Stochastic flows in the Brownian web and net

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joint with Emmanuel Schertzer and Rongfeng Sun

Outline

- Random motion in a random space-time environment

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- Construction using a marked Brownian web

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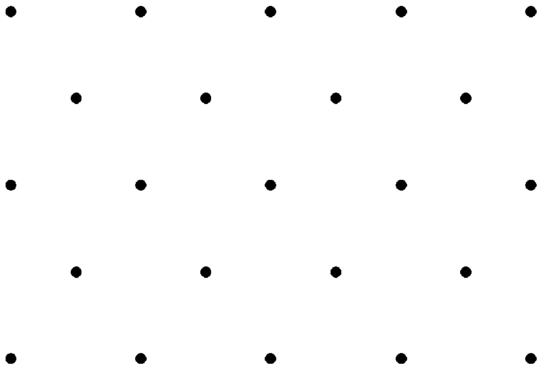
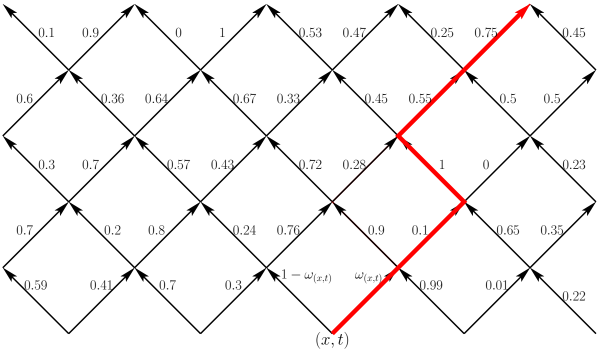
- Construction using a marked Brownian web
- The Brownian net

The odd integer lattice



Let $\mathbb{Z}^2_{\text{even}} := \{(x,t) \in \mathbb{Z}^2 : x + t \text{ is even}\}$.
Interpretation: x is space, t is time (upwards).

Random walk in random space-time environment



Fix a probability law μ on $[0,1]$.
Let $(\omega_z)_{z\in\mathbb{Z}^2_{\rm even}}$ be i.i.d. $[0,1]$-valued r.v.'s with law μ.

Random walk in random space-time environment

Conditional on $\omega = (\omega_z)_{z \in \mathbb{Z}^2_{\rm even}}$, let $p_{(x,s)} = (p(t))_{t \geq s}$ be a random walk in the random environment ω, started at time s in $p(s) = x$, such that

$$p(t+1) = \begin{cases} p(t) + 1 & \text{with probability } \omega_z, \\ p(t) - 1 & \text{with probability } 1 - \omega_z \end{cases}$$

We call $\mathbf{Q}^\omega_{(x,s)} := \mathbf{P}\big[(p_{(x,s)}(t))_{t \geq s} \in \cdot \,\big|\, \omega\big]$ the *quenched law* and $\int \mathbf{P}(\mathrm{d}\omega)\mathbf{Q}^\omega_{(x,s)}$ the *averaged law* of X.

Observation: Under the averaged law, $p_{(x,s)}$ is a random walk that jumps to the right (resp. left) with probability $\int \mu(\mathrm{d}q)q$ (resp. $\int \mu(\mathrm{d}q)(1-q)$).

Diffusive scaling limit

Let $Z_\varepsilon := \{(\varepsilon x, \varepsilon^2 s) : (x,s) \in \mathbb{Z}^2_{\rm even}\}$. Let $\varepsilon_k \to 0$ and let $\mathbf{Q}^{(k)}_{(x,s)}$ be the quenched law of

$$p^{(k)}_{(x,s)}(t) := p^k_{(\varepsilon_k^{-1}x, \varepsilon_k^{-2}s)}(\varepsilon_k^{-2}t) \qquad \big((x,s) \in Z_{\varepsilon_k}\big),$$

where p^k is a random walk in a random environment ω^k with law μ_k satisfying:

$$\begin{aligned} &\text{(i)} \quad \varepsilon_k^{-1} \int 2(q - \tfrac{1}{2})\mu_k(\mathrm{d}q) \underset{k\to\infty}{\longrightarrow} \beta, \\ &\text{(ii)} \quad \varepsilon_k^{-1} q(1-q)\mu_k(\mathrm{d}q) \underset{k\to\infty}{\Longrightarrow} \nu(\mathrm{d}q), \end{aligned}$$

for some $\beta \in \mathbb{R}$ and ν a finite measure on $[0,1]$. Then $\big(\mathbf{Q}^{(k)}_z\big)_{z \in Z_{\varepsilon_k}}$ converges in law to a collection $(\mathbb{Q}_z)_{z\in\mathbb{R}^2}$ of random probability laws describing a Markov process in a random environment.

n-point motions

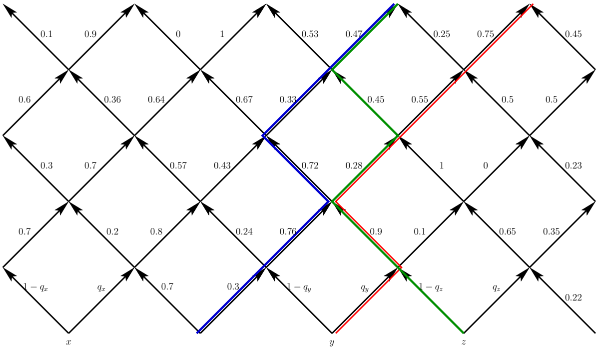
Conditional on the random environment ω, let $p_1, \ldots, p_n$ be independent random walks started from $x_1, \ldots, x_n$.

Observation Under the averaged law $\int \mathbf{P}(\mathrm{d}\omega)\mathbf{Q}^{\omega}_{(x_1,0)} \cdots \mathbf{Q}^{\omega}_{(x_n,0)}$, the process $\big(p_1(t), \ldots, p_n(t)\big)_{t\geq 0}$ is a Markov chain:

discrete n-point motion.

