,\mu_0)\geq\varepsilon\big] \underset{N\to\infty}{\longrightarrow} 0$ for all $\varepsilon>0$, and (3) holds.

(ii) $\big\|\mathbb{E}[(\mu_0^N)^{\otimes n}]-\mu_0^{\otimes n}\big\| \underset{N\to\infty}{\longrightarrow} 0$ for all $n\geq 1$.

Then

$$\mathbb{P}\Big[\sup_{0\leq t\leq T} d\big(\mu_t^N,\mathbf{T}_t(\mu_0)\big)\geq\varepsilon\Big] \underset{N\to\infty}{\longrightarrow} 0 \qquad (\varepsilon>0,\ T<\infty).$$

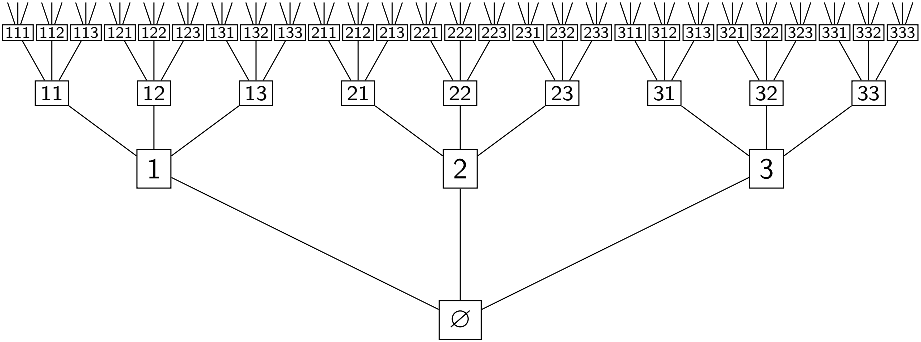
# The mean-field limit of the stochastic flow

**Question**

What is the mean-field limit of the stochastic flow $(\mathbf{X}_{s,t})_{s\leq t}$?

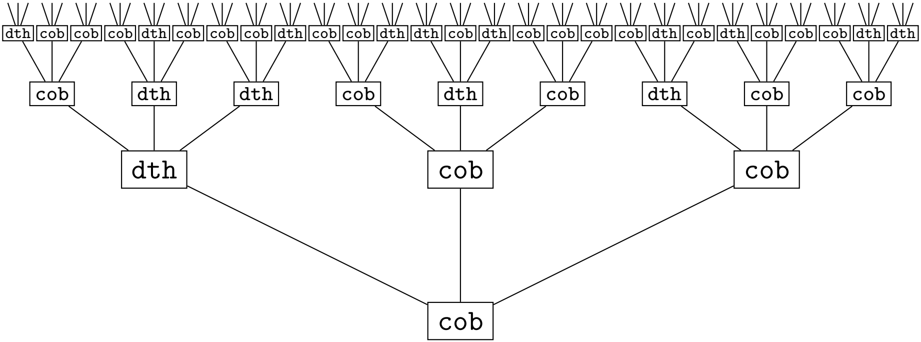
Fix $d \in \mathbb{N}_+ \cup \{\infty\}$ such that $\kappa(\omega) \leq d$ for all $\omega \in \Omega$. Let $\mathbb{T} = \mathbb{T}^d$ denote the space of all words $\mathbf{i} = i_1 \cdots i_n$ made from the alphabet $\{1, \ldots, d\}$ (if $d < \infty$) resp. $\mathbb{N}_+$ (if $d = \infty$).

# A recursive tree representation



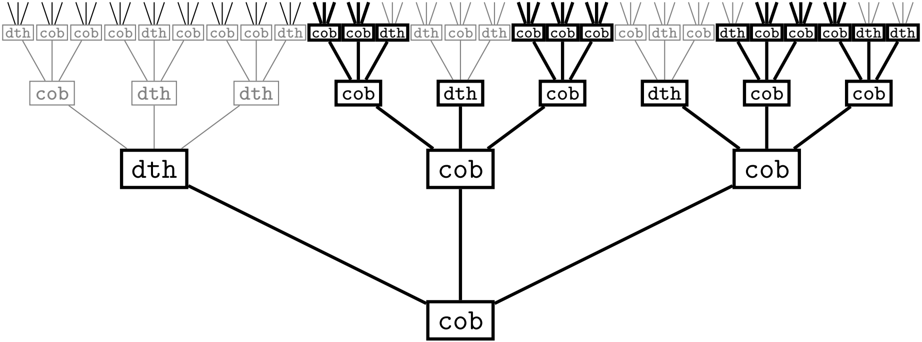

We view $\mathbb{T} = \mathbb{T}^d$ as a tree with root $\varnothing$, the word of length zero.

# A recursive tree representation




We attach i.i.d. $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ with law $|\mathbf{r}|^{-1}\mathbf{r}$ to each node, which translate into maps $(\gamma[\omega_{\mathbf{i}}])_{\mathbf{i}\in\mathbb{T}}$.

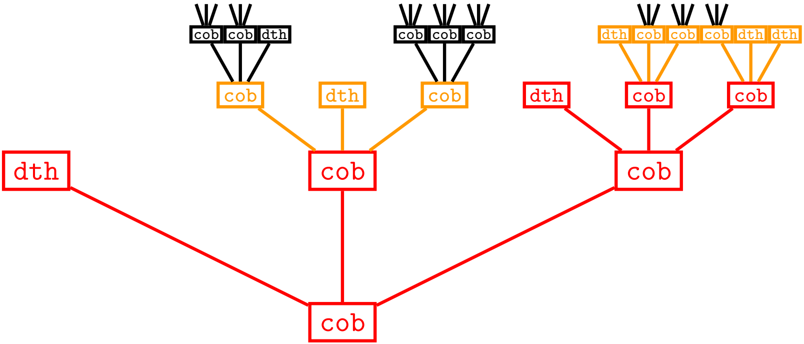
# A recursive tree representation




Let $\mathbb{S}$ be the random subtree of $\mathbb{T}$ defined as

$$\mathbb{S} := \{ i_1 \cdots i_n \in \mathbb{T} : i_m \leq \kappa(\omega_{i_1 \cdots i_{m-1}}) \ \forall 1 \leq m \leq n \}.$$

# A recursive tree representation

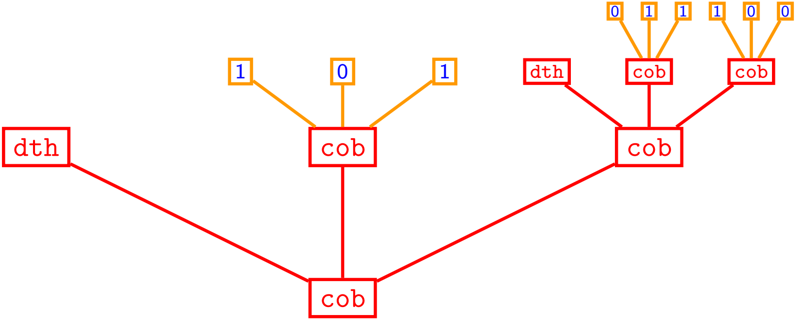



For any rooted subtree $\mathbb{U} \subset \mathbb{S}$, let

$$\nabla\mathbb{U} := \big\{ i_1 \cdots i_n \in \mathbb{S} : i_1 \cdots i_{n-1} \in \mathbb{U},\ i_1 \cdots i_n \notin \mathbb{U} \big\}$$

denote the boundary of $\mathbb{U}$ relative to $\mathbb{S}$.

