Stochastic flows in the Brownian web and net

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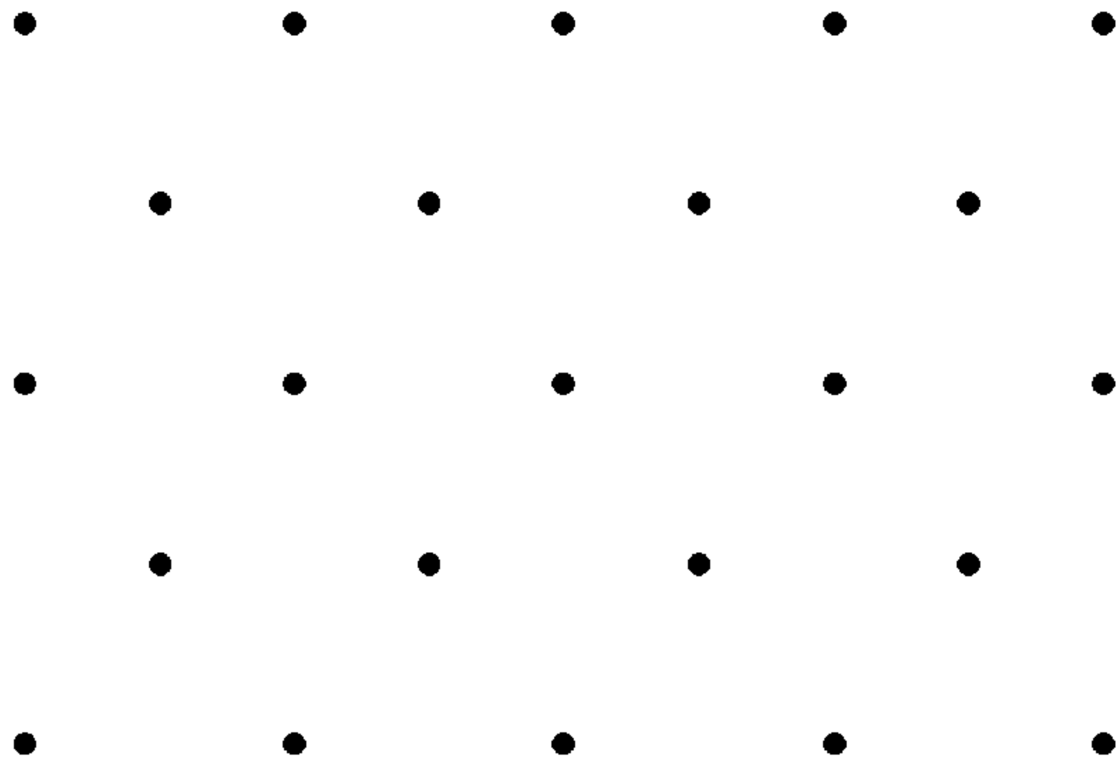
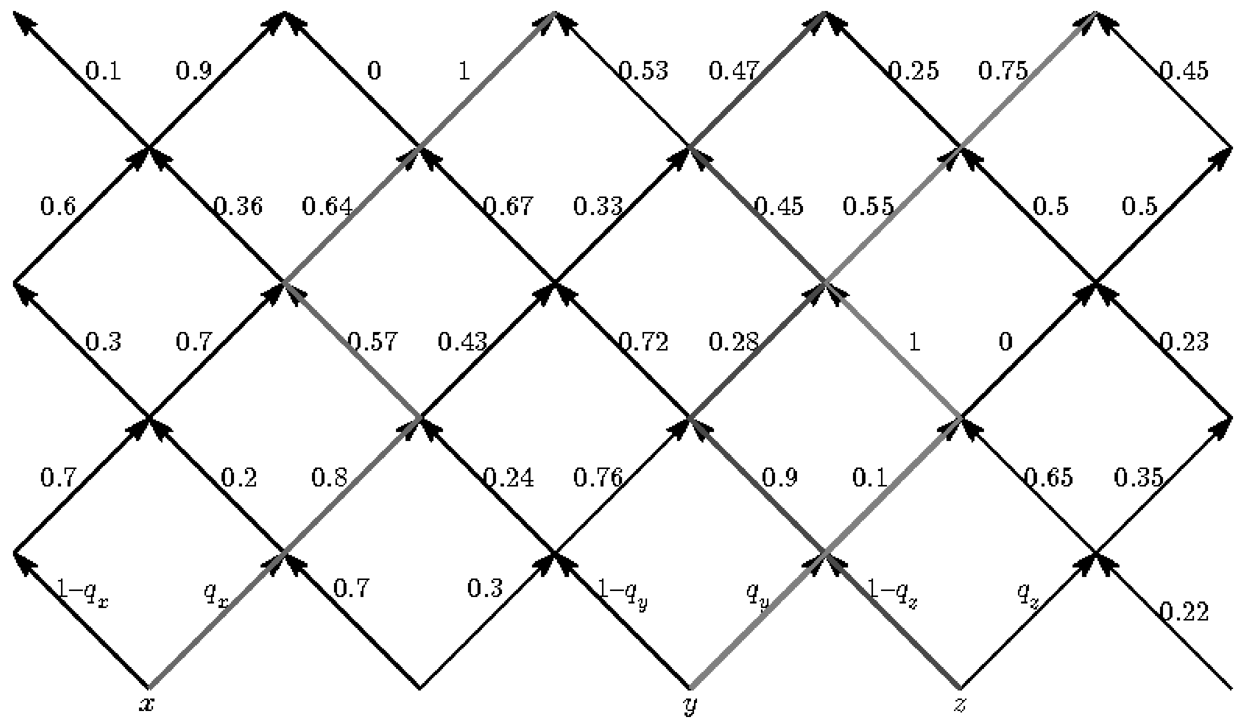
Howitt-Warren processes



Let $\mathbb{Z}^2_{\rm even} := \{(x,t) \in \mathbb{Z}^2 : x + t$ is even$\}$.
Interpretation: x is space, t is time (upwards).

Howitt-Warren processes



Fix a probability law μ on $[0, 1]$.

Let $(q_z)_{z\in\mathbb{Z}^2_{\rm even}}$ be i.i.d. $[0, 1]$-valued r.v.'s with law μ.

Howitt-Warren processes

Fix some probability measure ρ_0 on $\mathbb{Z}_{\rm even}$, and define inductively, for $(x,t) \in \mathbb{Z}^2_{\rm even}$:

$$\rho_t(x) := q_{(x-1,t-1)}\rho_{t-1}(x-1) + \big(1 - q_{(x+1,t-1)}\big)\rho_{t-1}(x+1).$$

Interpretation: in the time step from t to $t+1$, a $q_{(x,t)}$ fraction of the mass at x is sent to $x+1$ and the rest is sent to $x-1$.

Then $(\rho_t)_{t\geq 0}$ is a Markov chain taking values alternatively in the probability measures on $\mathbb{Z}_{\rm even}$ and $\mathbb{Z}_{\rm odd}$.

Howitt-Warren processes

Theorem [Le Jan & Raimond '04, Howitt & Warren '06]

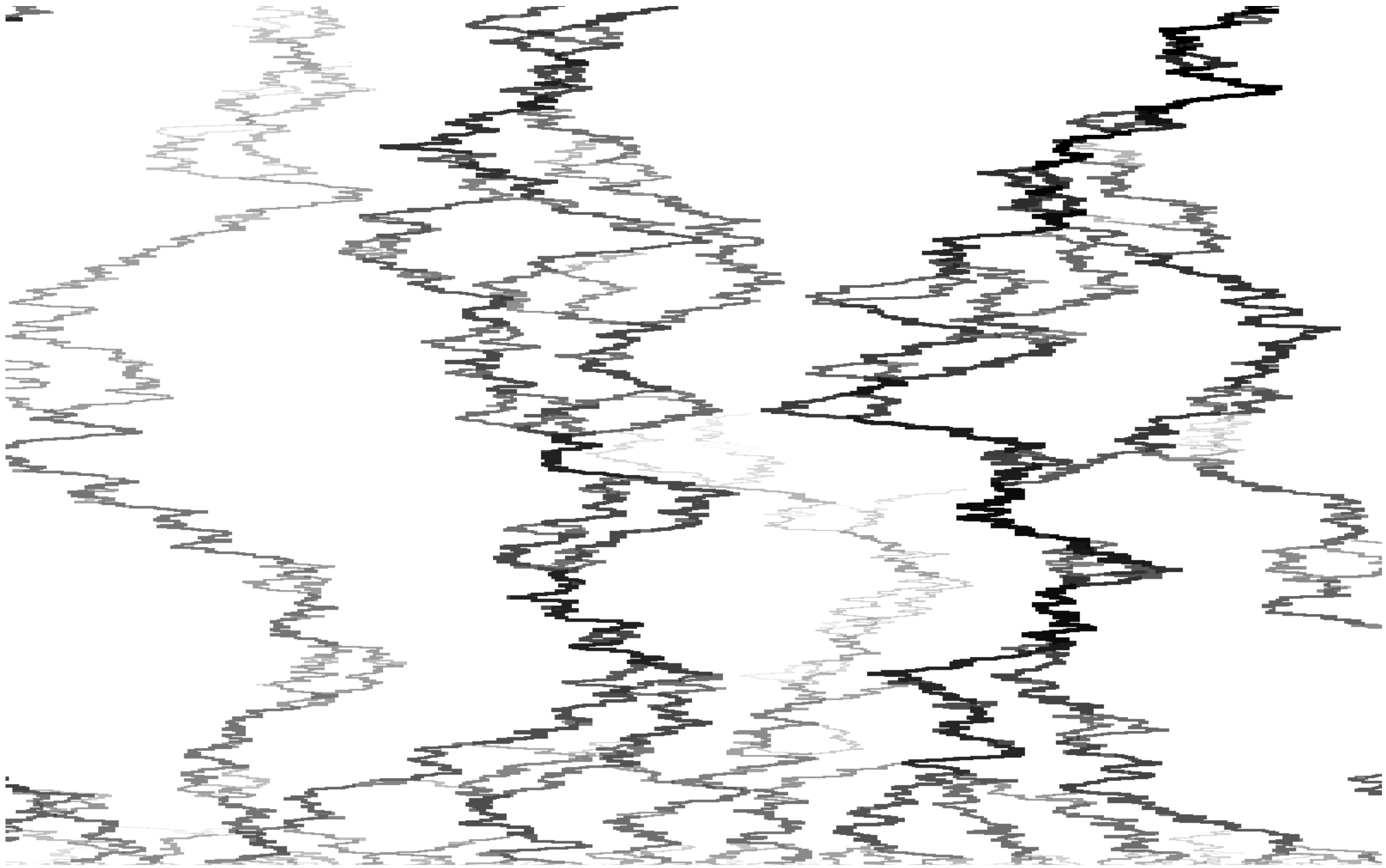
Let $\varepsilon_n \to 0$ and rescale diffusively: $\tilde{\rho}^{(n)}_{\varepsilon_n^2 t}(\varepsilon_n x) := \rho^{(n)}_t(x)$, where $\rho^{(n)}_t(x)$ are Markov chains defined by splitting laws μ_n satisfying:

$$\begin{array}{ll} \text{(i)} & \displaystyle \frac{1}{\varepsilon_n}\int 2(q-\tfrac{1}{2})\mu(\mathrm{d}q) \underset{n\to\infty}{\longrightarrow} \beta, \\ \text{(ii)} & \displaystyle \frac{1}{\varepsilon_n} q(1-q)\mu(\mathrm{d}q) \underset{n\to\infty}{\Longrightarrow} \nu(\mathrm{d}q), \end{array}$$

with $\beta \in \mathbb{R}$ and ν a finite measure on $[0,1]$.