In the continuum limit, $\int \mathbb{P}(\mathrm{d}\omega)\mathbb{Q}^{\omega}_{(x_1,0)} \cdots \mathbb{Q}^{\omega}_{(x_n,0)}$ is the law of a collection of Brownian motions with drift β and a form of sticky interaction described by ν.

n-point motions



An n-tuple of random walks, conditionally independent given the random environment $(\omega_z)_{z\in\mathbb{Z}^2_{\rm even}}$.

Howitt-Warren martingale problem

A continuous process $\vec{\pi}(t) = (\pi_1(t), \ldots, \pi_n(t))$ solves the Howitt-Warren martingale problem if the covariance process between π_i and π_j is given by

$$\langle \pi_i, \pi_j \rangle(t) = \int_0^t 1_{\{\pi_i(s) = \pi_j(s)\}} \mathrm{d}s \qquad (t \geq 0,\ i, j = 1, \ldots, n),$$

and, for each nonempty $\Delta \subset \{1, \ldots, n\}$,

$$f_\Delta\big(\vec{\pi}(t)\big) - \int_0^t \beta_+\big(g_\Delta(\vec{\pi}(s))\big) \mathrm{d}s$$

is a martingale with respect to the filtration generated by $\vec{\pi}$, where $\beta_+(m) := \beta + 2 \int \nu(\mathrm{d}q) \sum_{k=0}^{m-2} (1-q)^k$ and

$$f_\Delta(\vec{x}) := \max_{i \in \Delta} x_i \qquad \text{and} \qquad g_\Delta(\vec{x}) := \Big|\{i \in \Delta : x_i = f_\Delta(\vec{x})\}\Big|.$$

Stochastic flow of kernels

Setting

$$K_{s,t}(x,\mathrm{d}y) := \mathbb{Q}_{(x,s)}[\pi(t)\in\mathrm{d}y]$$

defines a collection of random probability kernels $(K_{s,t})_{s\leq t}$ on $\mathbb{R}$ satisfying:

(i) $K_{s,s}(x,\,\cdot\,)=\delta_x$ and $\displaystyle\int K_{s,t}(x,\mathrm{d}y)K_{t,u}(y,\,\cdot\,)=K_{s,u}(x,\,\cdot\,)$ a.s. for all $s\leq t\leq u$ and $x\in\mathbb{R}$.

(ii) For each $t_0<\cdots<t_n$, the random probability kernels $(K_{t_{i-1},t_i})_{i=1,\ldots,n}$ are independent.

(iii) $K_{s,t}$ and $K_{s+u,t+u}$ are equal in finite-dimensional distributions for each real $s\leq t$ and u.

Previous work

Theorem [Le Jan & Raimond '04]
Any consistent family of Feller processes on a compact metrizable space defines a "stochastic flow of kernels".

Theorem [Le Jan & Raimond '04]
Construction of the stochastic flow of kernels with $\beta = 0$ and $\nu(\mathrm{d}q) = \mathrm{d}q$ via its n-point motions, which are reversible, using Dirichlet form techniques.

Theorem [Howitt & Warren '06]
The discrete n-point motions, diffusively rescaled, converge to an $\mathbb{R}^n$-valued Markov process $\big(\pi_1(t), \ldots, \pi_n(t)\big)_{t \geq 0}$ given by a well-posed martingale problem.

A measure-valued process

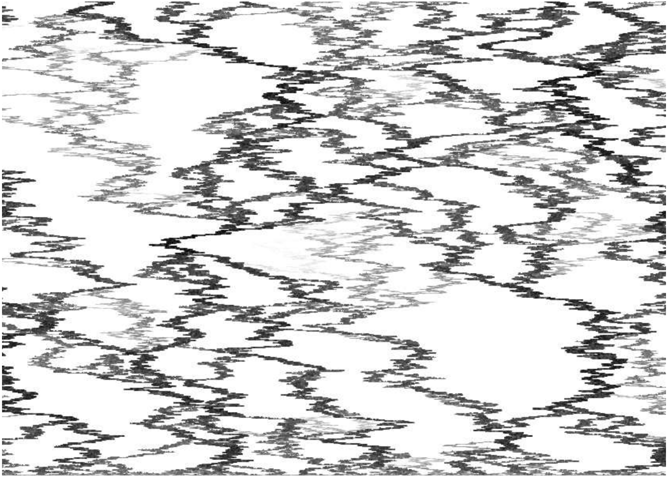
Let ρ_0 be a probability law on $\mathbb{R}$. Then

$$\begin{aligned}\rho_t &:= \int \rho_0(\mathrm{d}x) K_{0,t}(x,\,\cdot\,) \\ &= \int \rho_0(\mathrm{d}x) \mathbb{Q}_{(x,s)}[\pi(t) \in \cdot\,]\end{aligned}$$

defines a Markov process $(\rho_t)_{t\geq 0}$ taking values in the probability measures on $\mathbb{R}$.

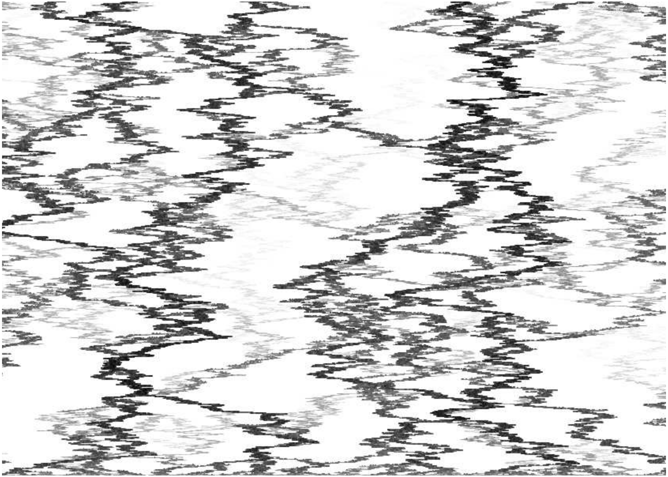
ρ_t is the random law at time t of a process π in a random environment ω, started in the initial law $\mathbb{P}[\pi(0) \in \cdot \,|\, \omega] = \rho_0$.

