On rebellious voter models

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Outline

- Definition of the models

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- Theoretical results

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- Theoretical results
- Numerical results

The Neuhauser-Pacala model

Neuhauser & Pacala (1999): Markov process in space $\{(x(i))_{i\in\mathbb{Z}^d} : x(i) \in \{0,1\}\}$, where spin $x(i)$ flips:

$$0 \mapsto 1 \text{ with rate } f_1(f_0 + \alpha_{01} f_1),$$
$$1 \mapsto 0 \text{ with rate } f_0(f_1 + \alpha_{10} f_0),$$

with

$$f_\tau(i) := \frac{\#\{j \in \mathcal{N}_i : x(j) = \tau\}}{\#\mathcal{N}_i} \quad \mathcal{N}_i := \{j : 0 < \|i-j\|_\infty \leq R\}.$$

the local frequency of type $\tau = 0, 1$.

Interpretation: *Interspecific competition rates* α_{01}, α_{10}. Organism of type 0 dies with rate $f_0 + \alpha_{01} f_1$ and is replaced by type sampled at random from distance $\leq R$.

The Neuhauser-Pacala model

Case $\alpha_{01} = \alpha_{10} = 1$ is pure voter model. Case $\alpha_{01}, \alpha_{10} < 1$ gives advantage to minority types.

Definitions: Type τ *survives* if started with a single site of type τ, there is a positive probability that there are sites of type τ at all times.
One has *coexistence* if there exists an invariant law concentrated on states with sites of both types.

Pure voter model: Neither type survives. One has coexistence iff $d \geq 3$.

Duality

In the *symmetric case* $\alpha_{01} = \alpha_{10} =: \alpha$ the Neuhauser-Pacala model X is dual to a system Y of branching-annihilating particles.

Dual model:
If $y(i) = 1$ there is a particle at i.
With rate α a particle at i jumps to a uniformly chosen site in $\mathcal{N}_i$.
With rate $1 - \alpha$ a particle at i gives birth to two new particles at independently, uniformly chosen sites in $\mathcal{N}_i$.
Two particles at the same site annihilate.

$$\mathbb{P}\big[|X_t Y_0| \text{ is odd}\big] = \mathbb{P}\big[|X_0 Y_t| \text{ is odd}\big] \qquad (t \geq 0)$$

whenever X and Y are independent. Here

$$|x| := \textstyle\sum_i x(i) \quad \text{and} \quad xy(i) := x(i)y(i).$$

The rebellious voter model

One-sided rebellious voter model Spin $x(i)$ flips:

$$0 \leftrightarrow 1 \text{ with rate } \alpha 1_{\{x(i-1)\neq x(i)\}} + (1-\alpha)1_{\{x(i-2)\neq x(i-1)\}}.$$