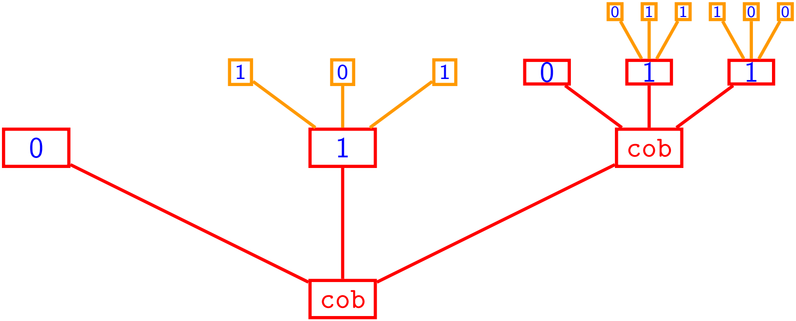
# A recursive tree representation




Given $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}$, we inductively define $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$ by

$$X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}]\big(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}\kappa(\omega)}\big) \qquad (\mathbf{i} \in \mathbb{U}).$$

# A recursive tree representation

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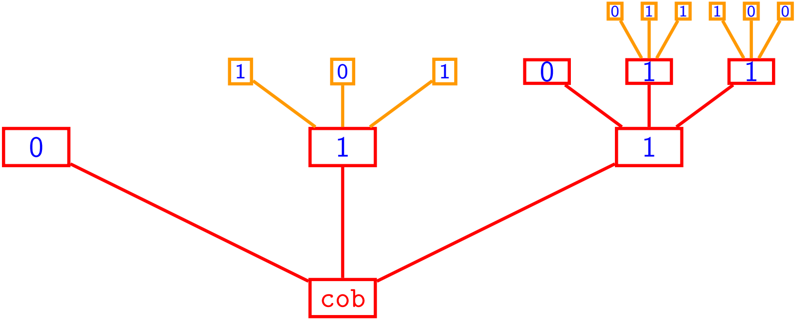


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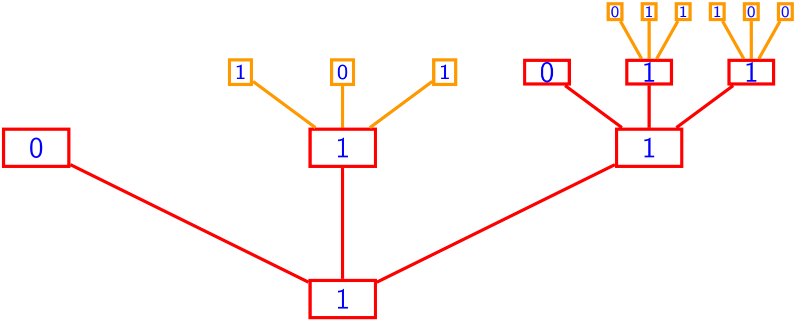


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$$X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}]\big(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}\kappa(\omega)}\big) \qquad (\mathbf{i} \in \mathbb{U}).$$

# A recursive tree representation

Define $G_{\mathbb{U}} : S^{\nabla\mathbb{U}} \to S$ by $G_{\mathbb{U}}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}\big) := X_{\varnothing}$.

$G_{\mathbb{U}}$ is the concatenation of the maps $\big(\gamma[\omega_{\mathbf{i}}]\big)_{\mathbf{i}\in\mathbb{U}}$ according to the tree structure of $\mathbb{U}$.

Let $|i_1 \cdots i_n| := n$ denote the length of a word $\mathbf{i}$ and set

$$\mathbb{S}_{(n)} := \{\mathbf{i} \in \mathbb{S} : |\mathbf{i}| < n\} \quad \text{and} \quad \nabla\mathbb{S}_{(n)} = \{\mathbf{i} \in \mathbb{S} : |\mathbf{i}| = n\}.$$

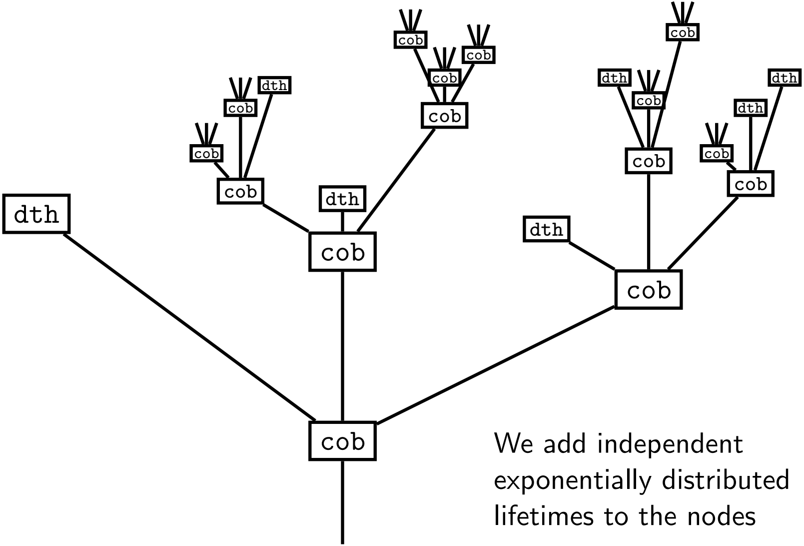
Aldous and Bandyopadyay (2005) observed that

$$\mathbf{T}^n(\mu) := \text{ the law of } G_{\mathbb{S}_{(n)}}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_{(n)}}\big),$$

where $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_{(n)}}$ are i.i.d. with law $\mu$ and independent of $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}_{(n)}}$, and

$$\mathbf{T}(\mu) := |\mathbf{r}|^{-1} \int_{\Omega} \mathbf{r}(\mathrm{d}\omega) \mathbf{T}_{\gamma[\omega]}(\mu).$$

# A recursive tree representation




We add independent exponentially distributed lifetimes to the nodes

# A recursive tree representation

Let $(\sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be i.i.d. exponentially distributed with mean $|\mathbf{r}|^{-1}$, independent of $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, and set

$$\tau^*_{\mathbf{i}} := \sum_{m=1}^{n-1} \sigma_{i_1\cdots i_m} \quad \text{and} \quad \tau^\dagger_{\mathbf{i}} := \tau^*_{\mathbf{i}} + \sigma_{\mathbf{i}} \qquad (\mathbf{i} = i_1\cdots i_n),$$
$$\mathbb{S}_t := \big\{\mathbf{i}\in\mathbb{S} : \tau^\dagger_{\mathbf{i}} \leq t\big\} \quad \text{and} \quad \nabla\mathbb{S}_t = \big\{\mathbf{i}\in\mathbb{S} : \tau^*_{\mathbf{i}} \leq t < \tau^\dagger_{\mathbf{i}}\big\}.$$