Howitt-Warren flows



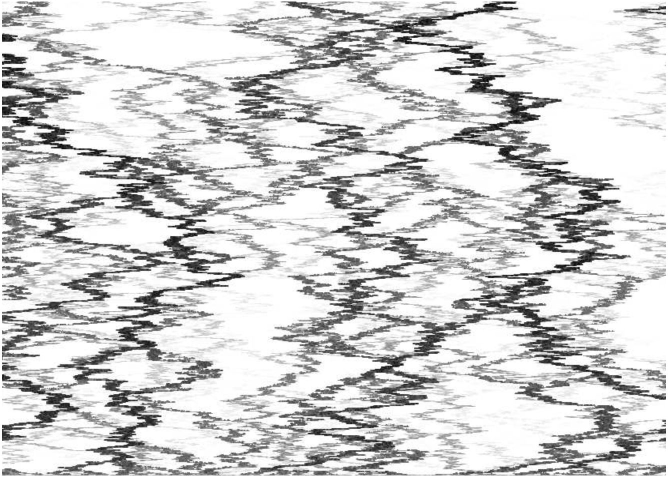
The equal splitting flow: $\beta = 0$ and $\nu = \delta_{1/2}$.

Howitt-Warren flows



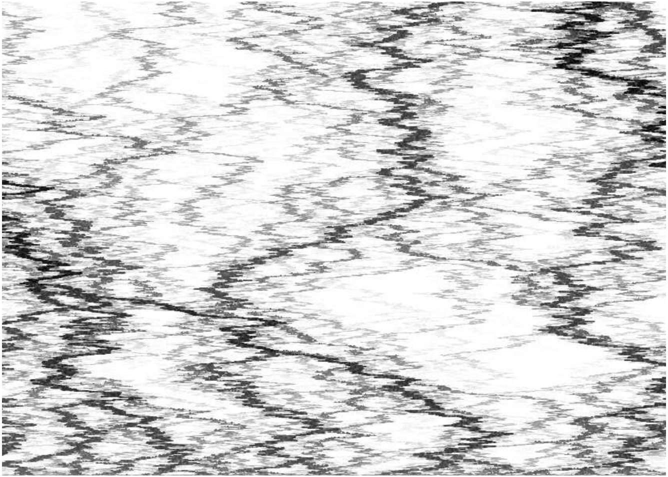
The process with $\beta = 0$ and $\nu(\mathrm{d}q) = 6q(1-q)\mathrm{d}q$.

Howitt-Warren flows



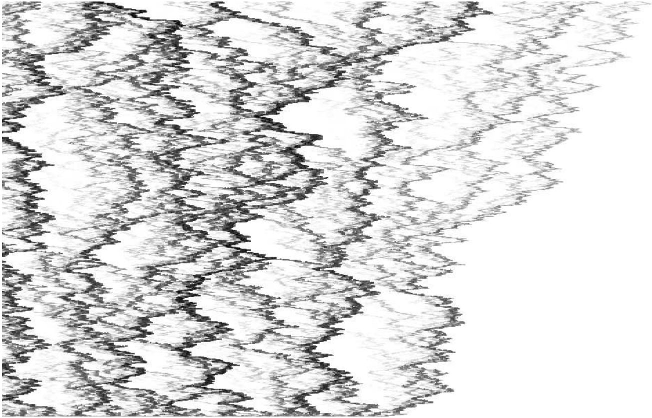
Le Jan-Raimond flow: $\beta = 0$ and $\nu(\mathrm{d}q) = \mathrm{d}q$ (reversible!).

Howitt-Warren flows



The erosion flow: $\beta = 0$ and $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

Howitt-Warren flows



One-sided erosion flow: $\beta = 0$ and $\nu = \delta_1$.

Left and right speed

Theorem [E. Schertzer, R. Sun & J.S. '10]
Set $\beta_+ := \beta + 2\int q^{-1}\nu(\mathrm{d}q)$. Assume $\sup(\mathrm{support}(\rho_0)) < \infty$.

(i) If $\beta_+ < \infty$, then $r_t := \sup(\mathrm{support}(\rho_t))$ is a Brownian motion with drift β_+.

(ii) If $\beta_+ = \infty$, then $\sup(\mathrm{support}(\rho_t)) = \infty$ for all $t > 0$.

Analogue statements hold for $\inf(\mathrm{support}(\rho_t))$, with β_+ replaced by $\beta_- := \beta - 2\int(1-q)^{-1}\nu(\mathrm{d}q)$.

Branching-coalescing point set

Theorem [E. Schertzer, R. Sun & J.S. '10] Assume $-\infty < \beta_- < \beta_+ < \infty$. Then

$$\xi_t := \mathrm{support}(\rho_t) \qquad (t \geq 0)$$

is a Markov process taking values in the closed subsets of $\mathbb{R}$.

(i) Reversible invariant law: the law of a Poisson point set with intensity $\beta_+ - \beta_-$.

(ii) For deterministic $t > 0$, a.s. ξ_t is a locally finite subset of $\mathbb{R}$.

(iii) There exists a dense set of random times $\tau > 0$ such that ξ_τ has no isolated points.

If $\beta_+ - \beta_- = \infty$ then $\mathrm{support}(\rho_t) = (-\infty, r_t]$ or $[l_t, \infty)$ or $\mathbb{R}$.

Atomicness

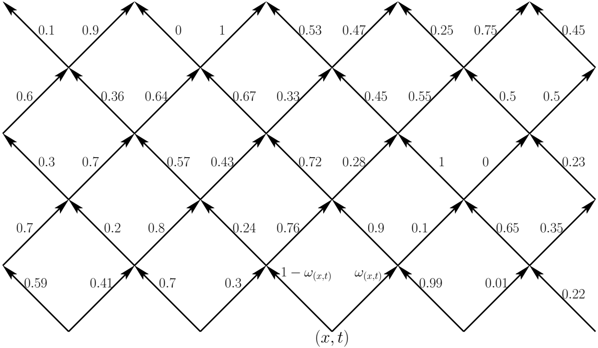
Theorem [E. Schertzer, R. Sun & J.S. '10]

(i) ρ_t is purely atomic at each deterministic $t > 0$.