Then $\tilde{\rho}^{(n)} \Rightarrow \rho$, where $(\rho_t)_{t\geq 0}$ is a Markov process taking values in the probability measures on $\mathbb{R}$, with dynamics characterized by β and ν.

Howitt-Warren processes



The equal splitting process: $\beta = 0$ and $\nu = \delta_{1/2}$.

Approximated with $\mu_n = (1 - \varepsilon_n)(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) + \varepsilon_n \delta_{1/2}$.

n-point motions

Given the random environment created by the $(q_z)_{z\in\mathbb{Z}^2_{\text{even}}}$, let $X_1(t),\ldots,X_n(t)$ be random walks started from $x_1,\ldots,x_n$ such that

$$X_k(t+1)=\begin{cases} X_k(t)+1 & \text{with probab. } q_{(X_k(t),t)}, \\ X_k(t)-1 & \text{with probab. } 1-q_{(X_k(t),t)}, \end{cases}$$

independently for each k and t.

Observation If we forget about the random environment, then $\big(X_1(t),\ldots,X_n(t)\big)_{t\geq 0}$ is a Markov chain:

discrete n-point motion.

n-point motions

Theorem [Howitt & Warren '06]
The discrete n-point motions, diffusively rescaled, converge to an $\mathbb{R}^n$-valued Markov process $\big(X_1(t), \ldots, X_n(t)\big)_{t \geq 0}$ characterized by β and ν. Each component is a Brownian motion with drift β. The Brownian motions interact with a form of sticky interaction described by ν.

Theorem [Le Jan & Raimond '04]
Any consistent family of Feller processes defines a probability-measure valued Markov process.

Theorem [Le Jan & Raimond '04]

The process with $\beta = 0$ and $\nu(\mathrm{d}q) = \mathrm{d}q$ is reversible, with explicit invariant law.

Path properties

Theorem [E. Schertzer, R. Sun & J.S. '09]
Case 1 Assume $\int \nu(\mathrm{d}q)q^{-1}(1-q)^{-1} < \infty$, $\nu \neq 0$. Then:

(i) $\mathrm{supp}(\rho_t)$ is locally finite at each deterministic $t > 0$.

(ii) There exist random times when ρ_t is purely non-atomic.

(iii) $\mathrm{supp}(\rho_t)$ is a Markov process.

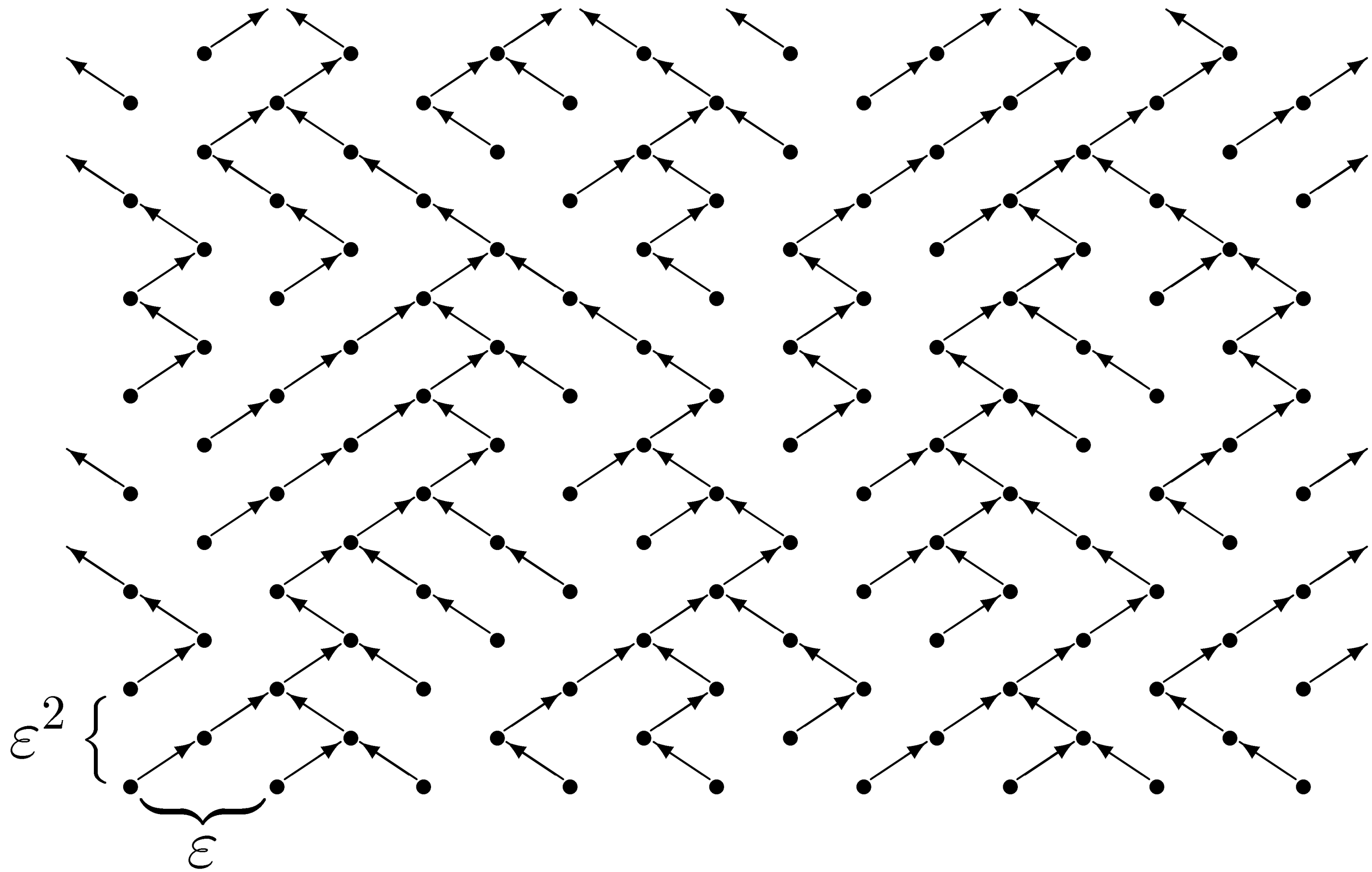
Case 2 Assume $\int_0^\varepsilon \nu(\mathrm{d}q)q^{-1} = \infty = \int_{1-\varepsilon}^1 \nu(\mathrm{d}q)(1-q)^{-1}$. Then:

(i)' ρ_t is purely atomic at each deterministic $t > 0$.

(iii)' $\mathrm{supp}(\rho_t) = \mathbb{R}$ at each $t > 0$.

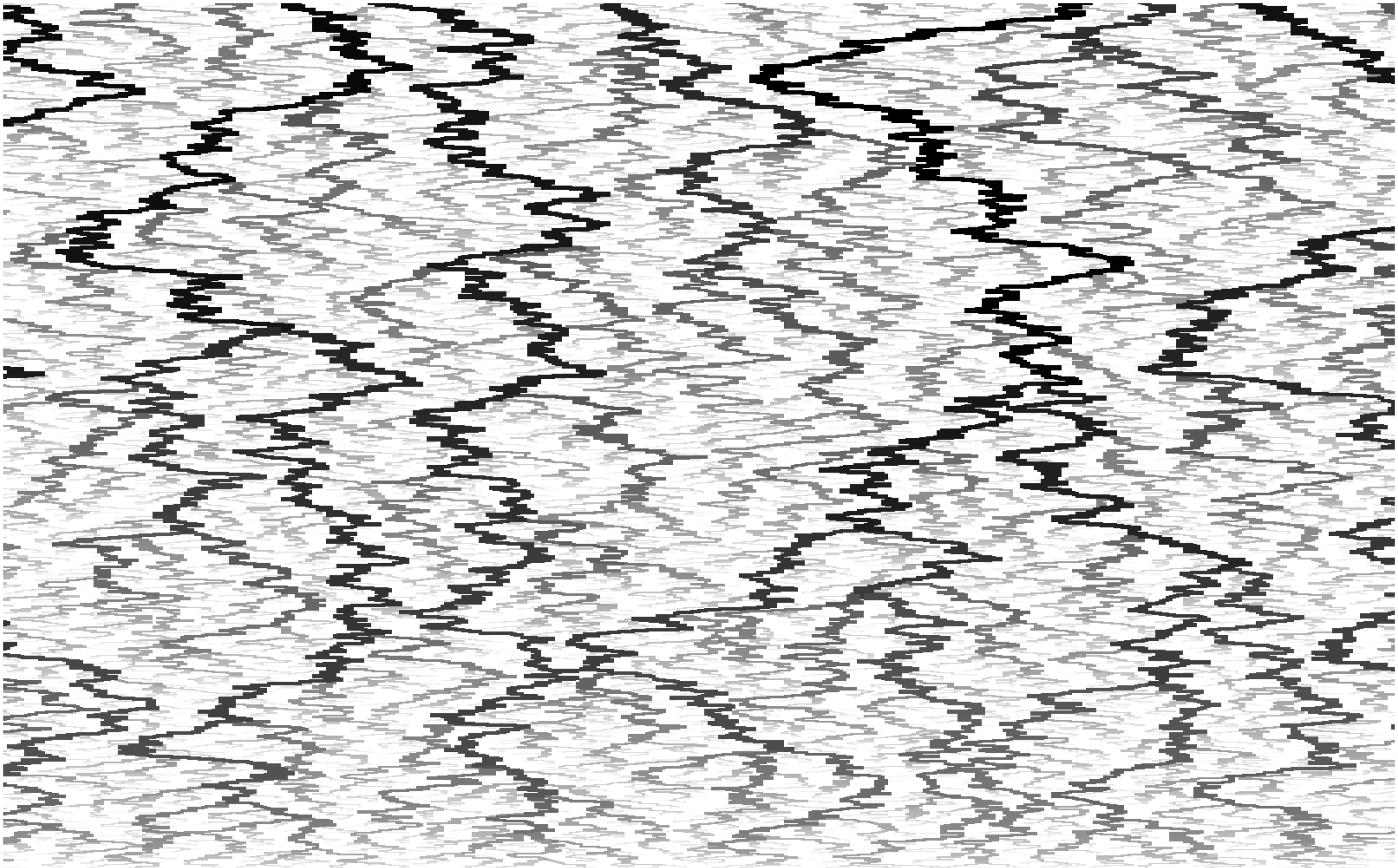
Conjecture: (ii) holds also in Case 2.