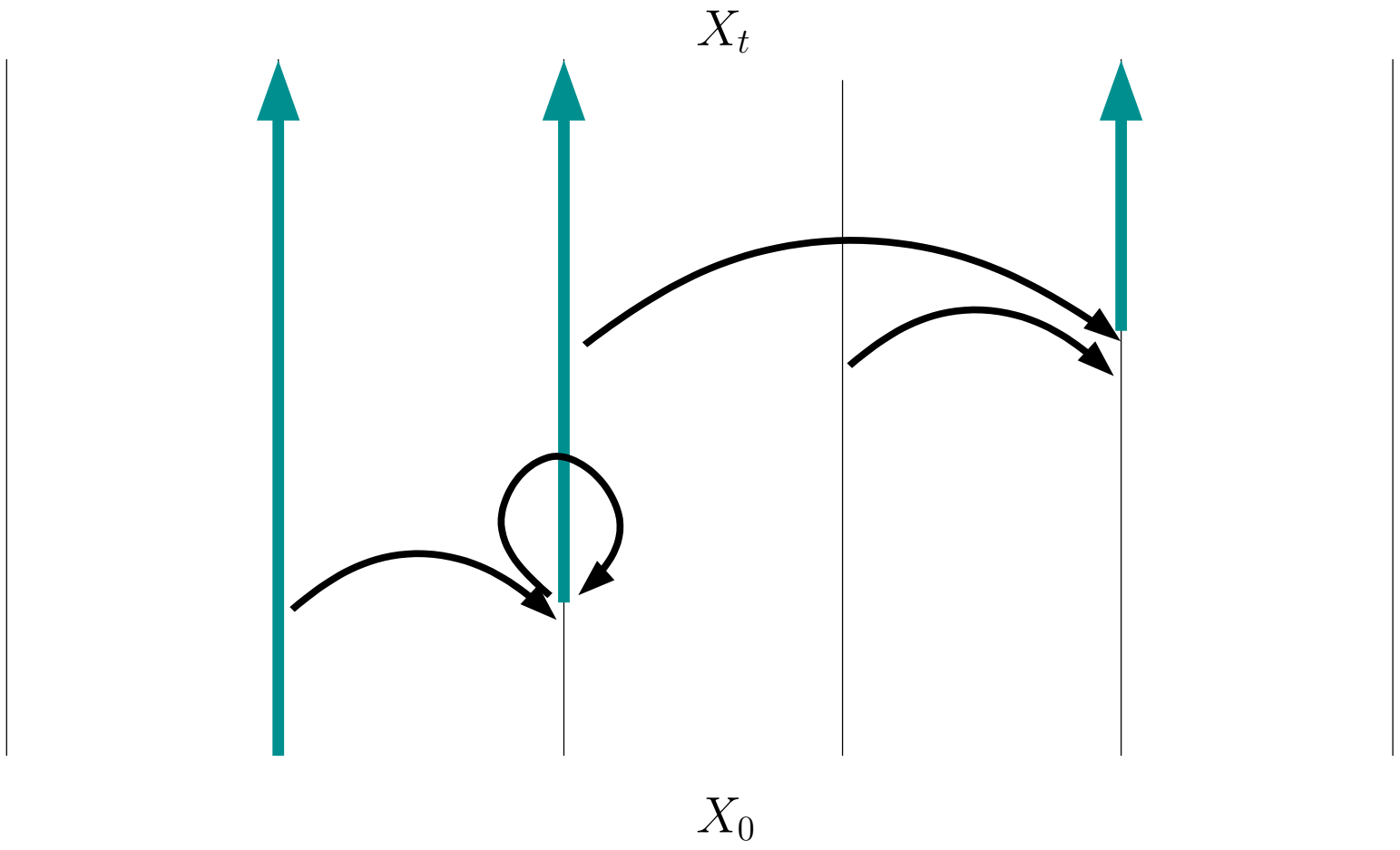
Two-sided rebellious voter model

$$0 \leftrightarrow 1 \text{ with rate } \tfrac{1}{2}\alpha 1_{\{x(i-1)\neq x(i)\}} + \tfrac{1}{2}(1-\alpha)1_{\{x(i-2)\neq x(i-1)\}}$$
$$\tfrac{1}{2}\alpha 1_{\{x(i)\neq x(i+1)\}} + \tfrac{1}{2}(1-\alpha)1_{\{x(i+1)\neq x(i+2)\}}.$$

Dual one-sided model Particles jump from i to $i-1$ with rate α and produce two new particles at $i-2, i-1$ with rate $1-\alpha$.

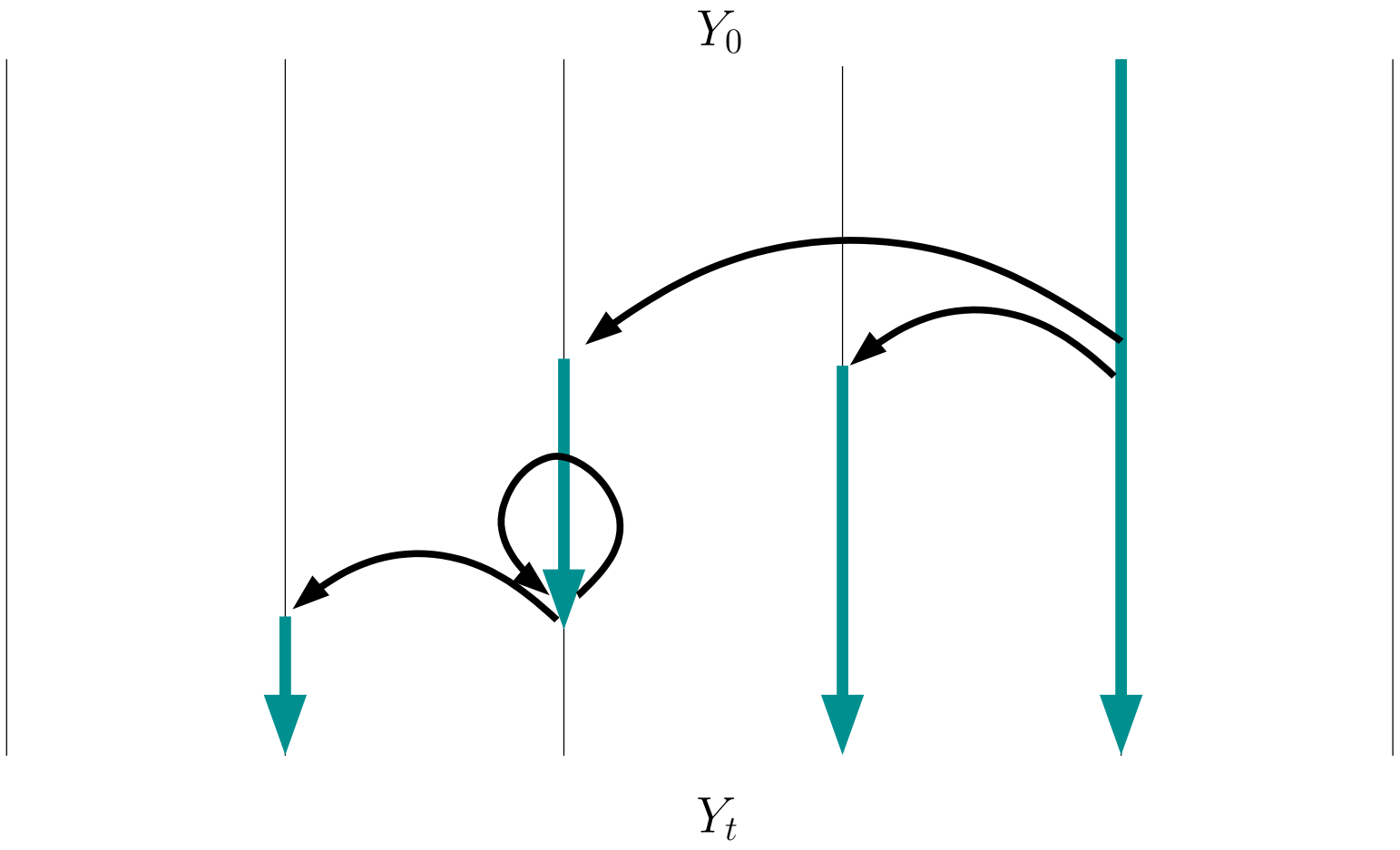
Dual two-sided model analoguous.