(ii) If $\int_{(0,1)} \nu(\mathrm{d}q) > 0$, then there exists a dense set of random times $\tau > 0$ at which ρ_τ is purely nonatomic.

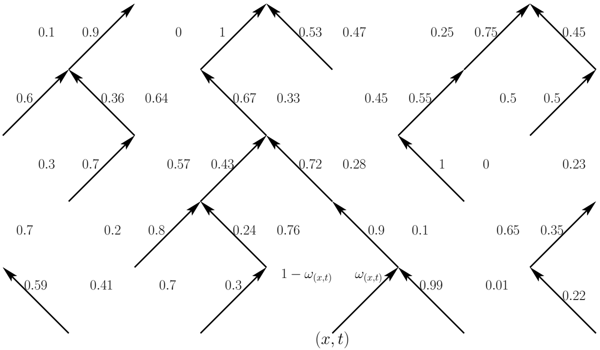
(iii) If $\int_{(0,1)} \nu(\mathrm{d}q) = 0$, then ρ_t is purely atomic at each $t \geq 0$ a.s.

Random environment



Conditional on the random environment $\omega = (\omega_z)_{z\in\mathbb{Z}^2_{\rm even}}$ ...

Discrete web



. . . we choose independent $\alpha = (\alpha_z)_{z\in\mathbb{Z}^2_{\rm even}}$ such that

$$\mathbb{P}[\alpha_z = +1 \,|\, \omega] = \omega_z.$$

A quenched law on the space of webs

Let $p^\alpha_{(y,s)} = p$ be the unique path started at $p(s) = y$ such that

$$p(t+1) = p(t) + \alpha_{(p(t),t)} \qquad (t \geq s).$$

Consider the collection of coalescing paths

$$\mathcal{U}^\alpha := \{p^\alpha_z : z \in \mathbb{Z}^2_{\rm even}\}.$$

We call

$$\mathbf{Q}^\omega := \mathbb{P}\big[\mathcal{U}^\alpha \in \cdot \,\big|\, \omega\big]$$

the *quenched law* of $\mathcal{U}^\alpha$. Then $\mathbf{Q}[p^\alpha_z \in \cdot\,] = \mathbf{Q}_z$.

Under the *averaged law* $\int \mathbf{P}(\mathrm{d}\omega)\mathbf{Q}^\omega$, the collection of paths $\mathcal{U}^\alpha$ is a *discrete web*.

Constuction based on a reference web

Let $\alpha^0 = (\alpha^0_z)_{z\in\mathbb{Z}^2_{\rm even}}$ be a 'reference' collection of i.i.d. $\{-1,+1\}$-valued random variables with $\mathbf{P}[\alpha^0_z = +1] = \theta_0$.

Conditional on α^0, let $\omega = (\omega_z)_{z\in\mathbb{Z}^2_{\rm even}}$ be a collection of independent $[0,1]$-valued random variables with

$$\mathbf{P}[\omega_z \in {\rm d}q \,|\, \alpha^0_z = -1] = \mu_{\rm l}({\rm d}q), \quad \mathbf{P}[\omega_z \in {\rm d}q \,|\, \alpha^0_z = +1] = \mu_{\rm r}({\rm d}q).$$

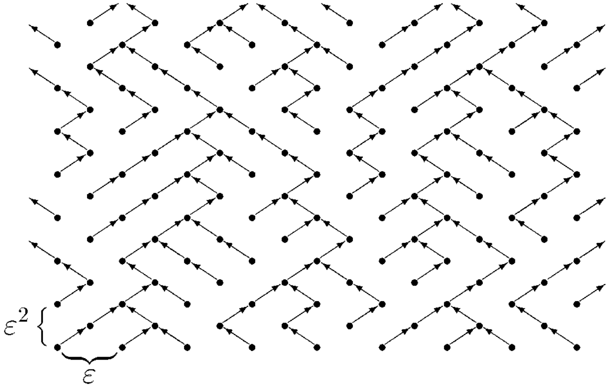
Conditional on (α^0,ω), let $\alpha = (\alpha_z)_{z\in\mathbb{Z}^2_{\rm even}}$ be a collection of independent $\{-1,+1\}$-valued random variables with

$$\mathbf{P}[\alpha_z = +1 \,|\, (\alpha^0,\omega)] = \omega_z.$$

Let $\mathcal{U}^0, \mathcal{U}$ be the *reference web* and *sample web* associated with α^0, α. Then $\mathbf{Q} = \mathbf{P}[\mathcal{U} \in \cdot \,|\, (\mathcal{U}^0,\omega)]$ is the quenched law with

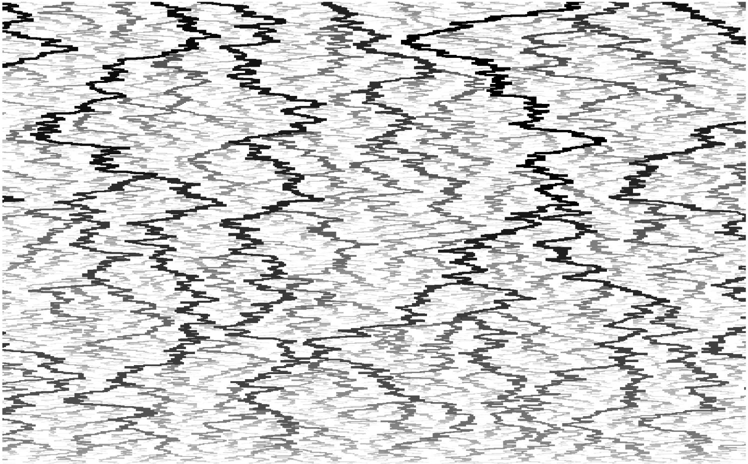
$$\mu = (1-\theta_0)\mu_{\rm l} + \theta_0\mu_{\rm r}.$$

Diffusive scaling



Rescaling discrete webs with $\mathbb{P}[\alpha_z = +1] = \frac{1}{2}(1 + \varepsilon\beta)$ ...

The Brownian web



. . . yields in the limit a Brownian web with drift β.