The Brownian web



Extreme case: $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, hence $q_z \in \{0, 1\}$. Coalescing random walks start from each point in $\mathbb{Z}^2_{\mathrm{even}}$.

The Brownian web



In the limit we obtain the Brownian web.

The Brownian web

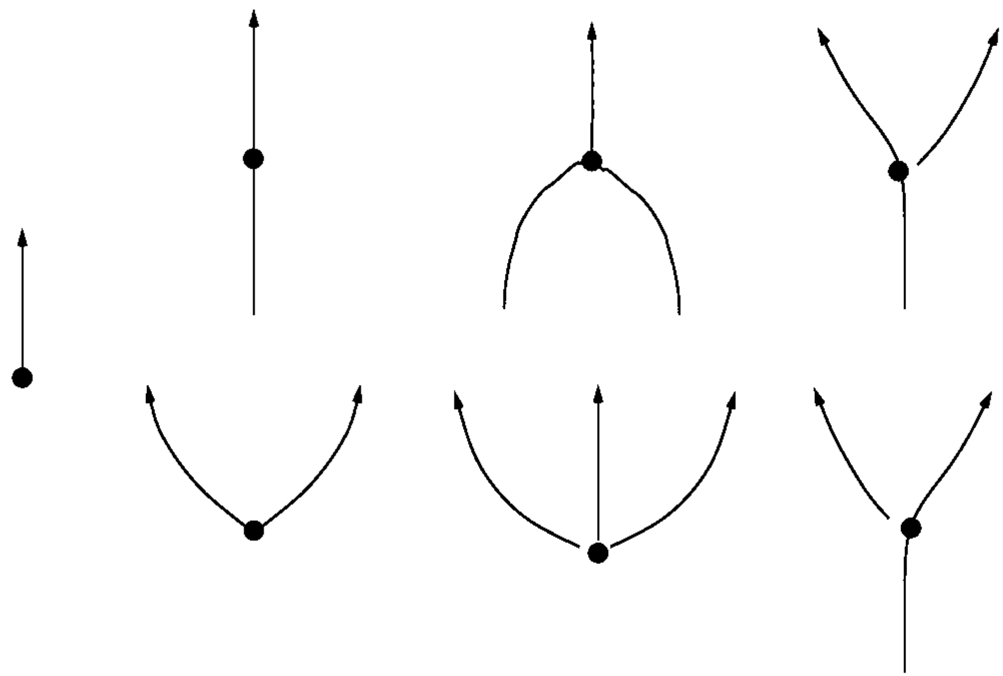
Introduced by Arratia '79, Tóth & Werner '98, and Fontes, Isopi, Newman & Ravishankar '02.
Formally, a Brownian web $\mathcal{W}$ is a compact set of paths, such that

(i) At deterministic $z \in \mathbb{R}^2$ there a.s. starts a unique path p_z.

(ii) Paths started at different points are coalescing Brownian motions.

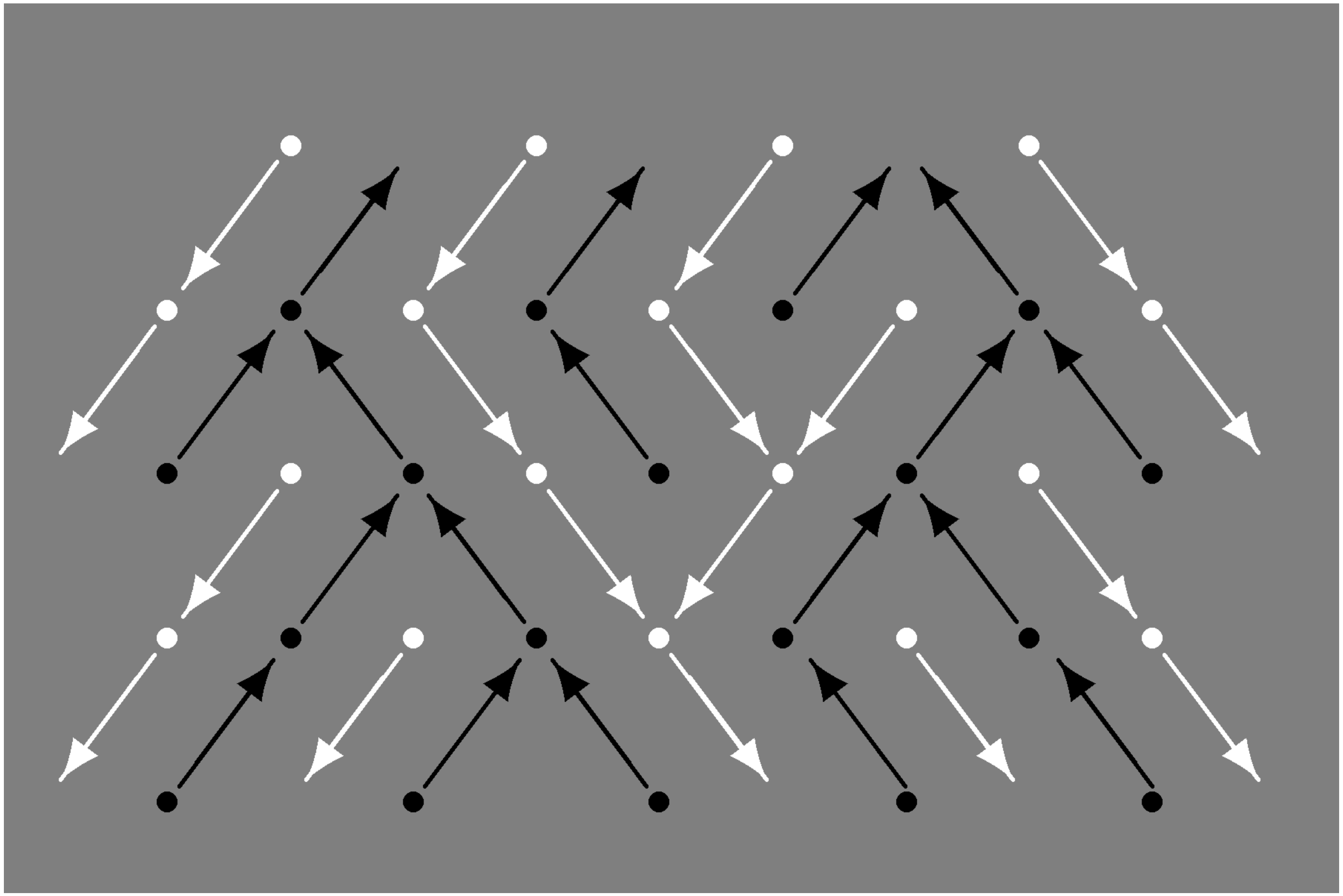
(iii) For any deterministic countable dense $\mathcal{D} \subset \mathbb{R}^2$, the web $\mathcal{W}$ is the closure of $\{p_z : z \in \mathcal{D}\}$.

The Brownian web



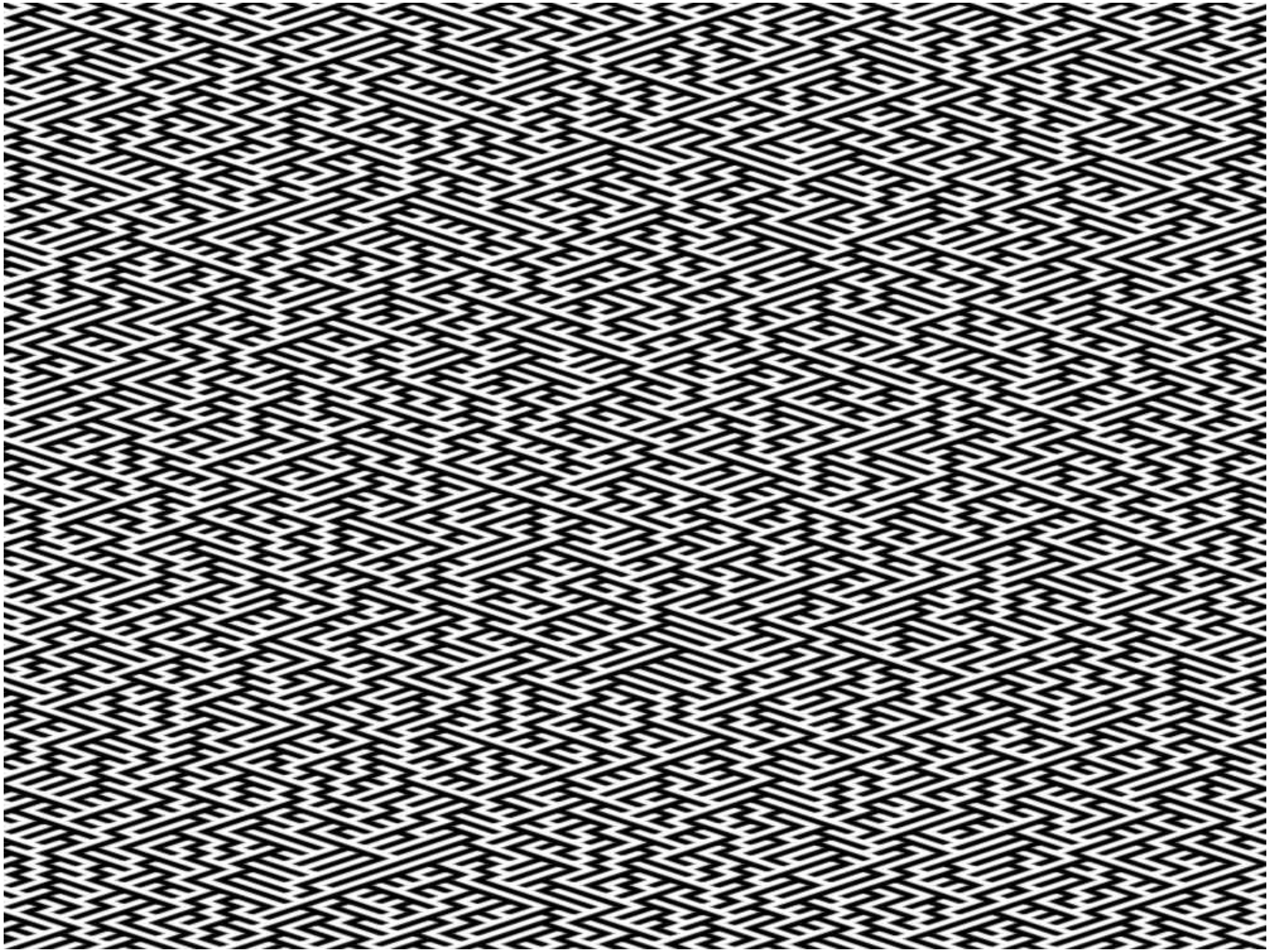
Special points of types $(0,1)$, $(1,1)/(0,2)$, $(2,1)/(0,3)$ and $(1,2)_{\mathrm{l}}/(1,2)_{\mathrm{r}}$.