Graphical representation



Graphical representation of the rebellious voter model.

Graphical representation



Graphical representation of the dual of the rebellious voter model.

Proof of duality

$$
\begin{aligned}
& |X_t Y_0| \text{ is odd} \\
\Leftrightarrow \quad & \# \text{ paths from } X_0 \text{ to } Y_0 \text{ is odd} \\
\Leftrightarrow \quad & |X_0 Y_t| \text{ is odd.}
\end{aligned}
$$

Consequences of duality

The branching-annihilating particle system Y *preserves parity*.

If X is started in product measure with intensity $1/2$ and $Y_0 = 1_{\{i,j\}}$, then

$$\mathbb{P}\big[X_t(i) \neq X_t(j)\big] = \mathbb{P}\big[|X_t Y_0| \text{ is odd}\big] = \mathbb{P}\big[|X_0 Y_t| \text{ is odd}\big]$$
$$= \tfrac{1}{2}\mathbb{P}\big[Y_s \neq 0\big] \underset{t\to\infty}{\longrightarrow} \tfrac{1}{2}\mathbb{P}\big[Y_s \neq 0 \ \forall s \geq 0\big].$$

Consequence: X has coexistence iff Y started with an even number of particles survives.

Similarly: X survives iff Y has a nontrivial invariant law.

First results

Recall: $\alpha = 1$ is pure voter, $1 - \alpha$ is branching rate of Y.

Neuhauser & Pacala '99 If $d \vee R > 1$, then one has coexistence and survival of both types for α suffiently close to zero.

In the special case $d = 1 = R$ ('disagreement voter model'), one has noncoexistence for all $\alpha > 0$.

Conjecture Except in the case $d = 1 = R$, one has coexistence for all $\alpha < 1$.

Cox, Perkins and Merle

Cox & Perkins '07 In dimensions $d \geq 3$ there exists some $0 < c < 1$ such that for $\alpha_{01} \wedge \alpha_{10}$ sufficiently close to *one* and $c\alpha_{01} \leq \alpha_{10} \leq c^{-1}\alpha_{01}$, one has coexistence and survival of both types.

Cox, Merle & Perkins '10 In dimensions $d = 2$ there exists some function $f : [0,1] \to [0,1]$ with $f(\alpha) < \alpha$ on $(0,1)$ such that for $\alpha_{01} \wedge \alpha_{10}$ sufficiently close to *one* and $\alpha_{01} \geq f(\alpha_{10})$, $\alpha_{10} \geq f(\alpha_{01})$, one has coexistence and survival of both types.

Proof As $\alpha \uparrow 1$, the process X started with a sparse configuration of ones, suitably rescaled, converges to supercritical super-Brownian motion. Comparison wih oriented percolation.

Morally, this implies coexistence for all $0 \leq \alpha < 1$ but not known if survival of Y is monotone in the branching rate $1 - \alpha$.

Dimension one

Corrected conjecture In dimension $d = 1$, there exists some $0 \leq \alpha_c < 1$ such that the symmetric model has coexistence for $\alpha < \alpha_c$ and noncoexistence for $\alpha_c < \alpha$.

Open problem Prove noncoexistence in any other case than 'trivial' $R = 1$.

Open problem Prove that noncoexistence is monotone in α.

Interface model

Interface model $(Y_t)_{t\geq 0}$ associated with $(X_t)_{t\geq 0}$ defined by

$$Y_t(i) := 1_{\{X_t(i)\neq X_t(i+1)\}} \qquad (i \in \mathbb{Z}).$$

$$X_t = \ldots 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ \ldots$$
$$Y_t = \ldots\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ \ldots$$

$$\begin{tikzcd} X \arrow[d, bend left, "\text{interface}"] \\ Y \arrow[u, bend left, "\text{dual}"] \end{tikzcd}$$

$$\begin{tikzcd} X' \arrow[r, "\text{interface}"] \arrow[d, leftrightarrow, "\text{dual}"'] & Y'' \arrow[d, leftrightarrow, "\text{dual}"] \\ Y' & X'' \arrow[l, "\text{interface}"'] \end{tikzcd}$$

voter models

X	rebellious
X'	disagreement
X''	swapping

random walks

Y	ADBARW
Y'	DBARW
Y''	SARW

Interface tightness

Definition A one-dimensional voter model X exhibits *interface tightness* if its interface model Y started with an odd number of particles is positively recurrent modulo translations.

Consequence System spends positive fraction of time in states with $|Y| = 1$.

Interface tightness for long-range voter models was proved by Cox and Durrett (1995) under a third moment condition on the infection rates. This was improved to a second moment condition, which is sharp, by Belhaouari, Mountford and Valle (2007).