Let $\mathcal{F}_t$ be the filtration

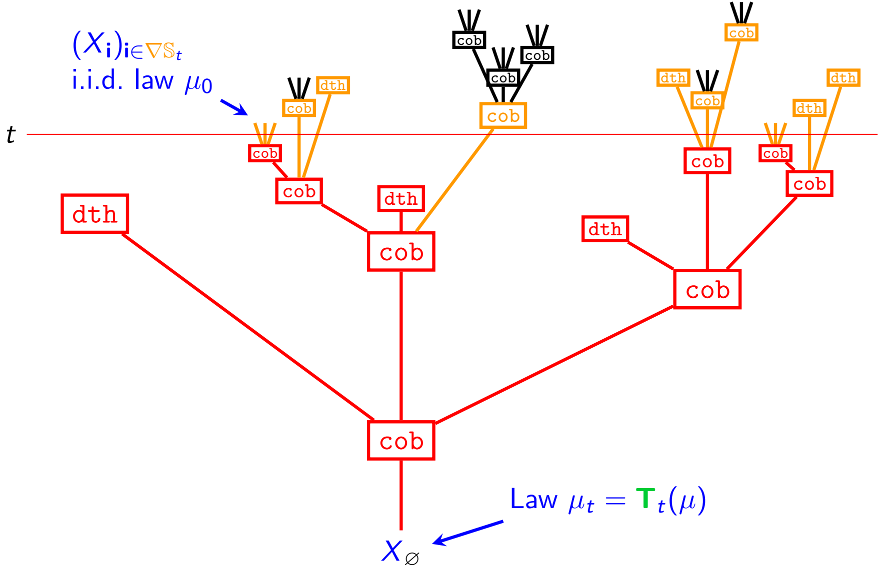
$$\mathcal{F}_t := \sigma\big(\nabla\mathbb{S}_t, (\omega_{\mathbf{i}}, \sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}_t}\big) \qquad (t \geq 0).$$

**Theorem [Mach, Sturm, S. '20]**

$$\mathbf{T}_t(\mu) := \text{ the law of } G_{\mathbb{S}_t}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}\big),$$

where $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}$ are i.i.d. with law $\mu$ and independent of $\mathcal{F}_t$.

# A recursive tree representation



# Recursive Tree Processes

A *Recursive Distributional Equation* is an equation of the form

$$X \stackrel{\mathrm{d}}{=} \gamma[\omega](X_1, \ldots, X_{\kappa(\omega)}) \qquad \text{(RDE)},$$

where $X_1, X_2, \ldots$ are i.i.d. copies of $X$, independent of $\omega$.

A law $\nu$ solves (RDE) iff

$$\text{(i)} \quad \mathbf{T}_t(\nu) = \nu \quad (t \geq 0) \qquad \text{or} \qquad \text{(ii)} \quad \mathbf{T}(\nu) = \nu.$$

We can view $\nu$ as the "invariant law" of a "Markov chain" where time has a tree-like structure.

In our example, solutions to the RDE are the Bernoulli distributions $\nu_{\rm low}$, $\nu_{\rm mid}$, $\nu_{\rm upp}$ with density $z_{\rm low}$, $z_{\rm mid}$, $z_{\rm upp}$.

# Recursive Tree Processes

For any rooted subtree $\mathbb{U} \subset \mathbb{T}$, let

$$\partial\mathbb{U} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_1 \cdots i_{n-1} \in \mathbb{U},\ i_1 \cdots i_n \notin \mathbb{U} \big\}$$

denote the boundary of $\mathbb{U}$ relative to $\mathbb{T}$.

For each solution $\nu$ of (RDE), there exists a *Recursive Tree Process (RTP)* $(\boldsymbol{\omega}_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, unique in law, such that:

(i) $(\boldsymbol{\omega}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ are i.i.d. with law $|\mathbf{r}|^{-1}\mathbf{r}$.

(ii) For finite $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(\mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}$ are i.i.d. with $\nu$ and independent of $(\boldsymbol{\omega}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$.

(iii) $\mathbf{X}_{\mathbf{i}} = \gamma[\boldsymbol{\omega}_{\mathbf{i}}]\big(\mathbf{X}_{\mathbf{i}1}, \ldots, \mathbf{X}_{\mathbf{i}\kappa(\boldsymbol{\omega}_{\mathbf{i}})}\big) \qquad (\mathbf{i} \in \mathbb{T}).$

If we add independent exponentially distributed lifetimes, then:

- Conditional on $\mathcal{F}_t$, the r.v.'s $(\mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}$ are i.i.d. with law $\nu$.

# Endogeny

Let $(\omega_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to a solution $\nu$ of the RDE.

Aldous and Bandyopadyay (2005) say that an RTP is *endogenous* if

$$\mathbf{X}_{\varnothing} \text{ is measurable w.r.t. the } \sigma\text{-field generated by } (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}.$$

They proved that endogeny is equivalent to *bivariate uniqueness*.

Warren (2005) links endogeny to *dynamical RTPs*.

Johnson, Podder & Skerman (2018) link endogeny to *pivotal vertices*.

# n-Variate processes

For each $n \geq 1$, a measurable map $g : S^k \to S$ gives rise to *n-variate map* $g^{(n)} : (S^n)^k \to S^n$ defined as

$$g^{(n)}(x_1, \ldots, x_k) = g^{(n)}(x^1, \ldots, x^n) := \big(g(x^1), \ldots, g(x^n)\big),$$

with $x = (x_i^m)_{i=1,\ldots,k}^{m=1,\ldots,n}$, $x_i = (x_i^1, \ldots, x_i^n)$, $x^m = (x_1^m, \ldots, x_k^m)$.

We define an *n-variate map*

$$\mathbf{T}^{(n)}(\mu^{(n)}) := |\mathbf{r}|^{-1} \int_\Omega \mathbf{r}(\mathrm{d}\omega) \mathbf{T}_{\gamma^{(n)}[\omega]}(\mu^{(n)}),$$

which acts on probability measures $\mu^{(n)}$ on $S^n$.

The *n-variate mean-field equation*

$$\tfrac{\partial}{\partial t}\mu_t^{(n)} = \int_\Omega \mathbf{r}(\mathrm{d}\omega)\big\{\mathbf{T}_{\gamma^{(n)}[\omega]}(\mu_t^{(n)}) - \mu_t^{(n)}\big\} \qquad (t \geq 0).$$

describes the mean-field limit of $n$ coupled processes that are constructed using the same stochastic flow $(\mathbf{X}_{s,u})_{s \leq u}$.

