Jan's open problems

The half-sphere SDE

Statement of the problem

It is known that for each $c \geq 0$ and for each initial condition $X_0 = x$ in the unit ball $\{x \in \mathbb{R}^d : |x| \leq 1\}$, the d-dimensional the Itô stochastic differential equation (SDE)

$$\mathrm{d}X_t = -cX_t\mathrm{d}t + \sqrt{2(1-|X_t|^2)}\mathrm{d}B_t \qquad (t \geq 0), \tag{1}$$

has a unique weak solution that stays inside the unit ball. The question is for what values of d and c this SDE has a unique strong solution.

What is known

Standard results show that in dimension $d = 1$ (1) has a unique strong solution for all $c \geq 0$. For dimensions $d \geq 2$, it has been proved in [Swa02] that strong uniqueness holds for $c = 0$ and for $c \geq 1$. This has been improved in [DeB04] where it was shown that strong uniqueness holds for $c > \frac{1}{2}(\sqrt{2}-1)$.

What is conjectured

It seems plausible that there exists a critical value $c_*(d)$ such that strong uniqueness does not hold for $0 < c < c_*(d)$ and on the other hand strong uniqueness holds for $c > c_*(d)$. But it is hard to tell, perhaps strong uniqueness holds for all $c \geq 0$ after all and we just don't know how to prove it.

How to attack the problem

Past experience has shown that numerical simulations can help quite a bit to find the right conjectures that one can then try to prove rigorously. One can construct finite state space Markov chains that in a natural way approximate the process consisting of two solutions to the SDE, driven by the same Brownian motion. For such a Markov chain, one can try to calculate the expected time before two solutions reach a fixed distance ε from each other, as a function of the initial state. This function solves a difference equation, that in the continuum limit becomes a differential equation. Thus, it seems that the problem is closely related to potential theory.

References

[DeB04] D. DeBlassie. Uniqueness for diffusions degenerating at the boundary of a smooth bounded set. *Ann. Probab.* 32(4) (2004), 3167–3190.

[Swa02] J.M. Swart. Pathwise uniqueness for a SDE with non-Lipschitz coefficients. *Stochastic Processes Appl.* 98 (2002), 131–149.