The dual Brownian web



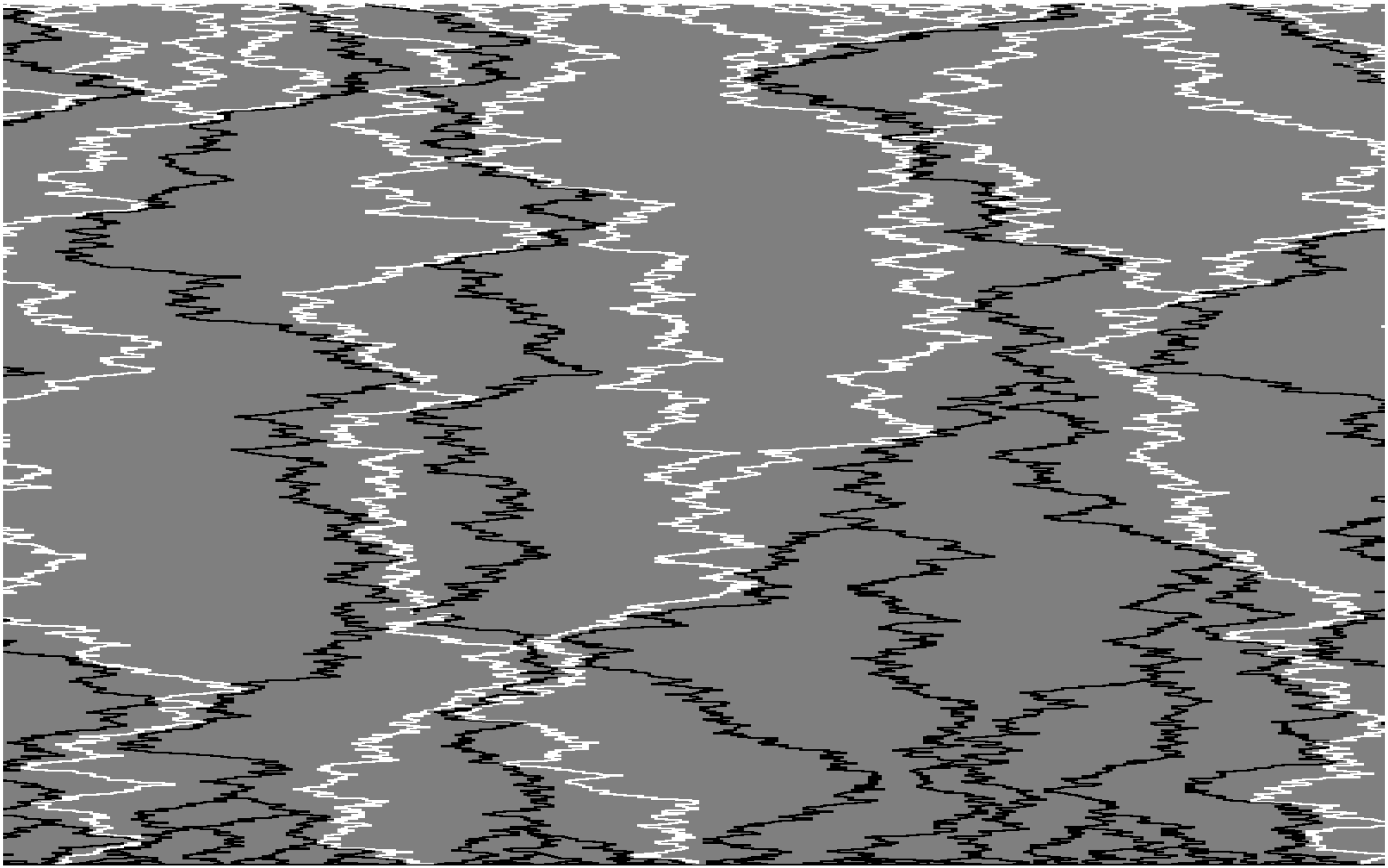
Forward and dual arrows.

The dual Brownian web



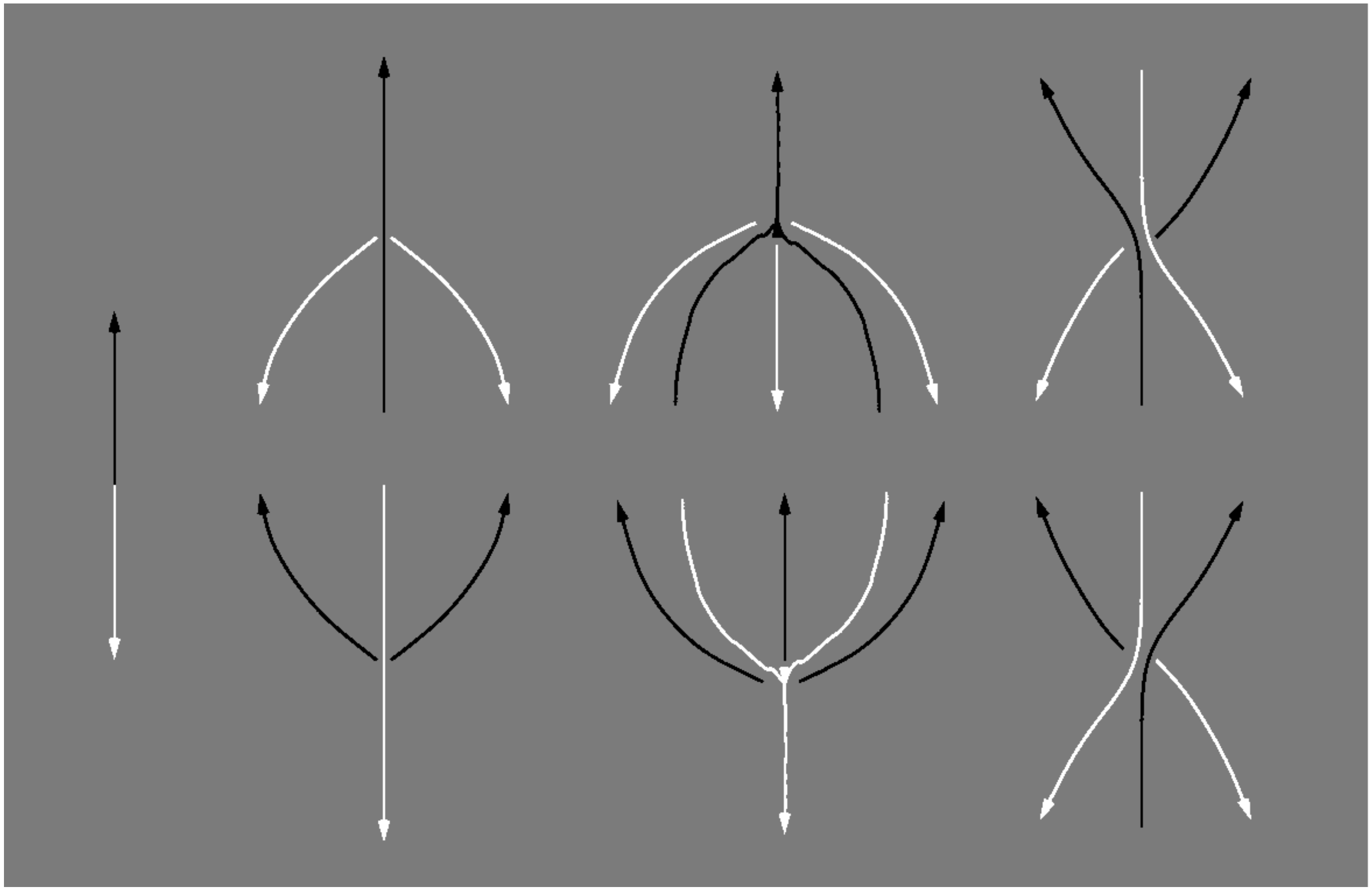
Approximation of the forward and dual Brownian web.

The dual Brownian web



Forward and dual paths started from fixed times.

The dual Brownian web



Special points of types $(0,1)$, $(1,1)/(0,2)$, $(2,1)/(0,3)$ and $(1,2)_{\mathrm{l}}/(1,2)_{\mathrm{r}}$.

Construction of Howitt-Warren processes

Observation
Fix random $(q_z)_{z\in\mathbb{Z}^2_{\rm even}}$ with law μ. Given this random environment, for $(x,t)\in\mathbb{Z}^2_{\rm even}$, draw an arrow to $(x+1,t+1)$ with probability $q_{(x,t)}$ and to $(x-1,t+1)$ with probability $1-q_{(x,t)}$. Let p_z be the unique path starting in z following the arrows. Then $(\rho_t)_{t\geq 0}$ is given (in law) by

$$\rho_t(y)=\sum_{x\in\mathbb{Z}_{\rm even}}\rho_0(x)\mathbb{P}\big[p_{(x,0)}(t)=y\,\big|\,(q_z)_{z\in\mathbb{Z}^2_{\rm even}}\big].$$

Construction of Howitt-Warren processes

Alternative discrete construction

Define weighted laws $\mu_{\rm l}(\mathrm{d}q) := \frac{1}{Z_{\rm l}}(1-q)\mu(\mathrm{d}q)$ and $\mu_{\rm r}(\mathrm{d}q) := \frac{1}{Z_{\rm r}} q\mu(\mathrm{d}q)$, where $Z_{\rm l}, Z_{\rm r}$ are normalizing constants.