The swapping voter model

The swapping voter model X'' has a mixture of voter and exclusion dynamics:

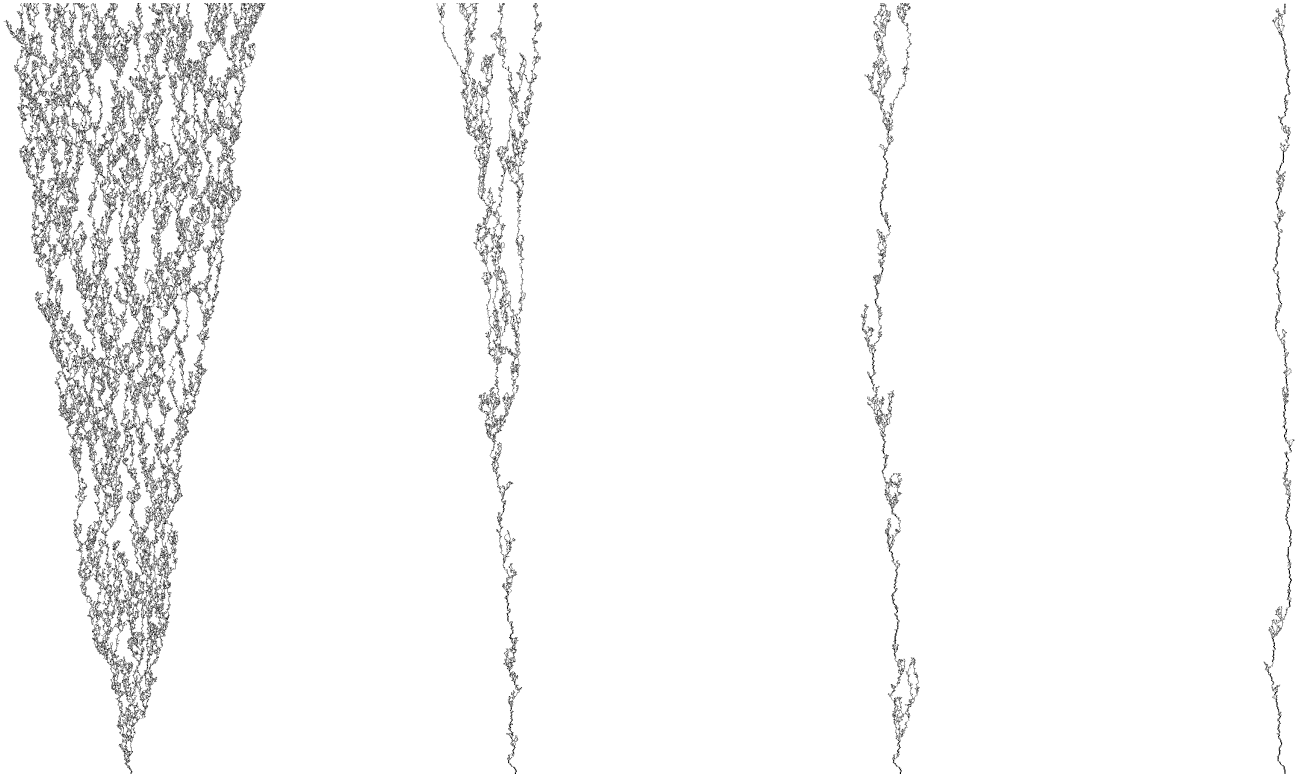
$$\begin{aligned} 01 &\to 11 \text{ with rate } \tfrac{1}{2}\alpha, \\ 01 &\to 00 \text{ with rate } \tfrac{1}{2}\alpha, \\ 01 &\leftrightarrow 10 \text{ with rate } 1-\alpha. \end{aligned}$$

For this model, the number of ones (resp. zeroes) is a martingale, hence in X'' both types die out for $\alpha > 0$.

The dual is a system of swapping and annihilating random walks (without branching), hence X'' exhibits noncoexistence for $\alpha > 0$.

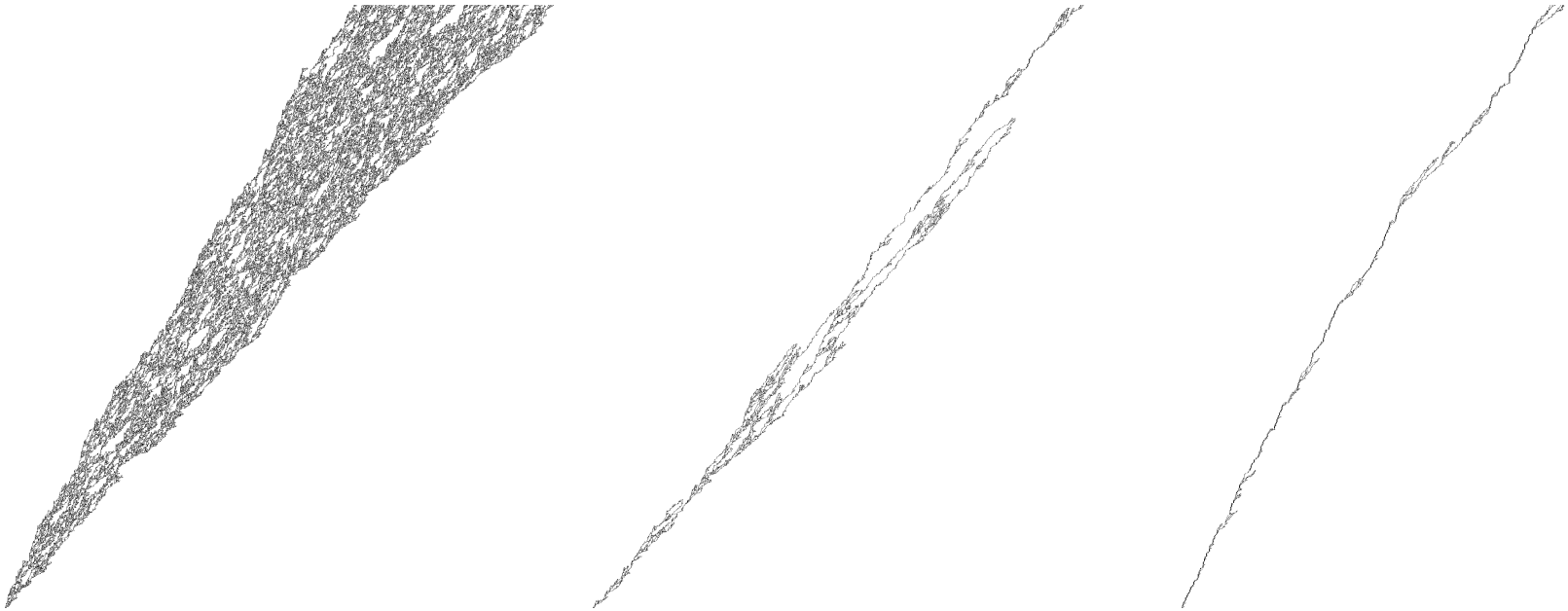
Interface tightness for X'' was proved in Sturm and S. (2008).