# n-Variate processes

| | |
|---|---|
| $\mathcal{P}(S)$ | space of probability measures on $S$. |
| $\mathcal{P}_{\rm sym}(S^n)$ | space of probability measures on $S^n$ that are symmetric under a permutation of the coordinates. |
| $S^n_{\rm diag}$ | $\{x \in S^n : x_1 = \cdots = x_n\}$ |
| $\mathcal{P}(S^n)_\mu$ | space of probability measures on $S^n$ whose one-dimensional marginals are all equal to $\mu$. |

- If $(\mu^{(n)}_t)_{t\geq 0}$ solves the $n$-variate equation, then its $m$-dimensional marginals solve the $m$-variate equation.
- $\mu^{(n)}_0 \in \mathcal{P}_{\rm sym}(S^n)$ implies $\mu^{(n)}_t \in \mathcal{P}_{\rm sym}(S^n)$ $(t \geq 0)$.
- $\mu^{(n)}_0 \in \mathcal{P}(S^n_{\rm diag})$ implies $\mu^{(n)}_t \in \mathcal{P}(S^n_{\rm diag})$ $(t \geq 0)$.
- If $\mathbf{T}(\nu) = \nu$, then $\mu^{(n)}_0 \in \mathcal{P}(S^n)_\nu$ implies $\mu^{(n)}_t \in \mathcal{P}(S^n)_\nu$.

# n-Variate processes

If $\nu = \mathbb{P}[X \in \cdot\,]$ solves the RDE $\mathbf{T}(\nu) = \nu$, then

$$\overline{\nu}^{(n)} := \mathbb{P}\big[(\underbrace{X, \ldots, X}_{n \text{ times}}) \in \cdot\,\big]$$

solves the *n-variate RDE* $\mathbf{T}^{(n)}(\nu^{(n)}) = \nu^{(n)}$.

Questions:

- Is $\overline{\nu}^{(n)}$ a stable fixed point of the *n*-variate equation?
- Is $\overline{\nu}^{(n)}$ the only solution in $\mathcal{P}_{\rm sym}(S^n)_\nu$ of the *n*-variate RDE?

# n-Variate processes

Recall that an RTP $(\boldsymbol{\omega}_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to a solution $\nu$ of the RDE is endogenous if

$$\mathbf{X}_{\varnothing} \text{ is measurable w.r.t. the } \sigma\text{-field generated by } (\boldsymbol{\omega}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}.$$

**Theorem [AB '05 & MSS '18]** The following statements are equivalent:

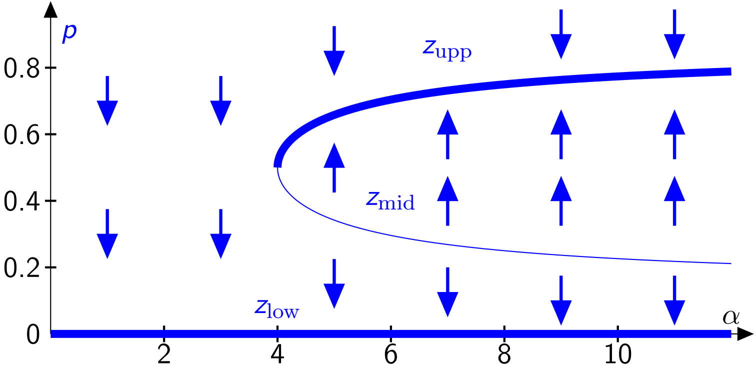
(i) The RTP corresponding to $\nu$ is endogenous.

(ii) $\mathbf{T}_t^{(n)}(\mu) \underset{t\to\infty}{\Longrightarrow} \overline{\nu}^{(n)}$ for all $\mu \in \mathcal{P}(S^n)_\nu$ and $n \geq 1$.

(iii) $\overline{\nu}^{(2)}$ is the only solution in $\mathcal{P}_{\rm sym}(S^2)_\nu$ of the bivariate RDE.

In our example, the RTPs for $\nu_{\rm low}, \nu_{\rm upp}$ are endogenous, but the RTP corresponding to $\nu_{\rm mid}$ is not.

# n-Variate processes




Fixed points of $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ for different values of $\alpha$.

*Cooperative branching with branching rate* $\alpha > 4$

The RDE $\mathbf{T}(\nu) = \nu$ has three solutions $\nu_{\rm low}, \nu_{\rm mid}$, and $\nu_{\rm upp}$, where $\nu_{\dots}$ is the probability measure on $\{0,1\}$ with mean $\nu_{\dots}(\{1\}) = z_{\dots}$ $(\dots = {\rm low}, {\rm mid}, {\rm upp})$, which

give rise to solutions $\overline{\nu}^{(2)}_{\rm low}, \overline{\nu}^{(2)}_{\rm mid}$, and $\overline{\nu}^{(2)}_{\rm upp}$ of the *bivariate RDE*.

**Proposition [Mach, Sturm, S. '20]** Apart from $\overline{\nu}^{(2)}_{\rm low}, \overline{\nu}^{(2)}_{\rm mid}, \overline{\nu}^{(2)}_{\rm upp}$, the *bivariate RDE* has one more solution $\underline{\nu}^{(2)}_{\rm mid}$ in $\mathcal{P}_{\rm sym}(S^2)$. The domains of attraction are:

$$
\begin{array}{ll}
\overline{\nu}^{(2)}_{\rm low}: & \big\{\mu^{(2)}_0 : \mu^{(1)}_0(\{1\}) < z_{\rm mid}\big\}, \\
\underline{\nu}^{(2)}_{\rm mid}: & \big\{\mu^{(2)}_0 : \mu^{(1)}_0(\{1\}) = z_{\rm mid},\ \mu^{(2)}_0 \neq \overline{\nu}^{(2)}_{\rm mid}\big\}, \\
\overline{\nu}^{(2)}_{\rm mid}: & \big\{\overline{\nu}^{(2)}_{\rm mid}\big\}, \\
\overline{\nu}^{(2)}_{\rm upp}: & \big\{\mu^{(2)}_0 : \mu^{(1)}_0(\{1\}) > z_{\rm mid}\big\}.
\end{array}
$$

# The higher-level equation

The $n$-variate map $\mathbf{T}^{(n)}$ is defined even for $n = \infty$, and $\mathbf{T}^{(\infty)}$ maps $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$ into itself.

By De Finetti's theorem, $(X_i)_{i\in\mathbb{N}_+}$ have a law in $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$ if and only if there exists a random probability measure $\xi$ on $S$ such that conditional on $\xi$, the $(X_i)_{i\in\mathbb{N}_+}$ are i.i.d. with law $\xi$.

Let $\rho := \mathbb{P}[\xi \in \cdot\,]$ the law of $\xi$. Then $\rho \in \mathcal{P}(\mathcal{P}(S))$.
In view of this, $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+}) \cong \mathcal{P}(\mathcal{P}(S))$.

The map $\mathbf{T}^{(\infty)} : \mathcal{P}_{\rm sym}(S^{\mathbb{N}_+}) \to \mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$
corresponds to a *higher-level map* $\check{\mathbf{T}} : \mathcal{P}(\mathcal{P}(S)) \to \mathcal{P}(\mathcal{P}(S))$.