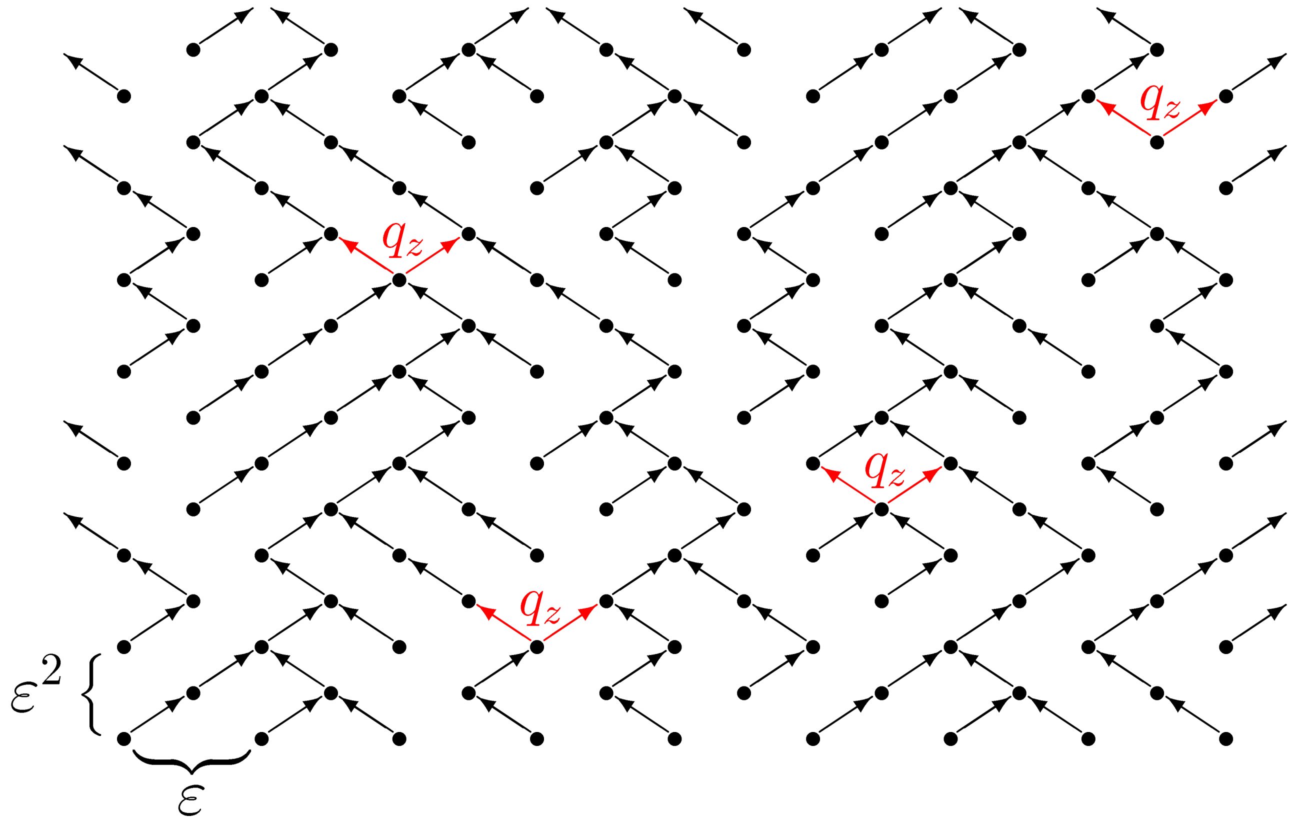
Fix a discrete 'reference' web W with drift $\int 2(q-\frac{1}{2})\mu(\mathrm{d}q)$.

Let $(q_z)_{z\in\mathbb{Z}^2_{\rm even}}$ be independent random variables, where q_z has law $\mu_{\rm l}$ (resp. $\mu_{\rm r}$) if the arrow at z points to the left (resp. right).

Define a 'modified' discrete web $\tilde W$ by drawing an arrow to $(x+1,t+1)$ with probability $q_{(x,t)}$ and to $(x-1,t+1)$ with probability $1-q_{(x,t)}$. Then

$$\rho_t(y) = \sum_{x\in\mathbb{Z}_{\rm even}} \rho_0(x)\mathbb{P}\big[p_{(x,0)}(t) = y \,\big|\, W, (q_z)_{z\in\mathbb{Z}^2_{\rm even}}\big].$$

Construction of Howitt-Warren processes



Construction of the modified discrete web.

Construction of Howitt-Warren processes

Construction when ν is concentrated on $(0,1)$.

Fix a reference Brownian web $\mathcal{W}$.

Let ℓ be the reflection local time between $\mathcal{W}$ and its dual.

Let $S_{\rm l}$ and $S_{\rm r}$ be the sets of points of type $(1,2)_{\rm l}$ and $(1,2)_{\rm r}$, respectively.

Let $\mathcal{M}_{\rm l}$ be a Poisson point set on $S_{\rm l}\times(0,1)$ with intensity $\ell(\mathrm{d}z)q^{-1}\nu(\mathrm{d}q)$, and let $\mathcal{M}_{\rm r}$ be a Poisson point set on $S_{\rm r}\times(0,1)$ with intensity $\ell(\mathrm{d}z)(1-q)^{-1}\nu(\mathrm{d}q)$.

Construction of Howitt-Warren processes

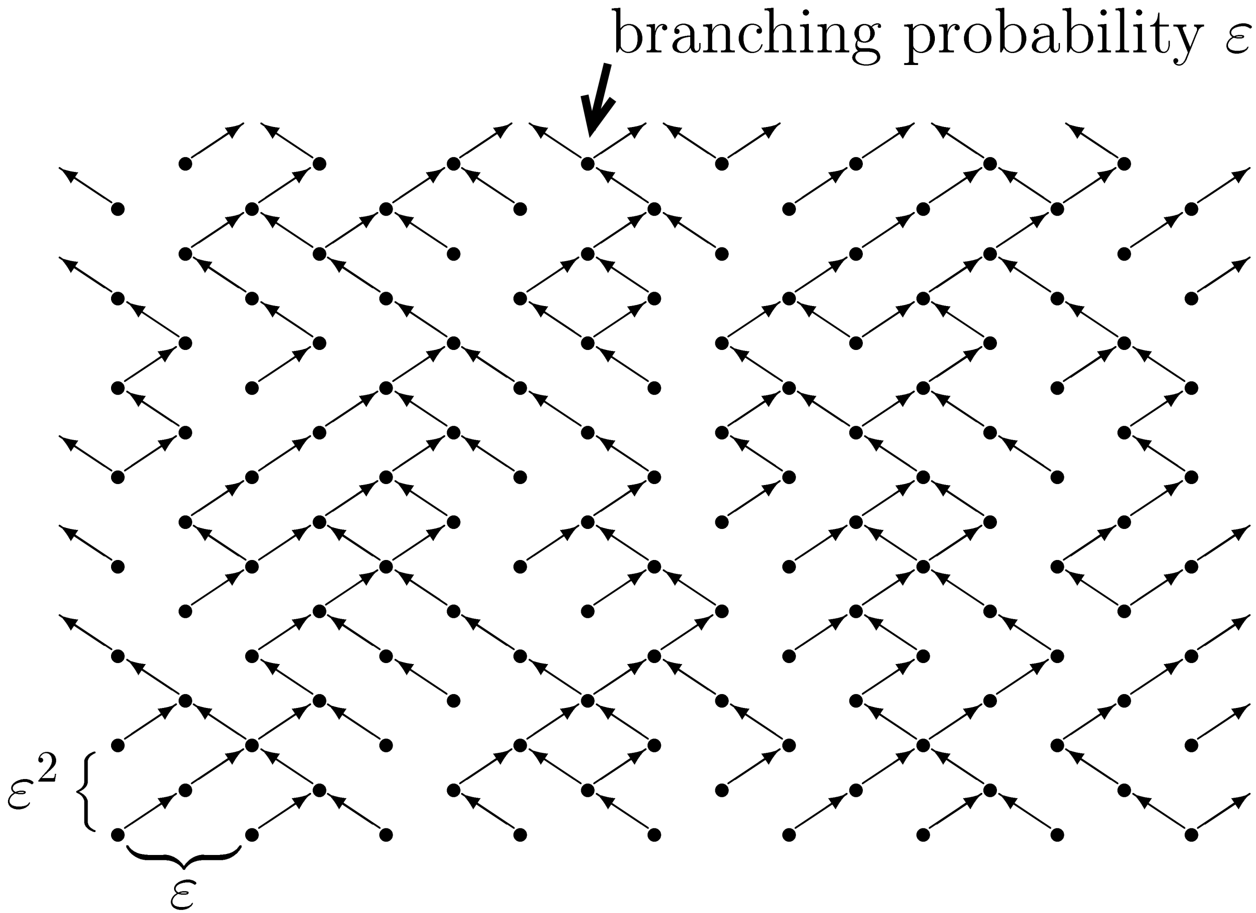
Construct a new web $\tilde{\mathcal{W}}$ as follows: Independently for each $(z,q) \in \mathcal{M}_\mathrm{l}$ we change z into a point of type $(1,2)_\mathrm{r}$ with probability q. Likewise, independently for each $(z,q) \in \mathcal{M}_\mathrm{r}$ we change z into a point of type $(1,2)_\mathrm{l}$ with probability $1-q$. Then

$$\rho_t(\mathrm{d}x) := \int \rho_0(\mathrm{d}y)\mathbb{P}\big[\tilde{p}_{(y,0)}(t) \in \mathrm{d}x \,\big|\, (\mathcal{W},\mathcal{M})\big]$$

defines a Howitt-Warren process, where $\tilde{p}_z$ denotes the a.s. unique path in $\tilde{\mathcal{W}}$ starting from a deterministic point z.

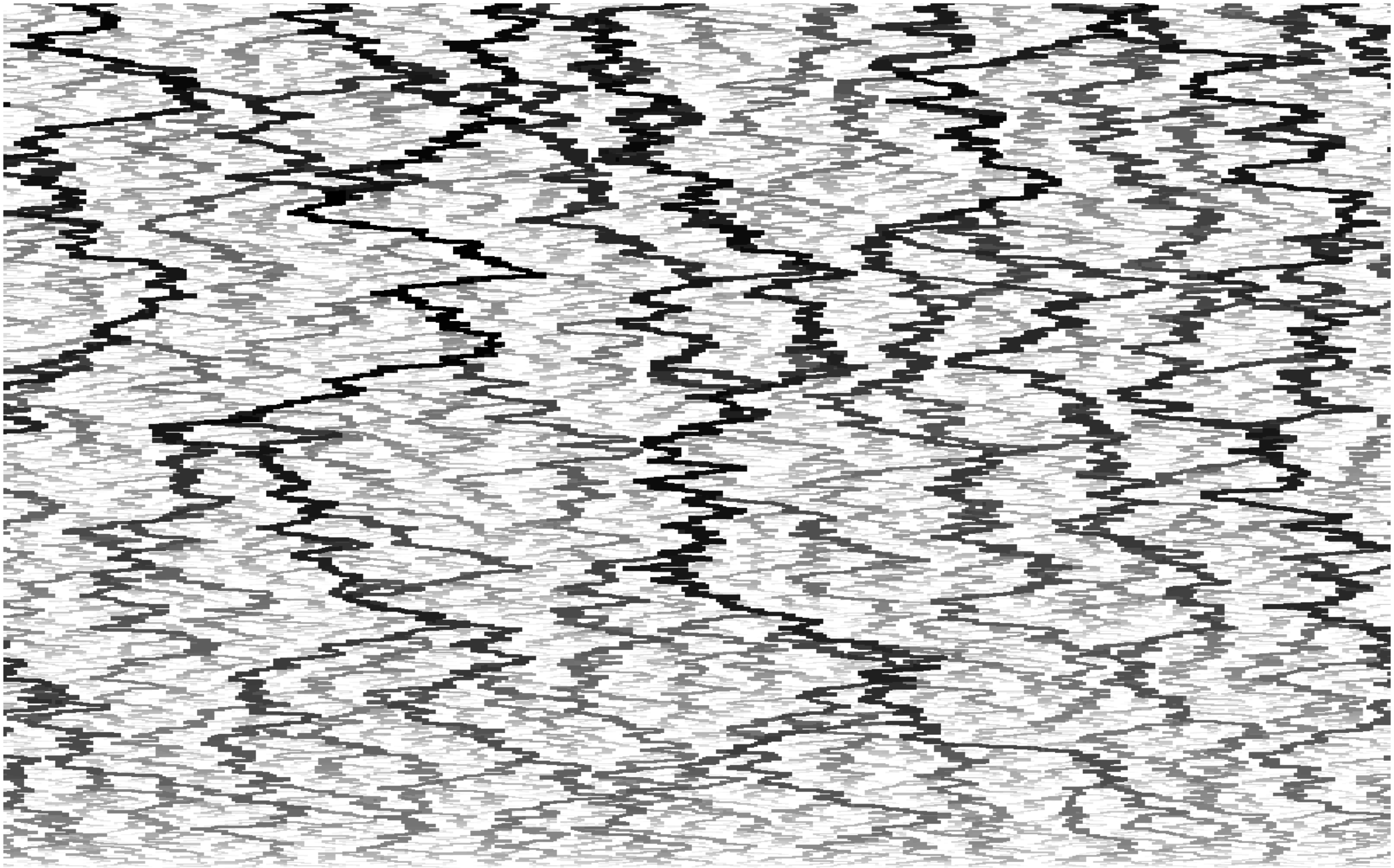
In the special case $\int \nu(\mathrm{d}q)q^{-1}(1-q)^{-1} < \infty$, the Howitt-Warren process can be embedded in a Brownian *net*.