Definition of the Brownian web

Introduced by Arratia '79, Tóth & Werner '98, and Fontes, Isopi, Newman & Ravishankar '02.

Formally, a Brownian web $\mathcal{W}$ is a compact set of paths, such that

- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique path $\pi_z \in \mathcal{W}(z)$.

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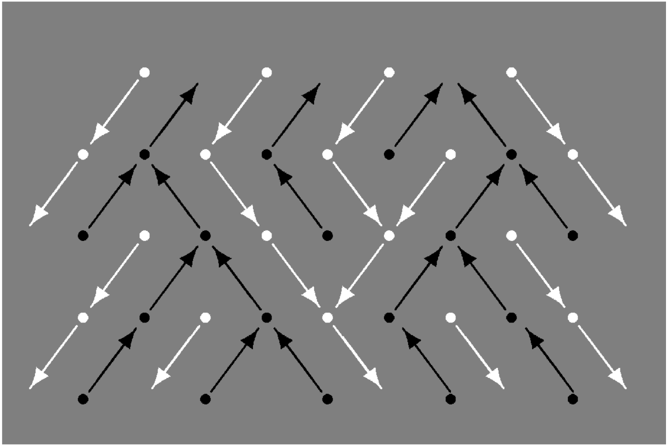
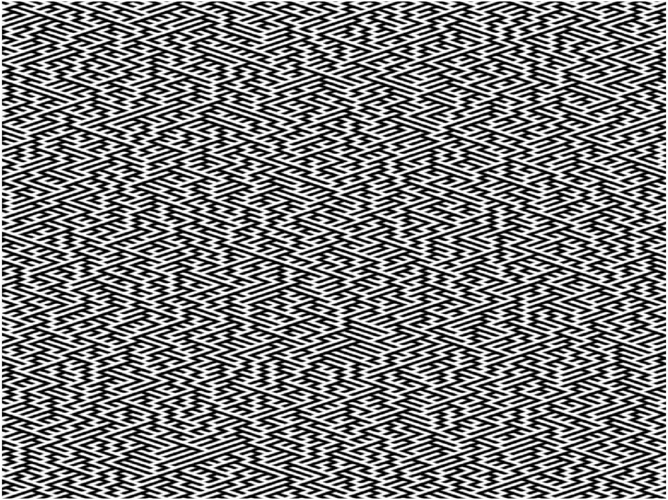
- For any deterministic countable dense subset $\mathcal{D} \subset \mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure of $\{\pi_z : z \in \mathcal{D}\}$.

A dual discrete web



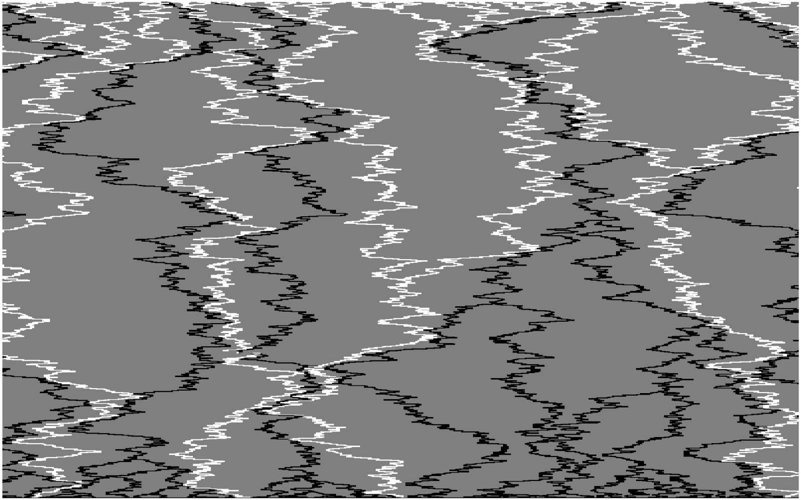
Forward and dual arrows.

Approximation of the dual Brownian web



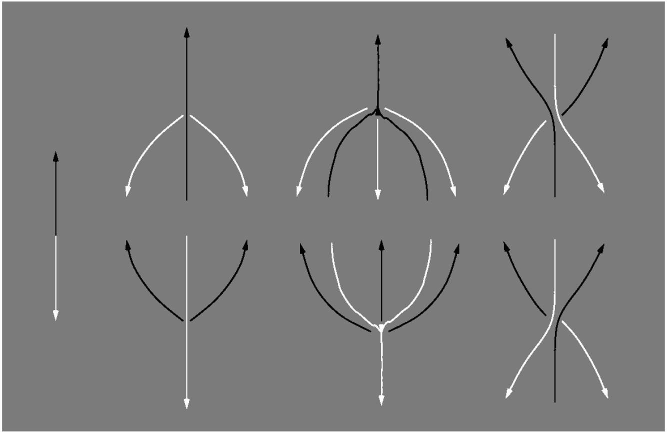
Approximation of the forward and dual Brownian web.

The dual Brownian web



Forward and dual paths started from fixed times.

Special points of the Brownian web



Special points of types $(0,1)$ $\begin{matrix}(1,1)\\(0,2)\end{matrix}$ $\begin{matrix}(2,1)\\(0,3)\end{matrix}$ $\begin{matrix}(1,2)_{\rm l}\\(1,2)_{\rm r}\end{matrix}$

Limits of modified webs

Idea: Modify a 'reference' Brownian web by 'switching' the orientation of points of type $(1,2)$.