# The higher-level equation

For any measurable map $g : S^k \to S$, define $\check{g} : \mathcal{P}(S)^k \to \mathcal{P}(S)$ by

$$\check{g} := \text{ the law of } g(X_1, \ldots, X_k), \quad \text{where } (X_1, \ldots, X_k) \text{ are independent with laws } \mu_1, \ldots, \mu_k.$$

Then

$$\check{\mathbf{T}}(\rho) := \text{ the law of } \check{\gamma}[\boldsymbol{\omega}](\xi_1, \ldots, \xi_{\kappa(\boldsymbol{\omega})}),$$

with $\boldsymbol{\omega}$ as before and $\xi_1, \xi_2, \ldots$ i.i.d. with law $\rho$.

Define *n-th moment measures*

$$\rho^{(n)} := \mathbb{E}\big[\underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}}\big] \quad \text{where } \xi \text{ has law } \rho.$$

**Proposition [MSS '20]** If $(\rho_t)_{t \geq 0}$ solves the *higher-level mean-field equation*, then its $n$-th moment measures $(\rho_t^{(n)})_{t \geq 0}$ solve the $n$-variate equation.

# The higher-level equation

Equip $\mathcal{P}(\mathcal{P}(S))_\nu = \{\rho : \rho^{(1)} = \nu\}$ with the *convex order*

$$\rho_1 \leq_{\rm cv} \rho_2 \quad \text{iff} \quad \int \phi \, \mathrm{d}\rho_1 \leq \int \phi \, \mathrm{d}\rho_2 \quad \forall \text{ convex } \phi.$$

**[Strassen '65]** $\rho_1 \leq_{\rm cv} \rho_2$ iff there exist a r.v. $X$ with law $\nu$ and $\sigma$-fields $\mathcal{H}_1 \subset \mathcal{H}_2$ s.t. $\rho_i = \mathbb{P}\big[\mathbb{P}[X \in \cdot \,|\mathcal{H}_i] \in \cdot\,\big]$ $(i = 1, 2)$.

Maximal and minimal elements: $\mathcal{H}_1 = \{\Omega, \emptyset\} \;\Rightarrow\; \rho_1 = \delta_\nu$.
$\mathcal{H}_2 = \sigma(X) \;\Rightarrow\; \rho_2 = \overline{\nu} := \mathbb{P}[\delta_X \in \cdot\,]$ with $\mathbb{P}[X \in \cdot\,] = \nu$.

$$\delta_\nu \leq_{\rm cv} \rho \leq_{\rm cv} \overline{\nu} \qquad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$

**Proposition [MSS '18]** $\check{\mathsf{T}}$ is monotone w.r.t. the convex order. There exists a solution $\underline{\nu}$ to the higher-level RDE s.t.

$$\check{\mathsf{T}}^n(\delta_\nu) \underset{n\to\infty}{\Longrightarrow} \underline{\nu} \quad \text{and} \quad \check{\mathsf{T}}_t(\delta_\nu) \underset{t\to\infty}{\Longrightarrow} \underline{\nu}$$

and any solution $\rho \in \mathcal{P}(\mathcal{P}(S))_\nu$ to the higher-level RDE satisfies

$$\underline{\nu} \leq_{\rm cv} \rho \leq_{\rm cv} \overline{\nu} \qquad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$

# The higher-level equation

**Proposition [MSS '18]**
Let $(\omega_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to $\gamma$ and $\nu$. Set

$$\xi_{\mathbf{i}} := \mathbb{P}\big[X_{\mathbf{i}} \in \cdot \,\big|\, (\omega_{\mathbf{ij}})_{\mathbf{j}\in\mathbb{T}}\big].$$

Then $(\omega_{\mathbf{i}}, \xi_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to $\check{\gamma}$ and $\underline{\nu}$.
Also, $(\omega_{\mathbf{i}}, \delta_{X_{\mathbf{i}}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to $\check{\gamma}$ and $\overline{\nu}$.

**Corollary** The RTP is endogenous iff $\underline{\nu} = \overline{\nu}$.

# The higher-level equation

**Theorem [Mach, Sturm, S. '20]** One has

$$\underline{\nu}_{\rm low} = \overline{\nu}_{\rm low}, \quad \underline{\nu}_{\rm upp} = \overline{\nu}_{\rm upp}, \quad \text{but} \quad \underline{\nu}_{\rm mid} \neq \overline{\nu}_{\rm mid}.$$

These are all solutions to the higher-level RDE.
Any solution $(\rho_t)_{t\geq 0}$ to the higher-level mean-field equation converges to one of these fixed points.
The domains of attraction are:

$$\begin{array}{ll} \overline{\nu}_{\rm low}: & \big\{\rho_0 : \rho_0^{(1)}(\{1\}) < z_{\rm mid}\big\}, \\ \underline{\nu}_{\rm mid}: & \big\{\rho_0 : \rho_0^{(1)}(\{1\}) = z_{\rm mid},\ \rho_0 \neq \overline{\nu}_{\rm mid}\big\}, \\ \overline{\nu}_{\rm mid}: & \big\{\overline{\nu}_{\rm mid}\big\}, \\ \overline{\nu}_{\rm upp}: & \big\{\rho_0 : \rho_0^{(1)}(\{1\}) > z_{\rm mid}\big\}. \end{array}$$

# The higher-level equation

The map $\mu \mapsto \mu(\{1\})$ defines a bijection $\mathcal{P}(\{0,1\}) \cong [0,1]$, and hence $\mathcal{P}(\mathcal{P}(\{0,1\})) \cong \mathcal{P}[0,1]$.

Then the higher-level RDE takes the form

$$\eta \stackrel{\mathrm{d}}{=} \chi \cdot \big(\eta_1 + (1-\eta_1)\eta_2\eta_3\big),$$

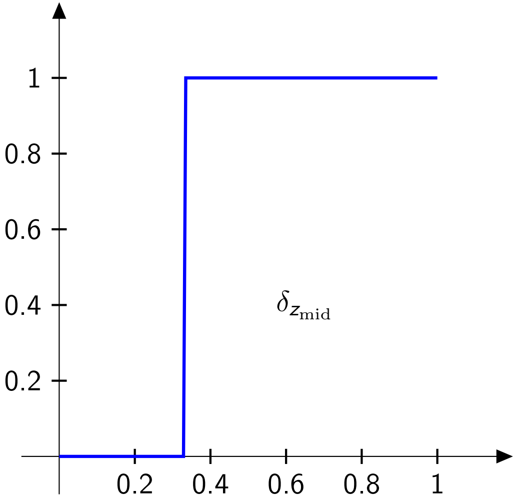
where $\eta$ takes values in $[0,1]$, $\eta_1, \eta_2, \eta_3$ are independent copies of $\eta$ and $\chi$ is an independent Bernoulli r.v. with $\mathbb{P}[\chi = 1] = \alpha/(\alpha+1)$.