The Brownian net



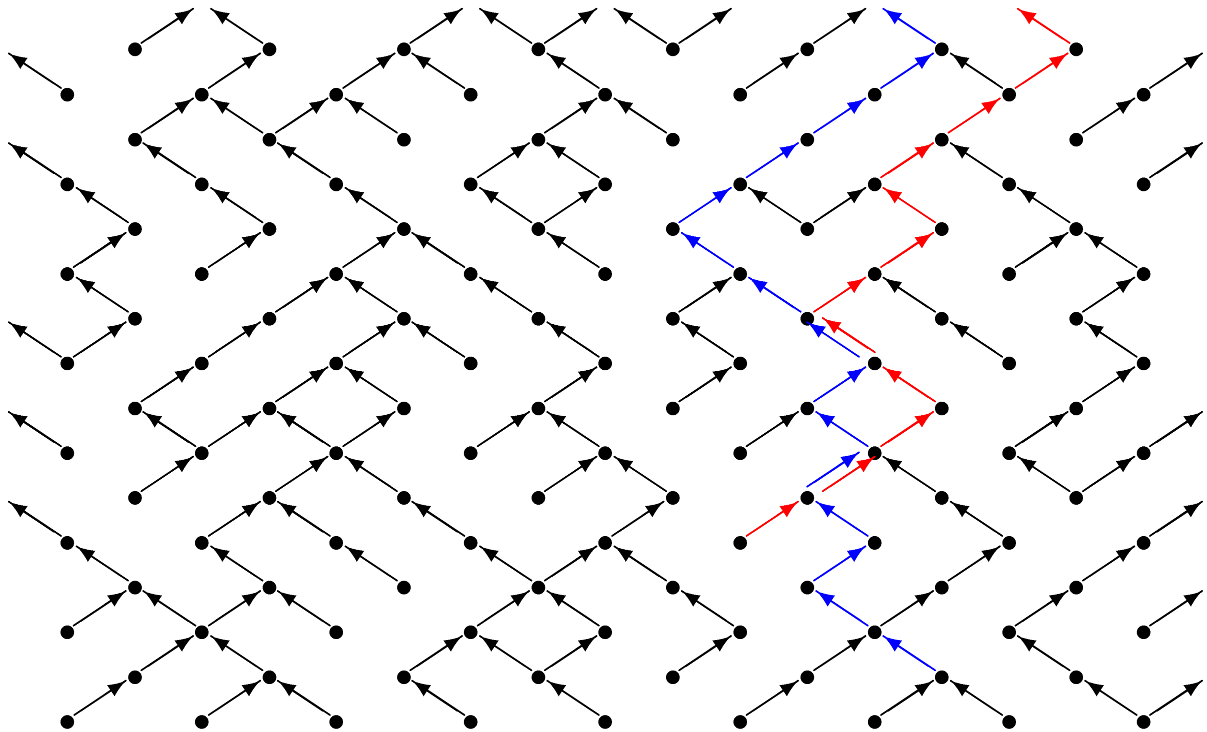
Discrete approximation of the Brownian net.

The Brownian net



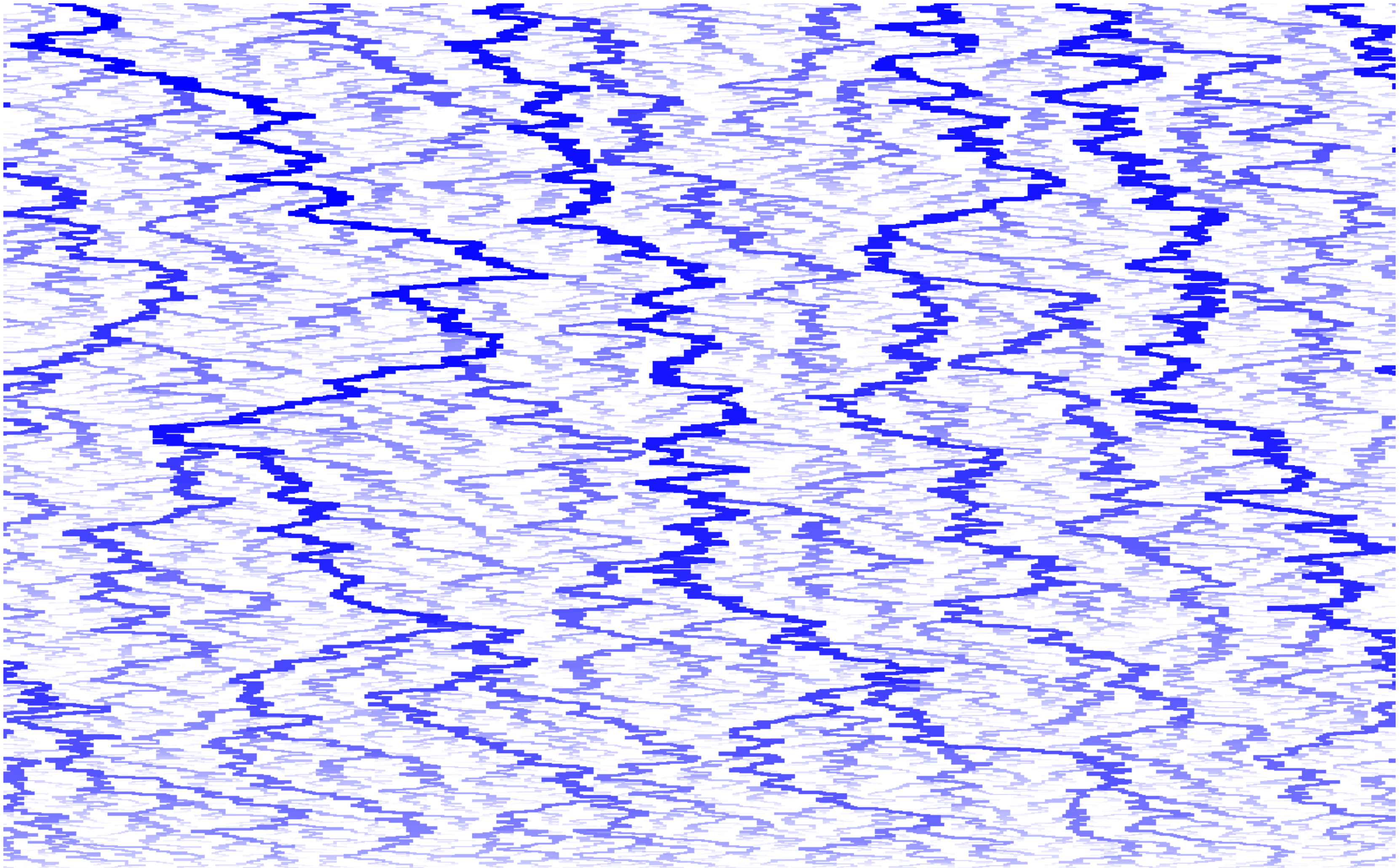
Brownian net.

The Brownian net



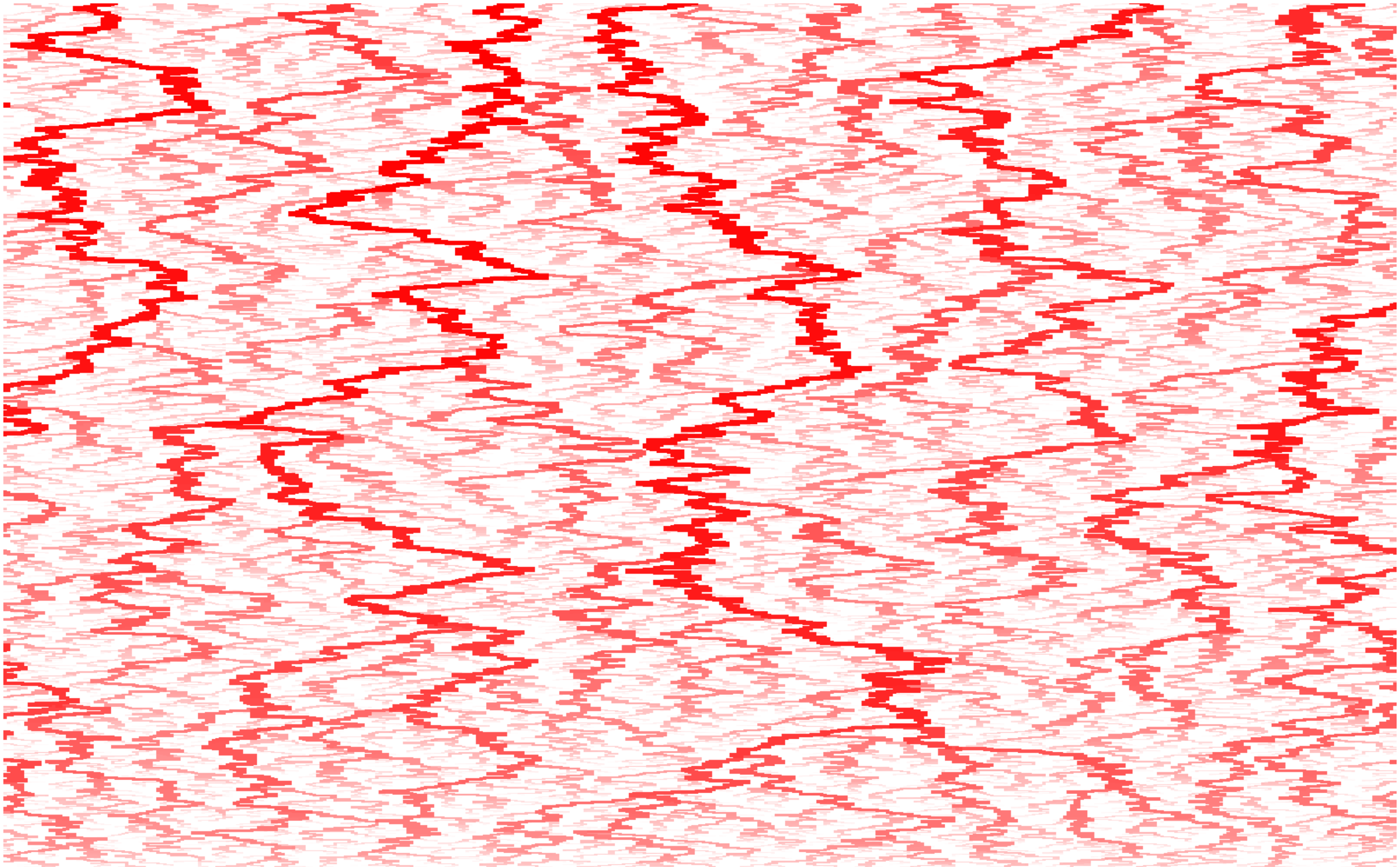
Draw left-most paths in blue and right-most paths in red.