Two-sided rebellious interface model



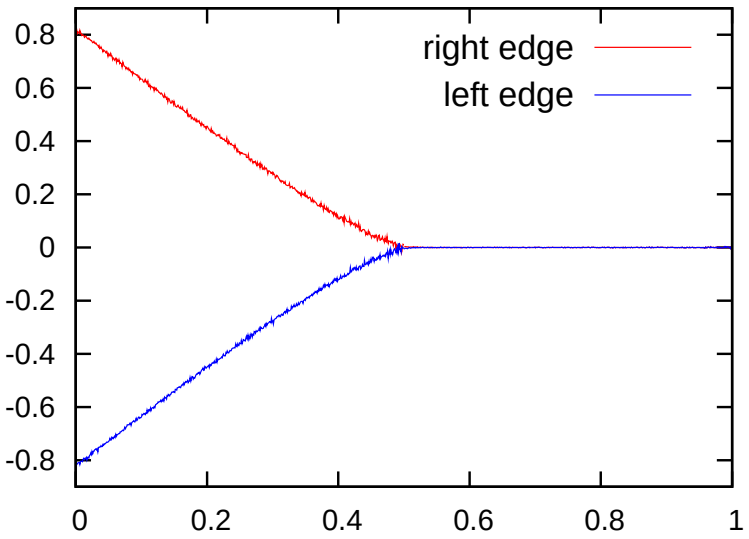
Interface process Y of the two-sided rebellious voter model for $\alpha = 0.4, 0.5, 0.51, 0.6$.

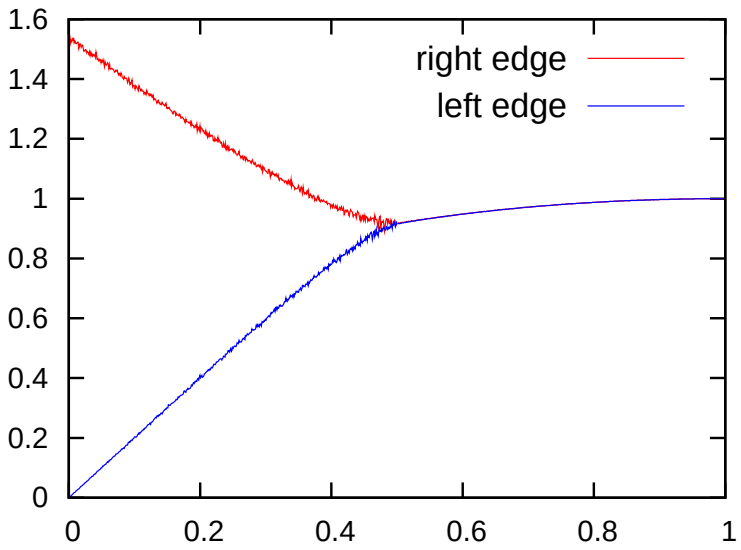
One-sided rebellious interface model



Interface process Y of the one-sided rebellious voter model for $\alpha = 0.3, 0.5, 0.6$.

Edge speeds





Edge speeds for the rebellious voter model (left) and its one-sided counterpart (right).

Theoretical results

Sturm and S. (2008):

Neuhauser-Pacala models and rebellious voter model:
If X exhibits coexistence, then there is a unique shift-invariant coexisting invariant law which is the limit law started from any shift-invariant coexisting initial law.

Rebellious voter model:
Coexistence for α sufficiently close to zero.
Complete convergence for α sufficiently close to zero.
Survival equivalent to coexistence.