This RDE has three "trivial" solutions

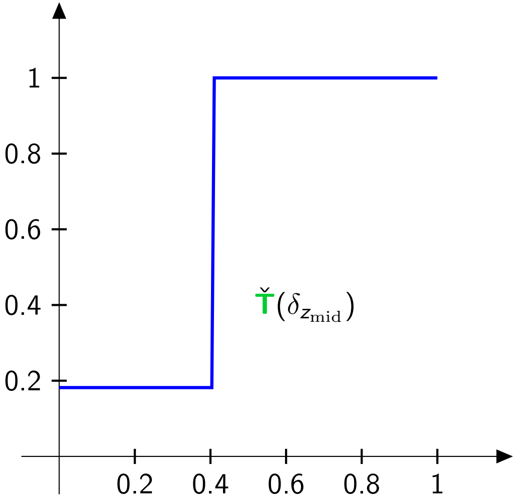
$$\overline{\nu}_{\ldots} = (1 - z_{\ldots})\delta_0 + z_{\ldots}\delta_1 \qquad (\ldots = \mathrm{low}, \mathrm{mid}, \mathrm{upp}),$$

and a nontrivial solution

$$\underline{\nu}_{\mathrm{mid}} = \lim_{n\to\infty} \check{\mathsf{T}}^n(\delta_{z_{\mathrm{mid}}}).$$


1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
$\delta_{z_{\text{mid}}}$

# Numerical results



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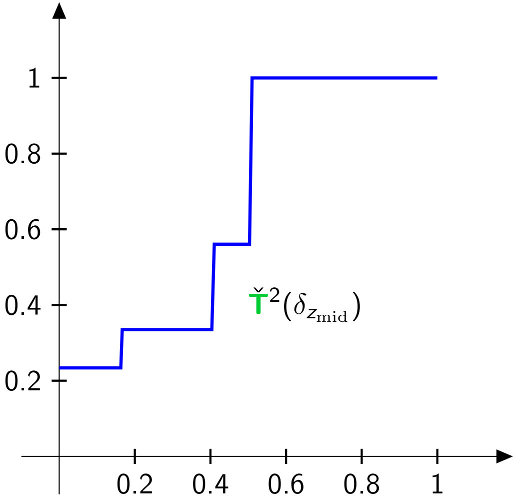

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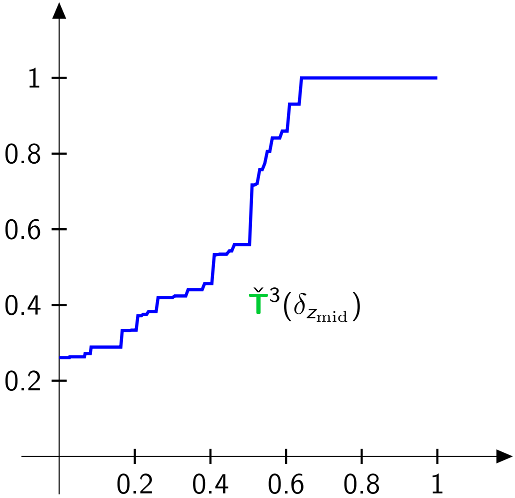

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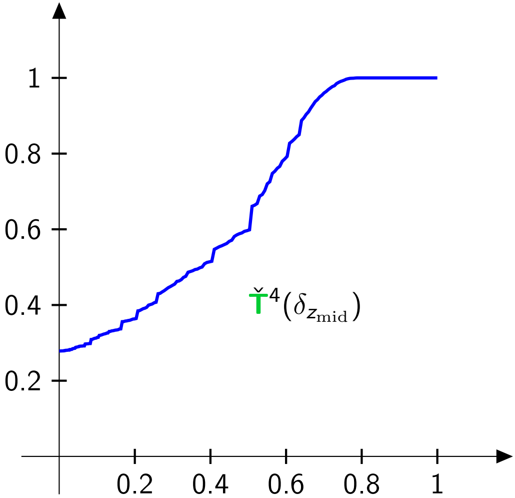

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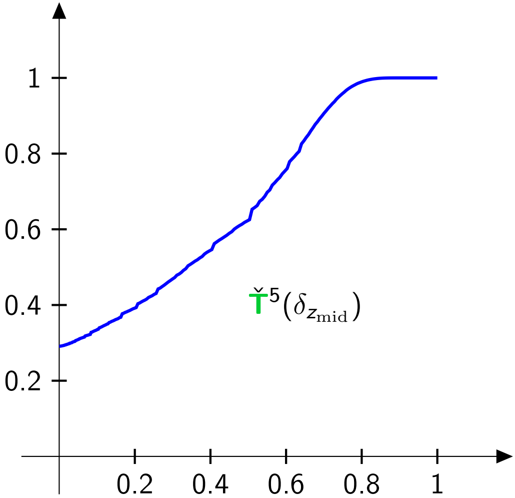

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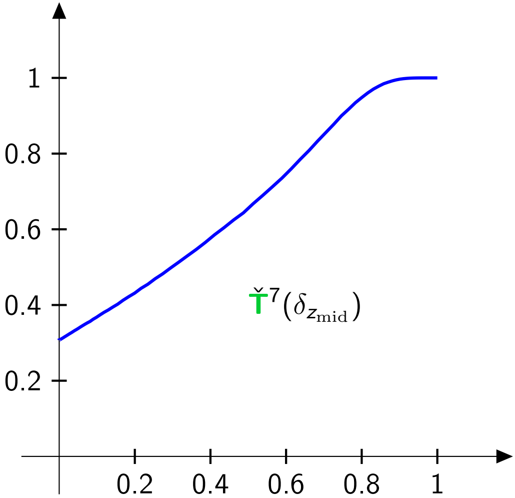

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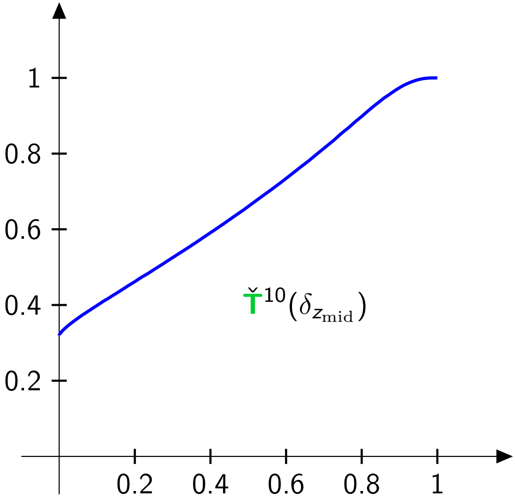

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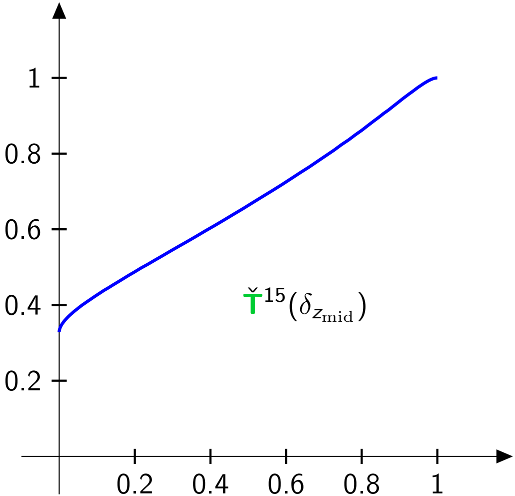