The Brownian net



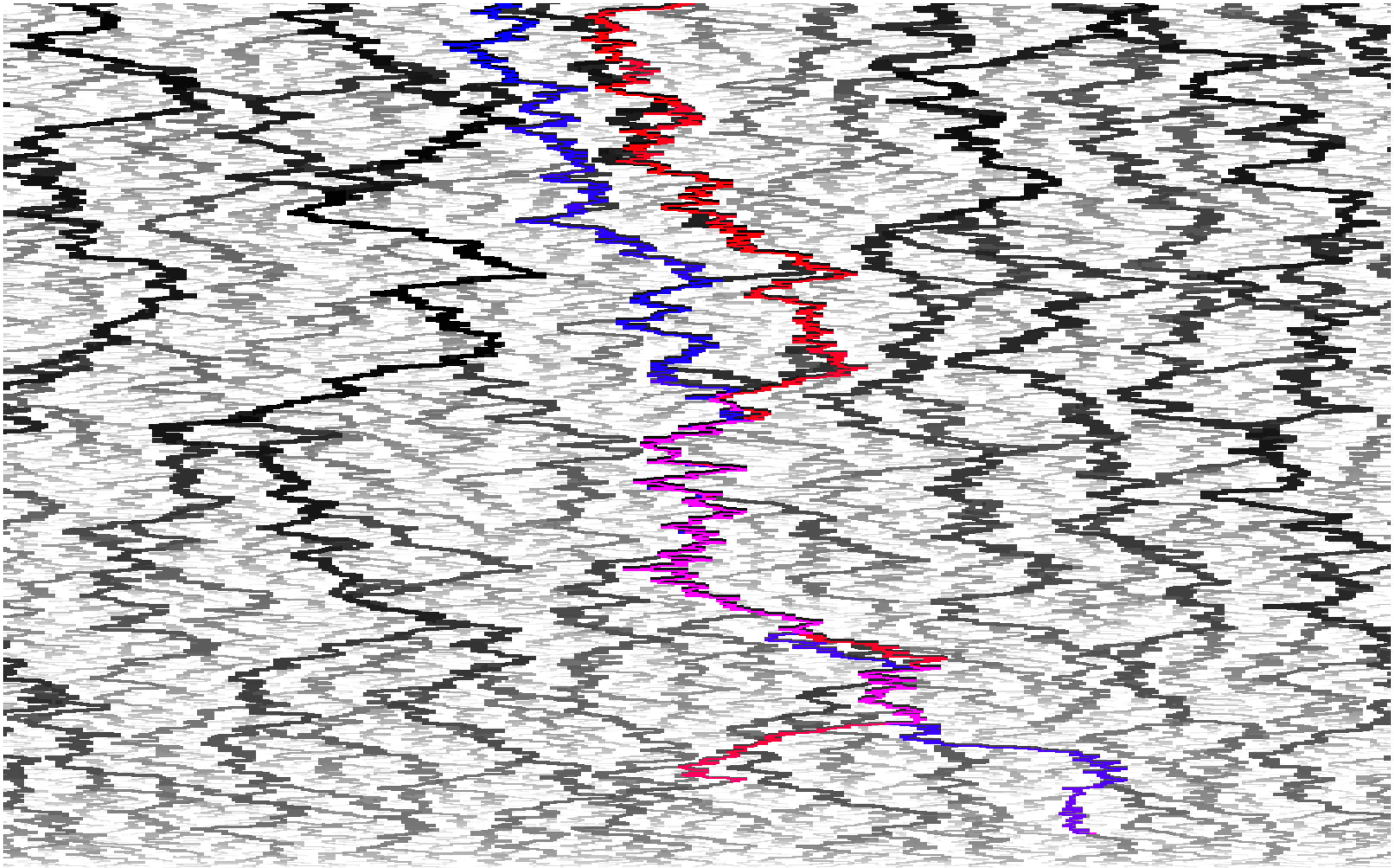
The left-most paths converge to a left Brownian web. . .

The Brownian net



. . . and the right-most paths to a right Brownian web.

The Brownian net



Left-most and right-most paths interact in a sticky way.

The Brownian net

The interaction between left-most and right-most paths is described by the stochastic differential equation (SDE):

$$\mathrm{d}L_t = 1_{\{L_t \neq R_t\}}\mathrm{d}B^{\mathrm{l}}_t + 1_{\{L_t = R_t\}}\mathrm{d}B^{\mathrm{s}}_t - \mathrm{d}t,$$

$$\mathrm{d}R_t = 1_{\{L_t \neq R_t\}}\mathrm{d}B^{\mathrm{r}}_t + 1_{\{L_t = R_t\}}\mathrm{d}B^{\mathrm{s}}_t + \mathrm{d}t,$$

where $B^{\mathrm{l}}_t, B^{\mathrm{r}}_t, B^{\mathrm{s}}_t$ are independent Brownian motions, and L_t and R_t are subject to the constraint that $L_t \leq R_t$ for all $t \geq T := \inf\{u \geq 0 : L_u \leq R_u\}$.

The Brownian net

Introduced by Sun & S. ’08 and by Newman, Ravishankar & Schertzer ’09.

Hopping construction A Brownian net $\mathcal{N}$ is a compact set of paths, such that

(i) At deterministic $z \in \mathbb{R}^2$ there a.s. starts a unique left-most path l_z and right-most paths r_z.

(ii) Paths started at different points are left-right coalescing Brownian motions.

(iii) If $\mathcal{D} \subset \mathbb{R}^2$ is countable and deterministic, then $\mathcal{N}$ is the closure of all paths that are finite concatenations of paths in $\{l_z : z \in \mathcal{D}\}$ and $\{r_z : z \in \mathcal{D}\}$.

Alternative constructions: *wedges, meshes* (Sun & S.), *marking* (Newman, Ravishankar & Schertzer).

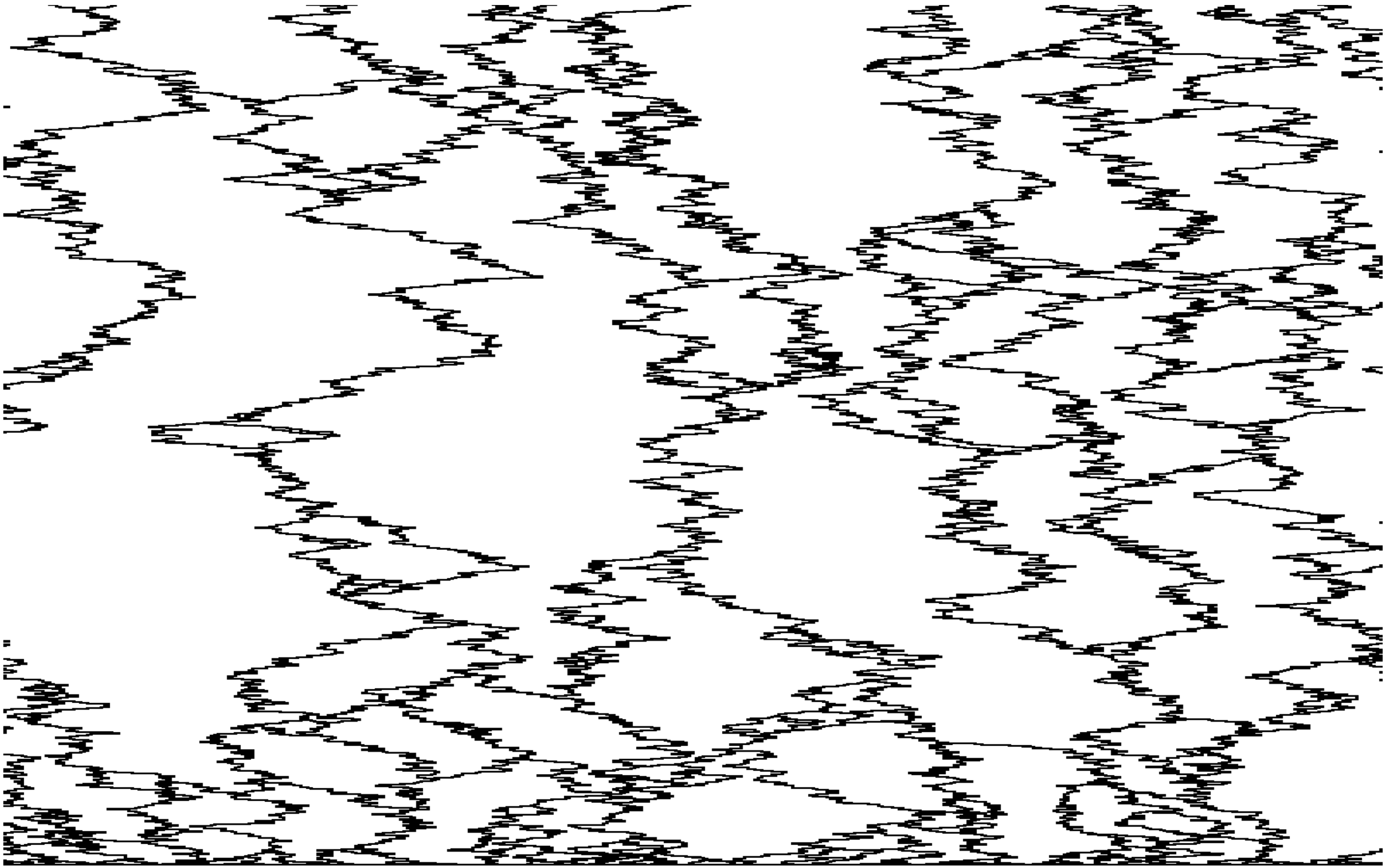
The Brownian net

Let $\mathcal{N}$ be a Brownian net. Let $\xi_0 \subset \mathbb{R}$ be closed. Then

$$\xi_t := \big\{x : \exists y \in \xi_0 \text{ s.t. } \exists \text{ path in } \mathcal{N} \text{ from } (y,0) \text{ to } (x,t)\big\}$$