Numerical results

S. and Vrbenský (2010):

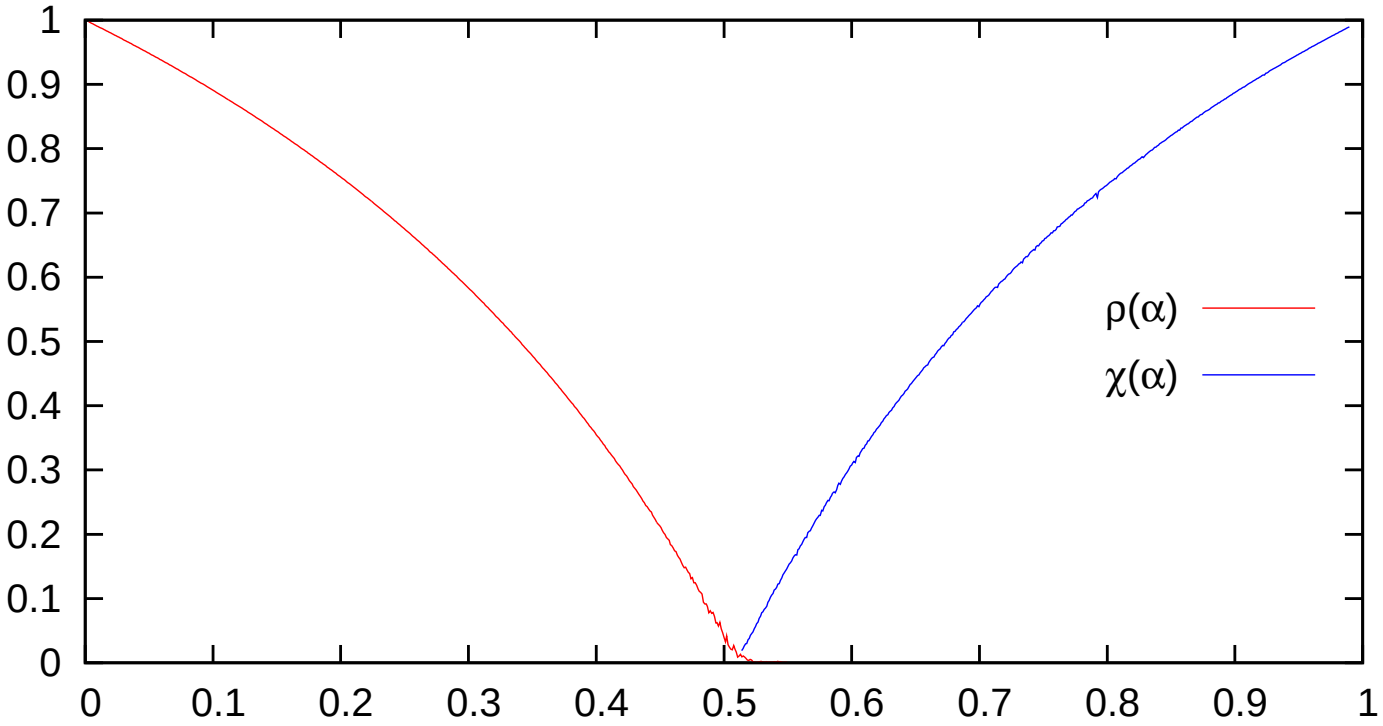
Start $X_0 = \ldots 00000100000 \ldots$, $Y_0 = \ldots 00000100000 \ldots$.
Define

$$\rho(\alpha) = \mathbb{P}[X_t \neq 0 \ \forall t \geq 0],$$

$$\chi(\alpha) = \lim_{t \to \infty} \mathbb{P}[|Y_t| = 1].$$

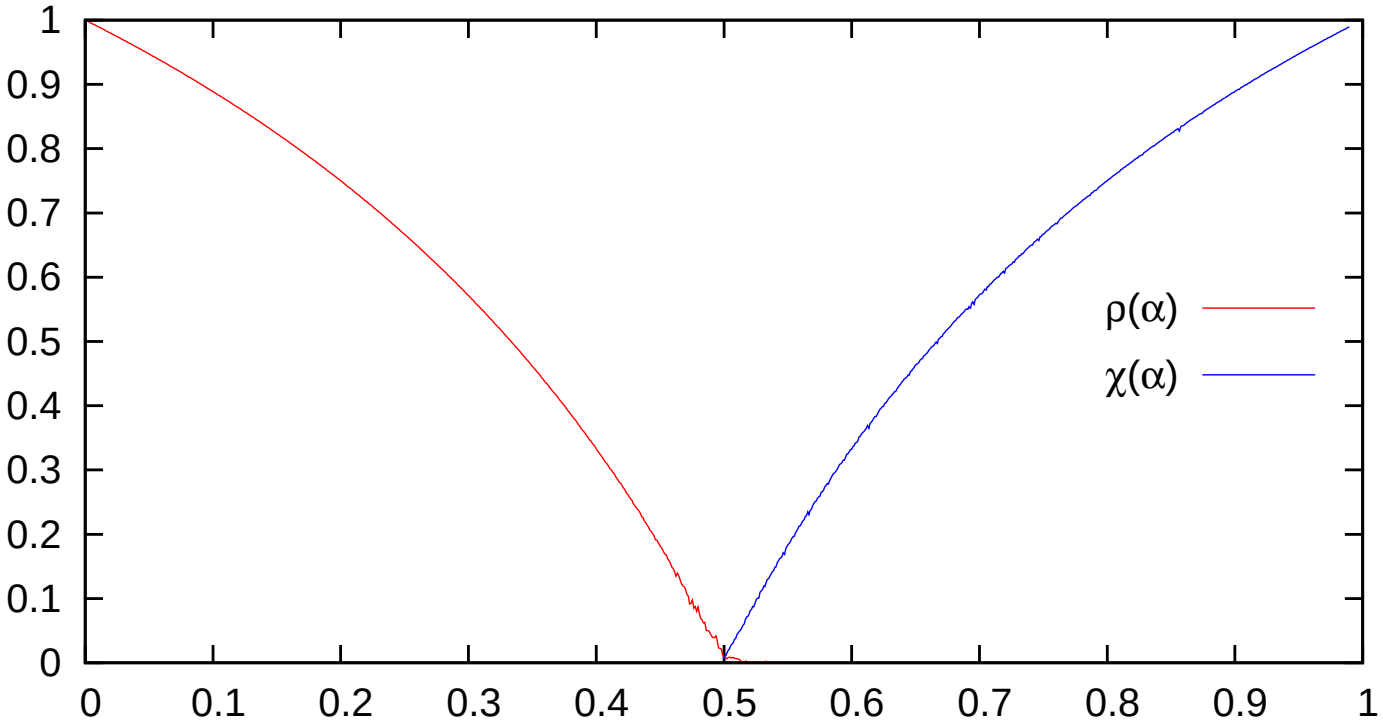
$\rho(\alpha) > 0$ iff ones survive,
$\chi(\alpha) > 0$ iff interface tightness.