Let $\varepsilon_k \to 0$ and let $(\mathcal{U}^{0\,(k)}, \omega^{(k)}, \mathcal{U}^{(k)})$ be a reference web, collection of $[0,1]$-valued r.v.'s and sample web as before. Let $\mu^k_{\rm l}$ resp. $\mu^k_{\rm r}$ be the conditional law of $\omega^{(k)}_z$ given $\alpha^{0\,(k)}_z = -1$ resp. $+1$. Assume that $\mathcal{U}^{0\,(k)}, \mathcal{U}^{(k)}$ converge to Brownian webs $\mathcal{W}_0, \mathcal{W}$ with drifts β_0, β and

$$\begin{array}{ll} \text{(i)} & \varepsilon_k^{-1} q \mu^k_{\rm l}(\mathrm{d}q) \underset{k\to\infty}{\Longrightarrow} \nu_{\rm l}(\mathrm{d}q), \\ \text{(ii)} & \varepsilon_k^{-1}(1-q)\mu^k_{\rm r}(\mathrm{d}q) \underset{k\to\infty}{\Longrightarrow} \nu_{\rm r}(\mathrm{d}q) \end{array}$$

for finite measures $\nu_{\rm l}, \nu_{\rm r}$ on $[0,1]$. Note that $\mu^k_{\rm l}(\mathrm{d}q)$ (resp. $\mu^k_{\rm r}(\mathrm{d}q)$) is close to δ_0 (resp. δ_1) so $\mathcal{U}^{(k)}$ is a small modification of $\mathcal{U}^{0\,(k)}$.

Intersection local time measure

Theorem [Newman, Ravishankar, Schertzer '08]
There exists a measure ℓ, concentrated on points of type $(1,2)$, such that for each path $\pi \in \mathcal{W}$ and dual path $\hat{\pi} \in \hat{\mathcal{W}}$ with starting times $\sigma_\pi, \hat{\sigma}_{\hat{\pi}}$:

$$\ell\big(\big\{z = (x,t) \in \mathbb{R}^2 : \sigma_\pi < t < \hat{\sigma}_{\hat{\pi}},\ \pi(t) = \hat{\pi}(t)\big\}\big)$$
$$= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \big|\big\{t \in \mathbb{R} : \sigma_\pi < t < \hat{\sigma}_{\hat{\pi}},\ |\pi(t) - \hat{\pi}(t)| \le \varepsilon\big\}\big|.$$

Modified Brownian web

Theorem [Newman, Ravishankar, Schertzer '08]
Let $\ell_{\rm l}, \ell_{\rm r}$ denote the restrictions of ℓ to the sets of points of type $(1,2)_{\rm l}$ resp. $(1,2)_{\rm r}$. Given a Brownian web $\mathcal{W}_0$ with drift β_0, let S be a Poisson point set with intensity

$$c_{\rm l}\ell_{\rm l} + c_{\rm r}\ell_{\rm r}.$$

Then the a.s. limit

$$\mathcal{W} := \lim_{\Delta_n \uparrow S} \mathrm{switch}_{\Delta_n}(\mathcal{W}_0)$$

exists and defines a Brownian web $\mathcal{W}$ with drift $\beta = \beta_0 + c_{\rm l} - c_{\rm r}$.

Construction of the quenched law

Theorem [E. Schertzer, R. Sun & J.S. '10]
Let $\nu_{\rm l}, \nu_{\rm r}$ be finite measures on $[0,1]$ such that

$$\nu(\mathrm{d}q) = (1-q)\nu_{\rm l}(\mathrm{d}q) + q\nu_{\rm r}(\mathrm{d}q).$$

Let $\mathcal{W}_0$ be a Brownian web with drift $\beta_0 := \beta - 2\int \mathrm{d}\nu_{\rm l} + 2\int \mathrm{d}\nu_{\rm r}$. Conditional on $\mathcal{W}_0$, let $\mathcal{M}$ be a Poisson point set on $\mathbb{R}^2 \times [0,1]$ with intensity

$$\ell_{\rm l}(\mathrm{d}z) \otimes 2\,1_{\{0<q\}} q^{-1} \nu_{\rm l}(\mathrm{d}q) + \ell_{\rm r}(\mathrm{d}z) \otimes 2\,1_{\{q<1\}} (1-q)^{-1} \nu_{\rm r}(\mathrm{d}q).$$

Set $\mathcal{M} := \{(z, \omega_z) : z \in M\}$. Interpretation: $(z, \omega_z) \in \mathcal{M}$ is a marked point. We call ω_z the *mark* of z.

Construction of the quenched law

Conditional on $\mathcal{W}_0, \mathcal{M}$, let $\alpha = (\alpha_z)_{z \in M}$ be independent $\{-1, +1\}$-valued random variables such that $\alpha_z = +1$ with probability ω_z. Set

$$A := \{z \in M : \alpha_z \neq \mathrm{sign}(z)\}$$

and let B be a Poisson point set with intensity $2\nu_{\mathrm{l}}(\{0\})\ell_{\mathrm{l}} + 2\nu_{\mathrm{r}}(\{1\})\ell_{\mathrm{r}}$. Define

$$\mathcal{W} := \lim_{\Delta_n \uparrow A \cup B} \mathrm{switch}_{\Delta_n}(\mathcal{W}_0).$$

Then

$$\mathbb{Q} = \mathbb{P}\big[\mathcal{W} \in \cdot \,\big|\, (\mathcal{W}_0, \mathcal{M})\big].$$

is the continuum quenched law.

Regular versions

Let π_z^+ denote the right-most path in $\mathcal{W}$ starting at z.

Let $\pi_z^\uparrow$ be the same, except in points of type $(1,2)_{\rm l}$, where $\pi_z^\uparrow$ is the left-most path.

Then

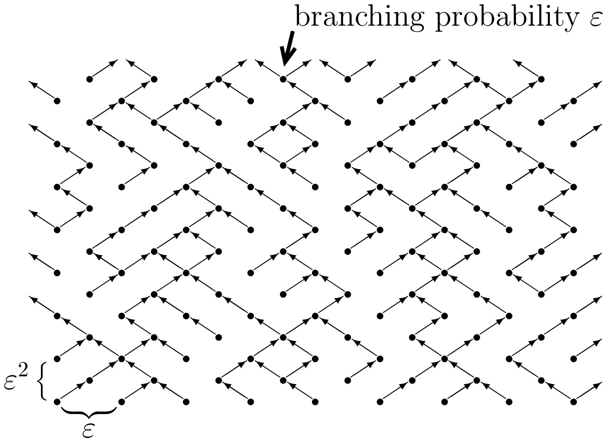
$$K^+_{s,t}(x,\mathrm{d}y) := \mathbb{P}[\pi^+_{(x,s)}(t) \in \mathrm{d}y],$$
$$K^\uparrow_{s,t}(x,\mathrm{d}y) := \mathbb{P}[\pi^\uparrow_{(x,s)}(t) \in \mathrm{d}y],$$

define versions of Howitt & Warren's stochastic flow of kernels.