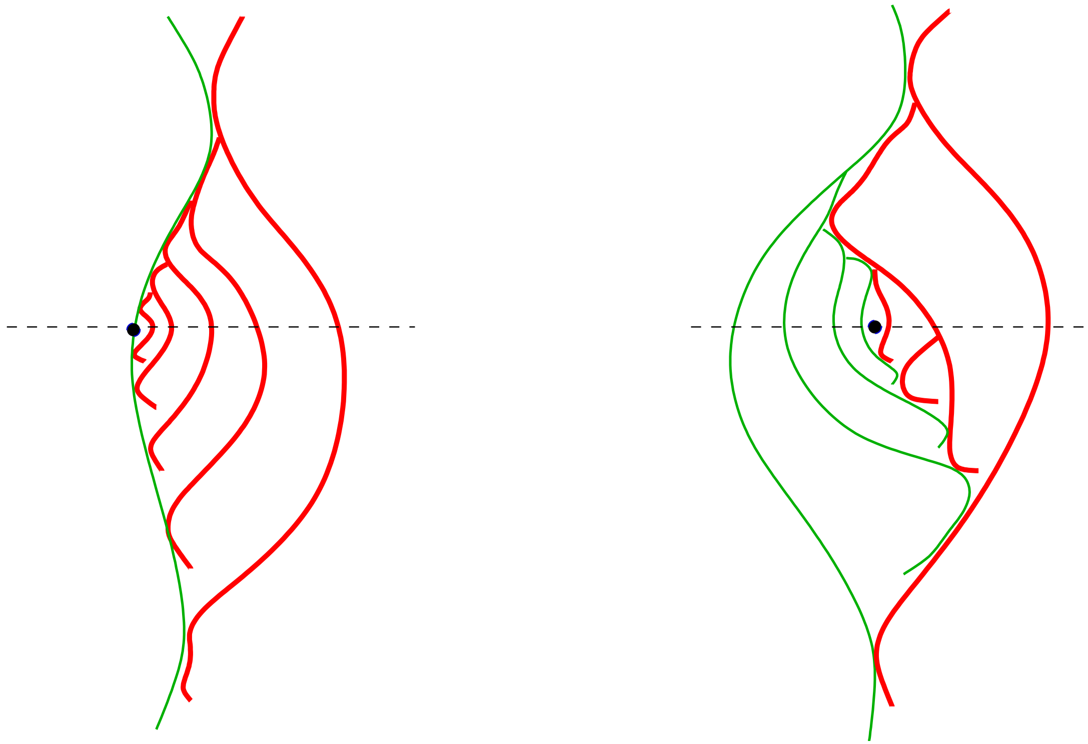
defines a Markov process taking values in the closed subsets of $\mathbb{R}$, called *branching-coalescing point set*. At deterministic times $t > 0$, the set ξ_t is locally finite. There exist random times when ξ_t has no isolated points.

The branching-coalescing point set



The branching-coalescing point set started in $\xi_0 = \mathbb{R}$.

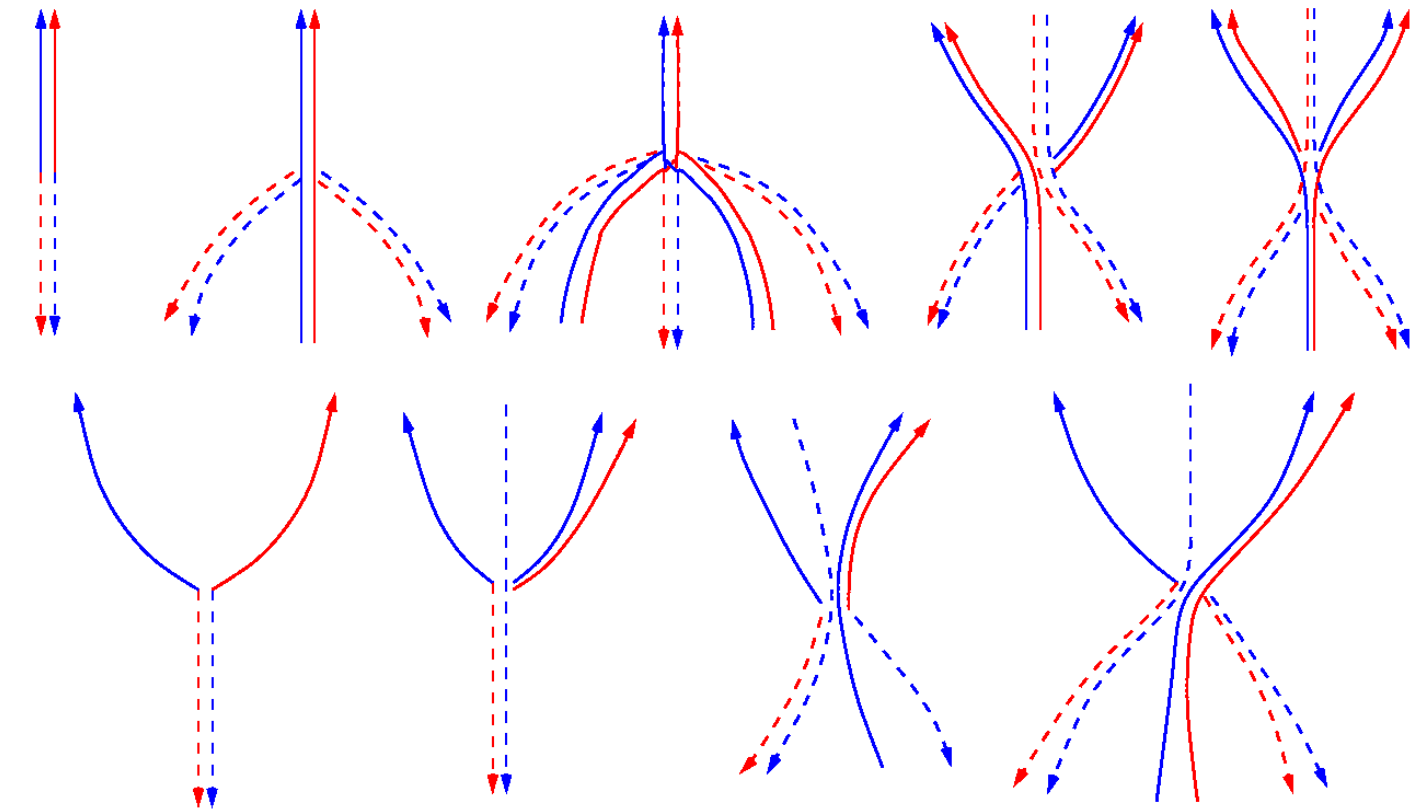
The Brownian net



(C_l) (C_n)

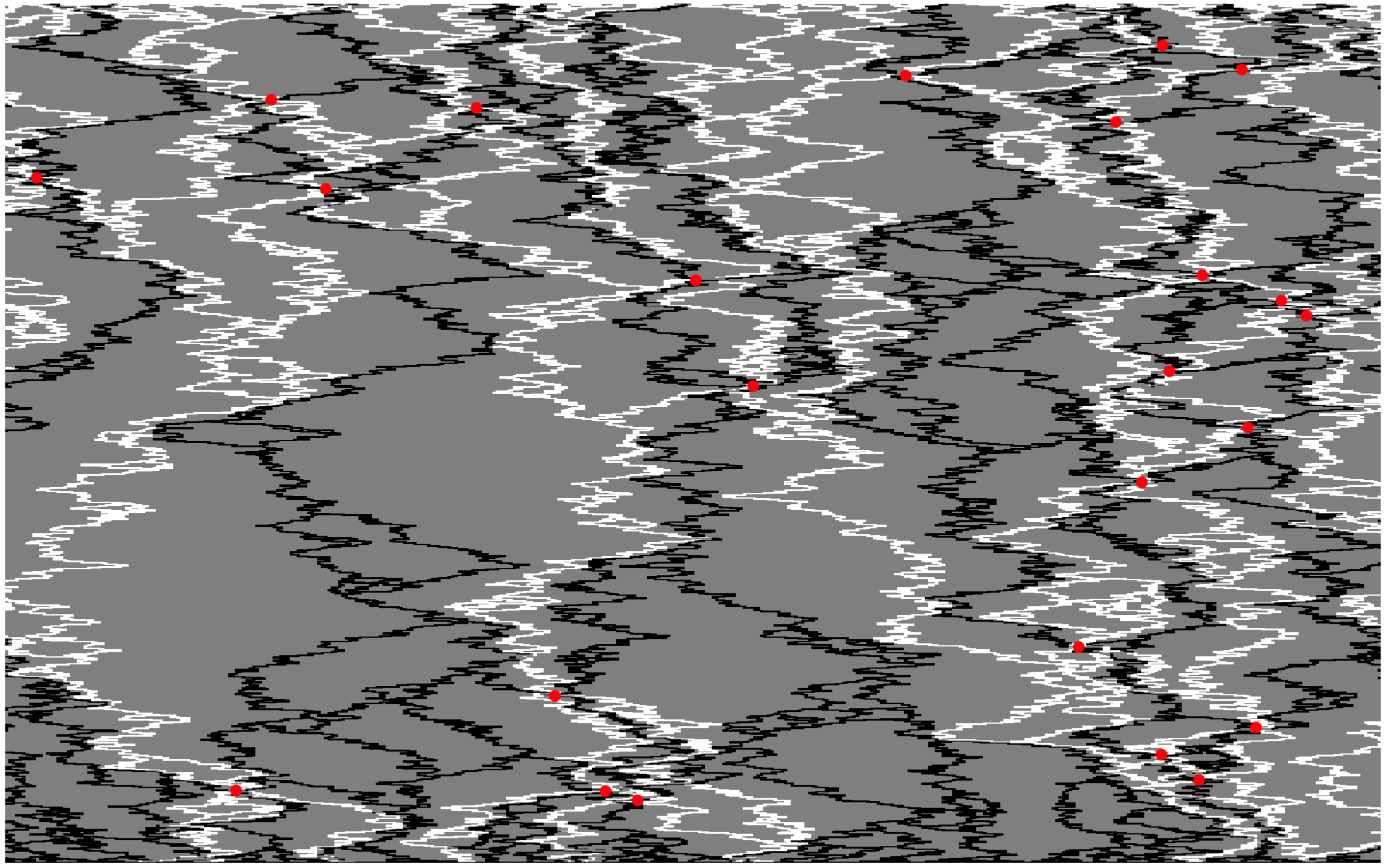
Cluster points of nested excursions between left-most and right-most paths give rise to random times when ξ_t has no isolated points and ρ_t is purely non-atomic.

The Brownian net



Modulo symmetry, there exist 9 types of special points of the Brownian net. [Schertzer, Sun & S. '09].

The Brownian net



‘Relevant’ separation points, where the forward Brownian net crosses its dual, are locally finite.