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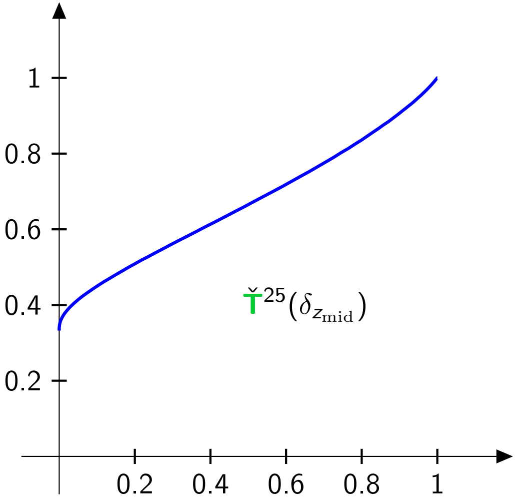

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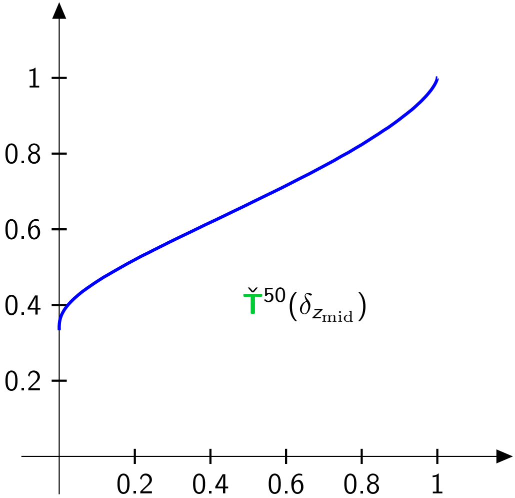

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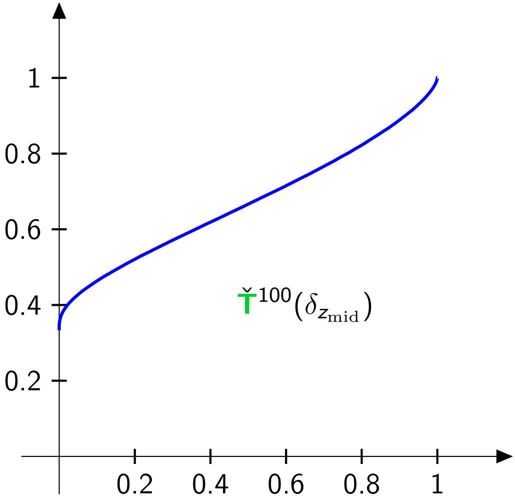

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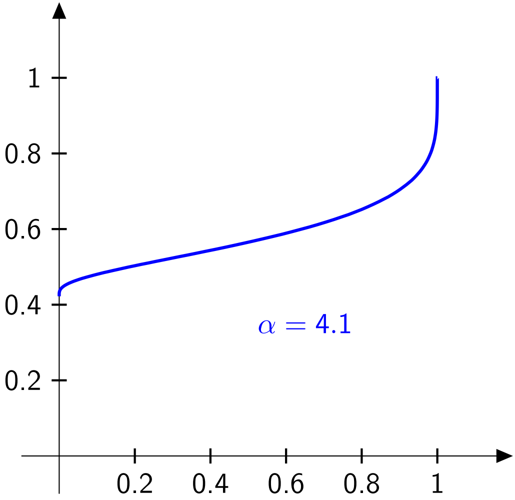

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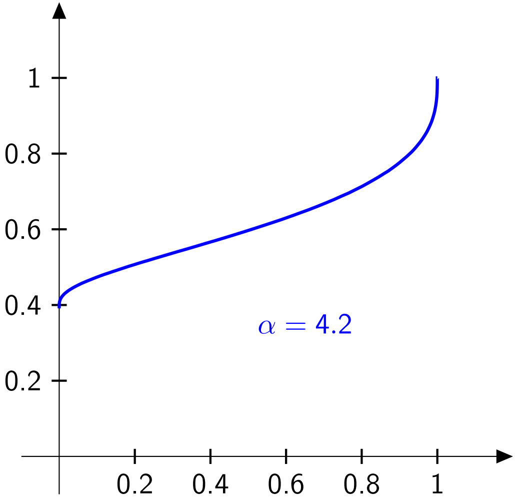

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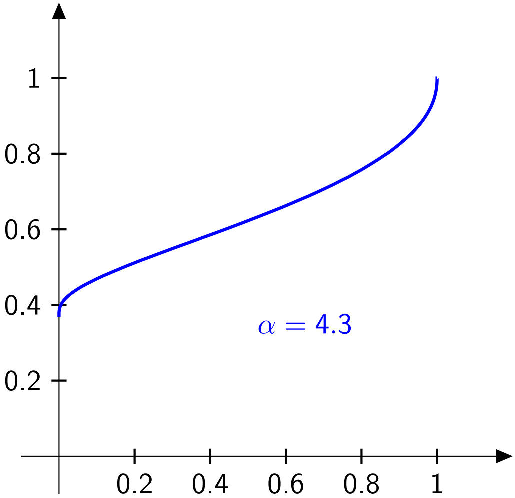

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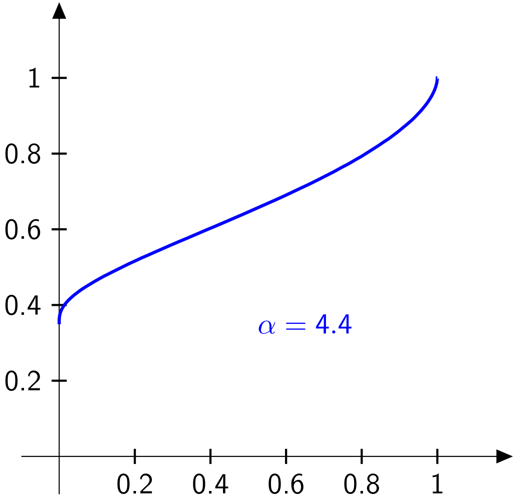

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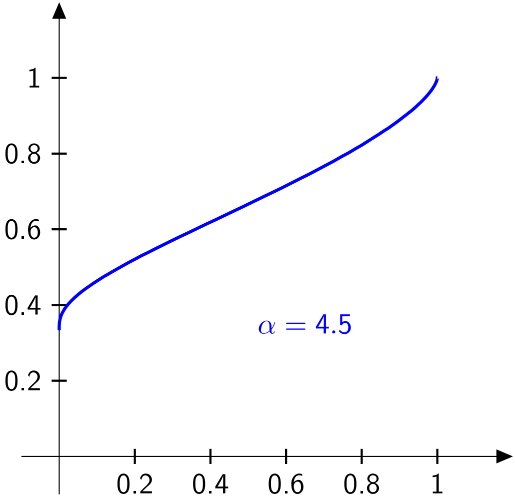