Moreover, $(K^\uparrow_{s,t})_{s\leq t}$ satisfies

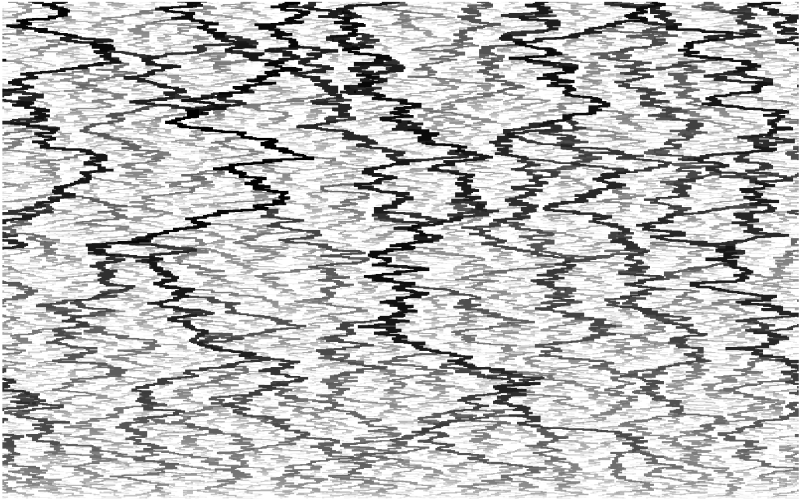
(i)' $\displaystyle\int K_{s,t}(x,\mathrm{d}y)K_{t,u}(y,\,\cdot\,) = K_{s,u}(x,\,\cdot\,)$ for all $s \leq t \leq u$ and $x \in \mathbb{R}$ a.s.

A discrete net



Discrete approximation of the Brownian net.

The Brownian net



Brownian net.

History

- Introduced in [Sun & S. '08] by means of a coupled left and right Brownian web.

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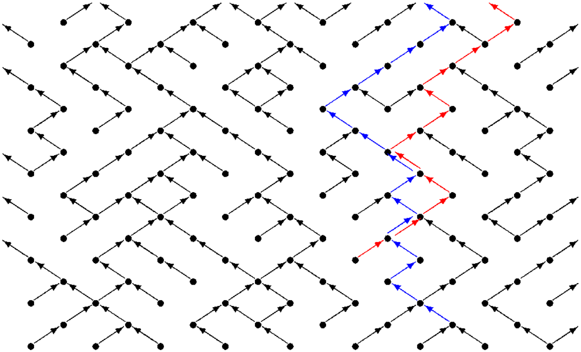
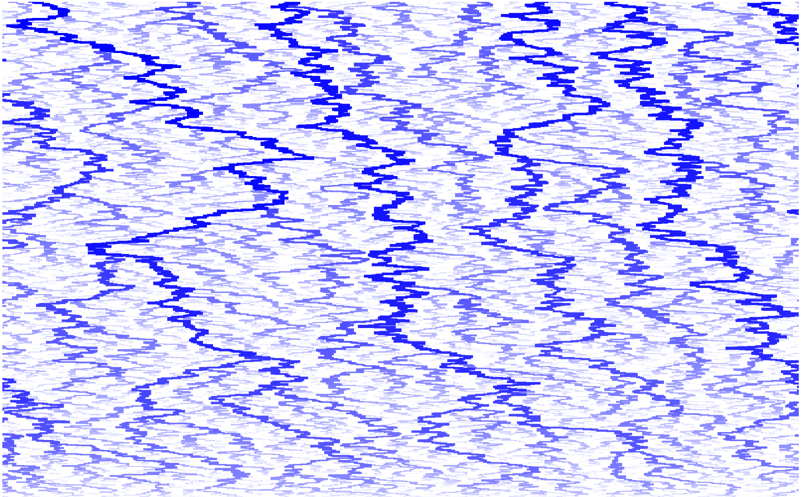
- Classification of special points in [Schertzer, Sun & S. '08].

Left and right paths



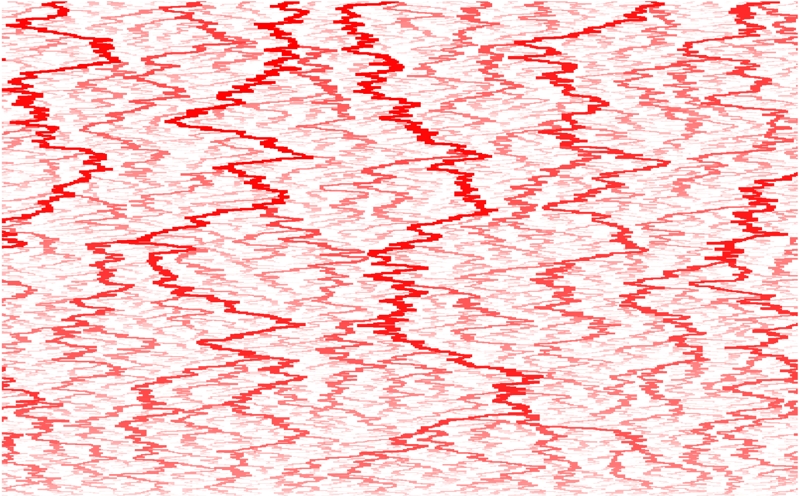
Draw left-most paths in blue and right-most paths in red.

The left Brownian web



The left-most paths converge to a left Brownian web. . .

The right Brownian web



. . . and the right-most paths to a right Brownian web.