Numerical data



The functions ρ and χ for the two-sided rebelious voter model.

Numerical data



The functions ρ and χ for the one-sided rebelious voter model.

Explicit formulas

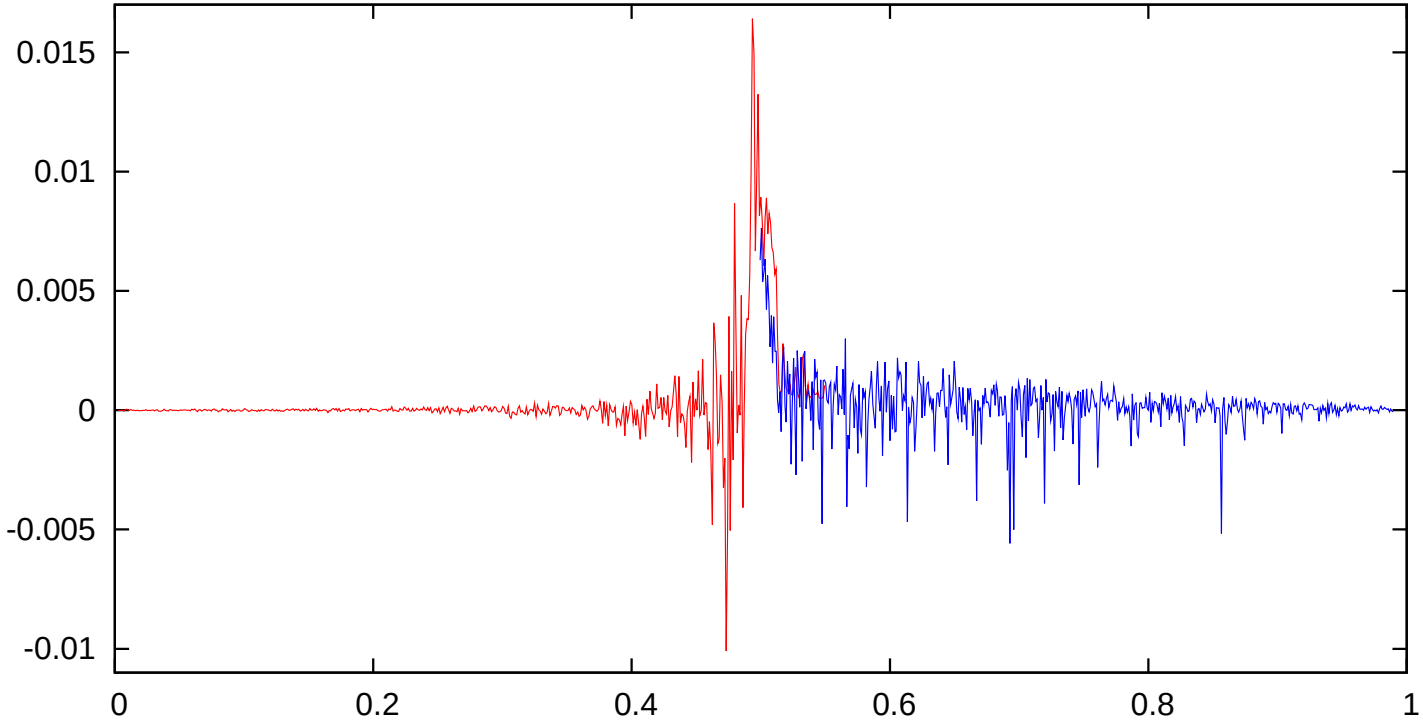
It seems that for the one-sided model, the functions ρ and χ are described by the explicit formulas:

$$\rho(\alpha) = 0 \vee \frac{1-2\alpha}{1-\alpha} \quad \text{and} \quad \chi(\alpha) = 0 \vee \big(2 - \frac{1}{\alpha}\big).$$

In particular, one has the symmetry $\rho(1-\alpha) = \chi(\alpha)$ and the critical parameter seems to be given by $\alpha_{\rm c} = 1/2$.

Explanation?

