Jan's open problems

The autodiffusive equation

Statement of the problem

The *autodiffusive equation* is the PDE

$$\frac{\partial}{\partial t}a(t,x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}a(t,x) + a(t,x) \qquad (t \geq 0,\ x \in \overline{D}), \tag{1}$$

where $D \subset \mathbb{R}^d$ is a bounded open set and $\overline{D}$ is its closure. The function a is defined on $[0,\infty) \times \overline{D}$ and takes values in the space M_+^d of symmetric non-negative definite $d \times d$ real matrices. To make the problem more concrete, take $D = (0,1)^2$ and take for $a(0,\cdot)$ the function

$$a(0,x) := \begin{pmatrix} x_2 x_1(1-x_1) & 0 \\ 0 & x_1 x_2(1-x_2) \end{pmatrix}. \tag{2}$$

The open problems are then:

- Show that the PDE (1), interpreted in the right way, has a unique solution with the initial state (2).
- Show that the solution of (1) with initial state (2) converges to a limit $a(\infty,\cdot\,) := \lim_{t\to\infty} a(t,\cdot\,)$. Show that $a(\infty,\cdot\,)$ is a fixed point of (1) and determine its domain of attraction.

What is known

In [FS05], the autodiffusive equation (1) is studied for initial states of the form

$$a(0,x) := \begin{pmatrix} x_1(1-x_1) & 0 \\ 0 & p(x_1)x_2(1-x_2) \end{pmatrix}, \tag{3}$$

where $p : [0,1] \to [0,\infty)$ is Lipschitz continuous with $p(0) = 0$ and $p(1) > 0$. It turns out that if $a(0,\cdot\,)$ is of this form, then it is possible to give a rigorous definition of (1) and show that its solutions stay in the class of diffusion functions a of this form. Moreover, it is shown that solutions converge as $t \to \infty$ to a unique fixed point in this class.

What is conjectured

In [FS05], numerical results are shown for the solution of (1) with initial condition (2). These numerical data come from a naive approximation scheme for which it is not known whether it converges to the true solution of (1), whatever that is. However, the data suggest that the PDE is somehow well-defined and that the solution of (1) with initial condition (2) converges as $t \to \infty$ to a fixed point. It is conjectured that this fixed point is unique in a huge class of diffusion functions on $[0,1]^2$. It is believed that what matters is the "effective boundary" of a diffusion function a, which is defined as $\{x \in \overline{D} : a(x) = 0\}$. For the initial state in (2), this effective boundary is

$$\{x \in [0,1]^2 : x_1 = 0 \text{ or } x_2 = 0\} \cup \{(1,1)\}. \tag{4}$$

It is believed that (1) has a unique attractive fixed point in the class of diffusion functions with this effective boundary.

How to attack the problem

For diffusion functions of the form (3), there is a stochastic representation of solutions to the autodiffusive equation (1), which is why in [FS05] fairly complete results could be proved for this case. In general, it seems there is no hope to find a similar stochastic representation and one has to rely on PDE methods. Even proving that solutions, interpreted in the right way, exist and are unique is a difficult problem since the interesting cases are when the diffusion function is not uniformly elliptic. As a start, one could study the equation on compact manifolds for uniformly elliptic diffusion functions, though it is a question how much this helps towards solving the interesting initial states like (2). It should be noted that by a simple transformation, one can get rid of the linear term $a(t,x)$ on the right-hand side of (1), so if the aim is only to prove existence and uniqueness of solutions, then one can do away with this term. For the fixed points, it is important however.

References

[FS05] K. Fleischmann and J.M. Swart (2005). Renormalization analysis of catalytic Wright-Fisher diffusions. *Electronic J. Probab.* 11 (2006) paper no. 24, 585–654.