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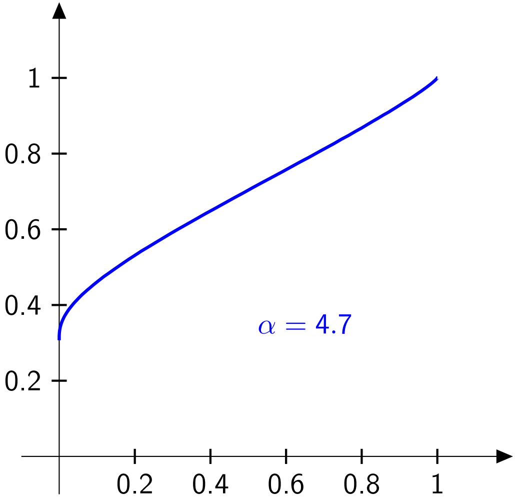

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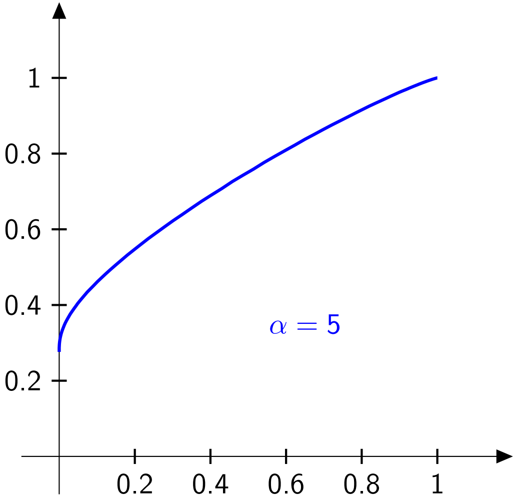

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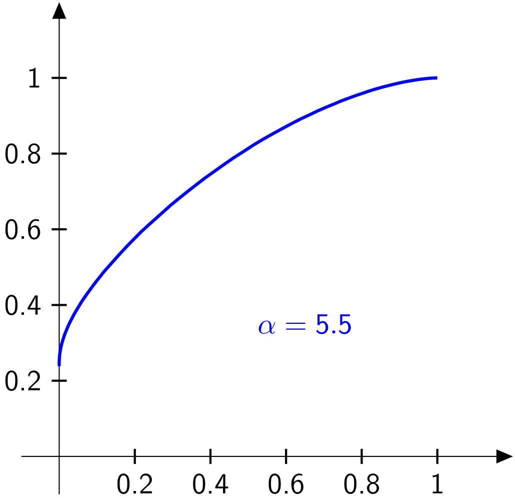

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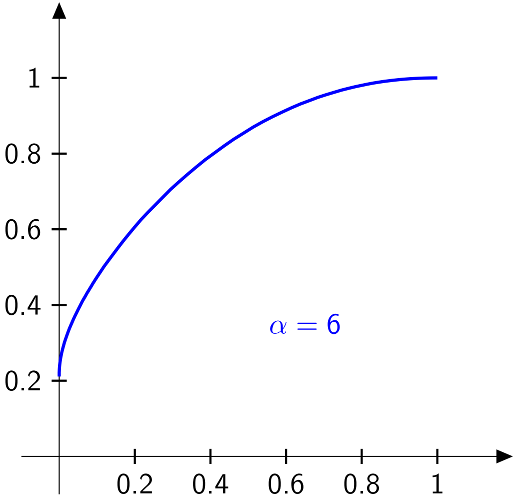

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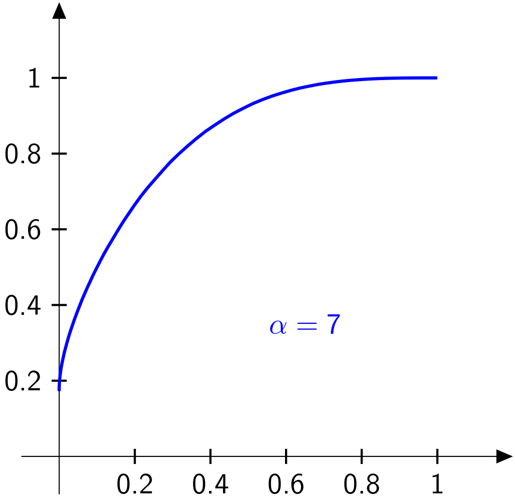

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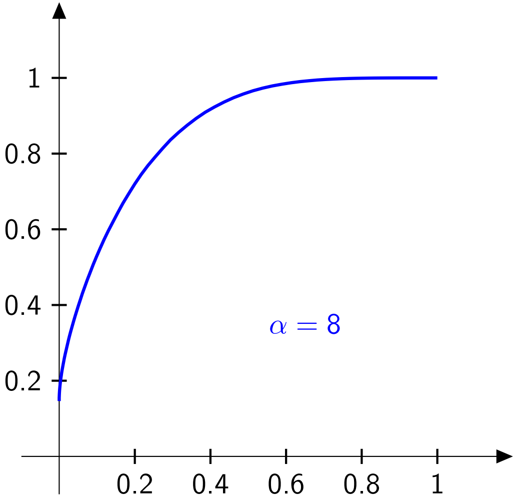

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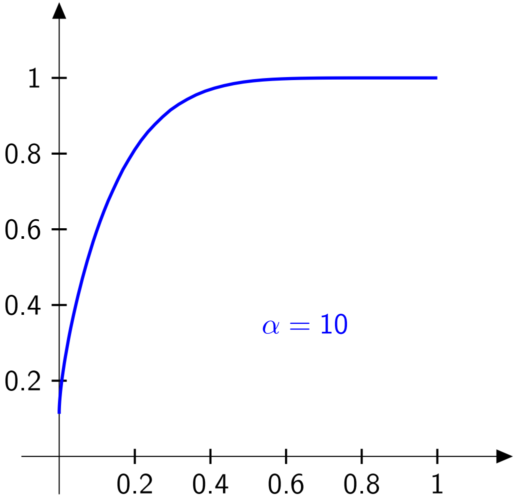

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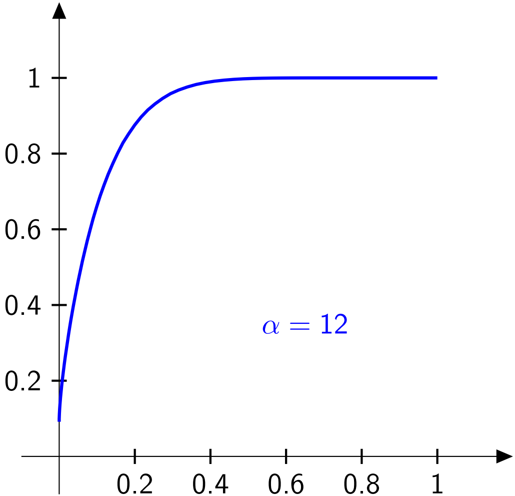

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