Marking construction

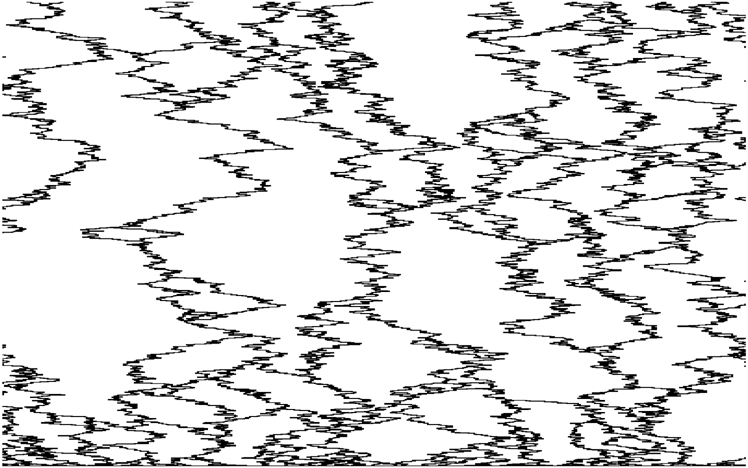
Let $\mathcal{W}_0$ be a 'reference' Brownian web with drift β_0. Let S_l and S_r be independent Poisson point sets with intensities $c_\mathrm{l}\ell_\mathrm{l}$ and $c_\mathrm{r}\ell_\mathrm{r}$, respectively. Then

$$\begin{aligned}
&\text{(i)} \quad \mathcal{N} := \lim_{n\to\infty} \mathrm{hop}_{\Delta_n \uparrow S_\mathrm{l}\cup S_\mathrm{r}}(\mathcal{W}_0), \\
&\text{(ii)} \quad \mathcal{W}^\mathrm{l} := \lim_{n\to\infty} \mathrm{switch}_{\Delta_n \uparrow S_\mathrm{r}}(\mathcal{W}_0), \\
&\text{(iii)} \quad \mathcal{W}^\mathrm{r} := \lim_{n\to\infty} \mathrm{switch}_{\Delta_n \uparrow S_\mathrm{l}}(\mathcal{W}_0)
\end{aligned} \tag{1}$$

defines a Brownian net $\mathcal{N}$ and associated left-right Brownian web $(\mathcal{W}^\mathrm{l}, \mathcal{W}^\mathrm{r})$ with left and right speeds $\beta_- = \beta_0 - c_\mathrm{r}$, $\beta_+ = \beta_0 + c_\mathrm{l}$,. The *standard Brownian net* has $\beta_- = -1, \beta_+ = +1$.

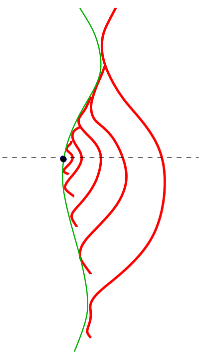
Here hop_Δ means that at points in Δ of type $(1,2)$, we allow incoming paths to continue along both outgoing paths.

The branching-coalescing point set

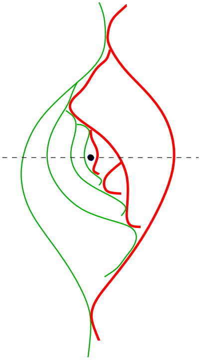


The branching-coalescing point set $(\xi_t)_{t\geq 0}$ started in $\xi_0 = \mathbb{R}$.

Cluster points of nested excursions



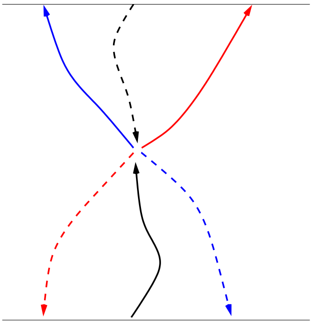
$(\mathrm{C_l})$



$(\mathrm{C_n})$

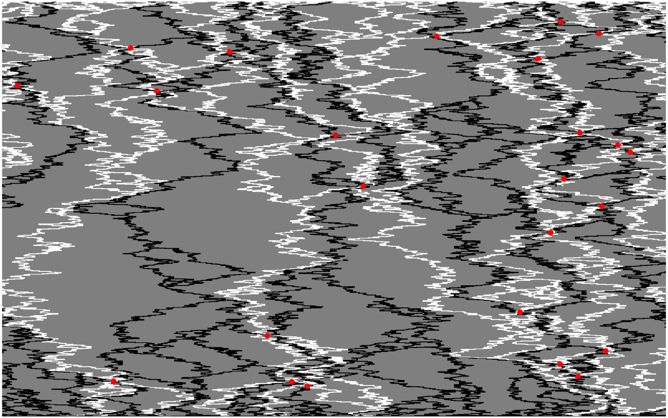
[SSS'09] Cluster points of nested excursions between left-most and right-most paths give rise to random times when ξ_t has no isolated points.

Relevant separation points



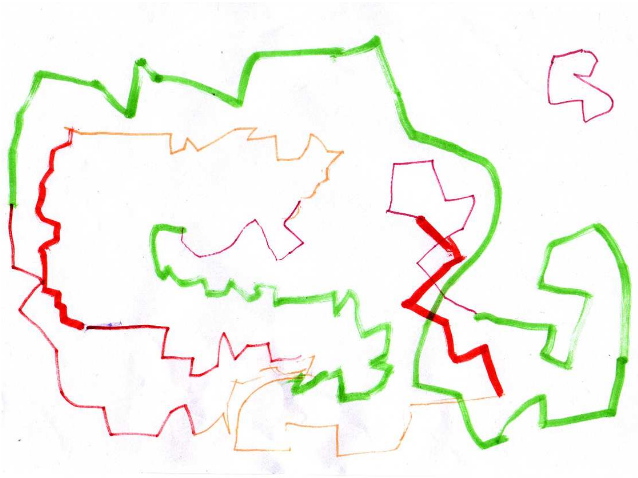
By definition, a separation point $z = (x, t)$ with $S < t < U$ is S, U-*relevant* if there is a path $\pi \in \mathcal{N}$ entering z starting at time S, and there are $l \in \mathcal{W}^{\mathrm{l}}(z)$, $r \in \mathcal{W}^{\mathrm{r}}(z)$ such that $l < r$ on (t, U).

Relevant separation points



[SSS'09] 'Relevant' separation points, where the forward Brownian net crosses its dual, are locally finite.

The End



Thank you!