Numerical data



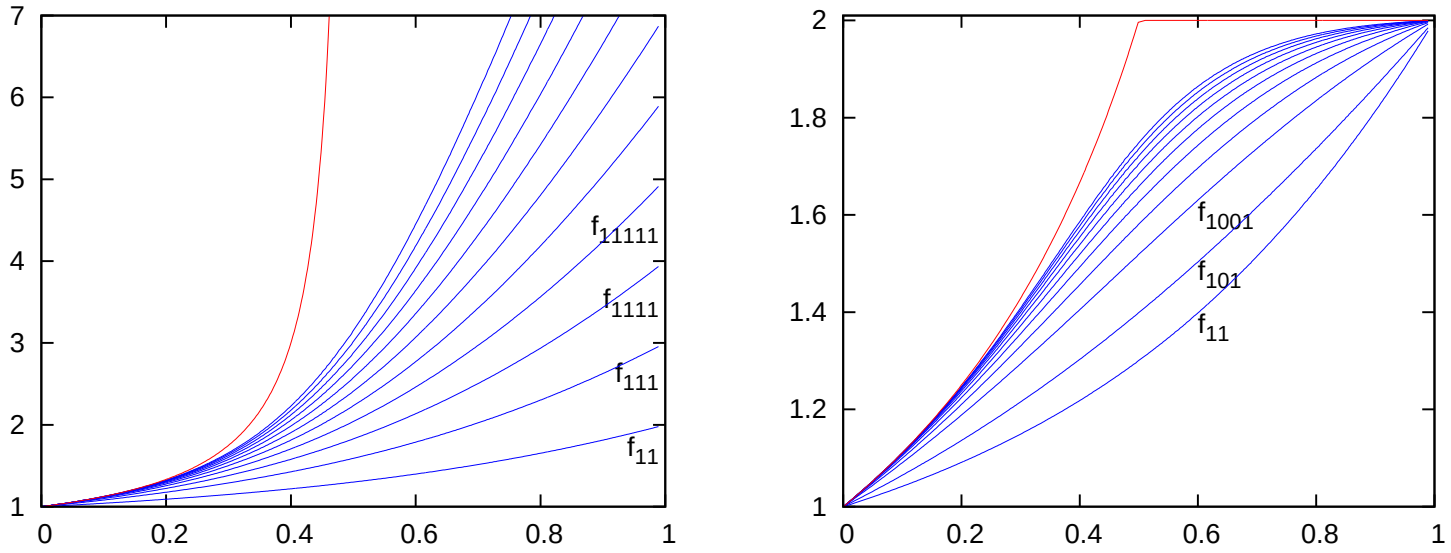
Differences of ρ and χ with presumed explicit formulas.

Harmonic functions

Recall that for the voter model, the number of ones is a martingale, hence $f_x := |x|$ is a harmonic function.

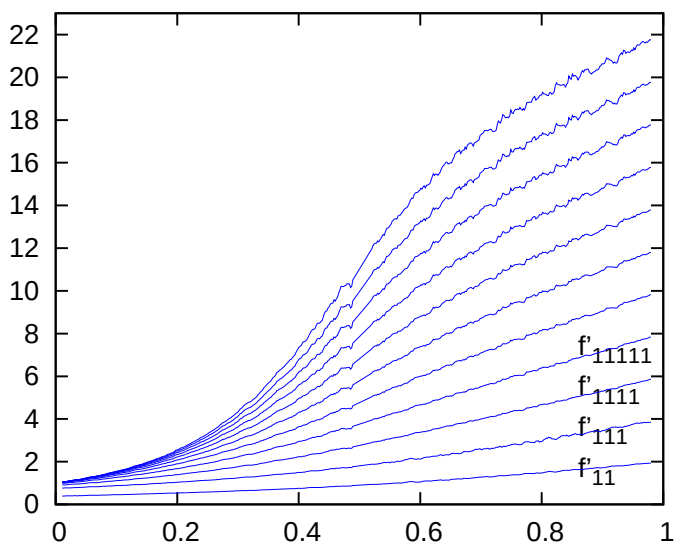
Numerically, we can find a harmonic function $f_x(\alpha)$ for all values of α.

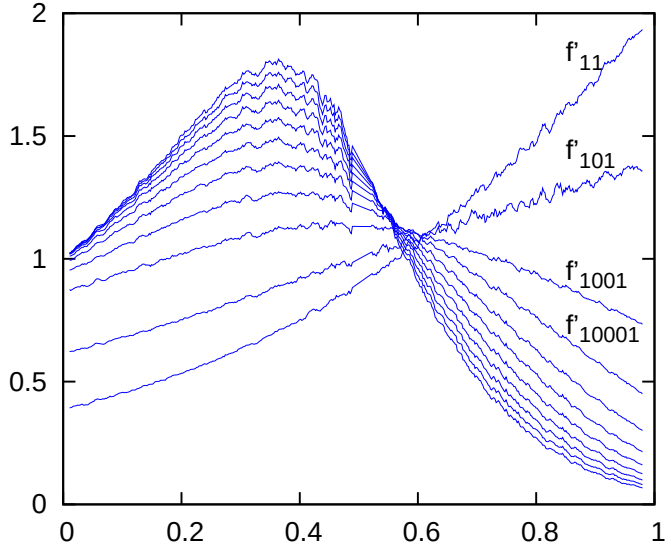
Harmonic functions



Numerical data for $f_x(\alpha)$ for the one-sided model.

Harmonic functions





Derivatives $\frac{\partial}{\partial \alpha} f_x(\alpha)$.