Jan's open problems

Sharpness of the phase transition for local survival

Statement of the problem

Contact processes on Cayley graphs with fixed infection rates and variable death rate have two critical values $\delta'_{\rm c} \leq \delta_{\rm c}$ of the death rate. Here $\delta_{\rm c}$ is the critical death rate for global survival and $\delta'_{\rm c}$ is the critical death rate for local survival. It is known that $\delta'_{\rm c} = \delta_{\rm c}$ on some graphs such as $\mathbb{Z}^d$ while $\delta'_{\rm c} < \delta_{\rm c}$ on other graphs such as regular trees, but it is not known exactly for which graphs the two critical rates are equal.

It is known that contact processes exhibit *sharpness of the phase transition* in the sense that the expected size of the population decreases exponentially fast in the whole subcritical[1] regime $\delta > \delta_{\rm c}$. The open problem is to prove a local version of this result, saying that the probability that the origin is infected decreases exponentially fast in the whole subcritical regime $\delta > \delta'_{\rm c}$. This is the informal statement of the problem. We now formulate a precise conjecture.

Let Λ be a countable group with group action $(i,j) \mapsto ij$, inverse operation $i \mapsto i^{-1}$, and unit element 0 (also refered to as the origin). A contact process is a Markov process $\eta = (\eta_t)_{t\geq 0}$ taking values in the subsets of a countable set Λ, with the following description. If $i \in \eta_t$, then we say that the site i is infected at time t; otherwise it is vacant. Infected sites i infect vacant sites j with *infection rate* $a(i,j) \geq 0$, and infected sites become vacant with *death rate* $\delta \geq 0$. We assume that the infection rates satisfy $a(i,i) = 0$ $(i \in \Lambda)$ and

$$\begin{aligned} &\text{(i)} \quad a(i,j) = a(ki,kj) \qquad\qquad (i,j,k \in \Lambda),\\ &\text{(ii)} \quad |a| := \sum_{i\in\Lambda} a(0,i) < \infty, \end{aligned} \tag{1}$$

which means they are invariant under the left action of the group and summable. The formal generator of the process is given by

$$\begin{aligned} Gf(A) := &\sum_{i,j\in\Lambda} a(i,j) 1_{\{i\in A\}} 1_{\{j\notin A\}} \{f(A\cup\{j\}) - f(A)\}\\ &+\delta \sum_{i\in\Lambda} 1_{\{i\in A\}} \{f(A\backslash\{i\}) - f(A)\}. \end{aligned} \tag{2}$$

We call this the (Λ,a,δ)-*contact process*. In particular, we can consider the case that a is of the form

$$a(i,j) = \begin{cases} \lambda & \text{if } j = ik \text{ for some } k \in \Delta,\\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where $\lambda \geq 0$ is a constant and Δ is a finite set that generates Λ and satisfies $0 \notin \Delta$ and $i^{-1} \in \Delta$ for all $i \in \Delta$. Then the (Λ,a,δ)-contact process is the contact process with infection rate λ and death rate δ on the Cayley graph $G = (V,E)$ with vertex set $V := \Lambda$ and edge set $E := \big\{\{i,j\} : i^{-1}j \in \Delta\big\}$.

The state space of a (Λ,a,δ)-contact process is the set $\mathcal{P}(\Lambda)$ of all subsets of Λ. By identifying a set with its indicator function, we can identify $\mathcal{P}(\Lambda)$ with $\{0,1\}^\Lambda$. We equip the latter with

[1] This is called "subcritical" because of the tradition to fix the death rate δ and vary the infection rate λ, which means that the condition $\delta > \delta'_{\rm c}$ translates into $\lambda < \lambda'_{\rm c}$. In our setting, it will be more convenient to fix the infection rates and vary the death rate.

the product topology and through the identification $\mathcal{P}(\Lambda) \cong \{0,1\}^\Lambda$ also put this topology on $\mathcal{P}(\Lambda)$, making it into a compact metrisable space. We let $(\eta_t^A)_{t\geq 0}$ denote the process started in the initial state $\eta_0^A = A$. It is well-known that

$$\mathbb{P}\big[\eta_t^\Lambda \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}, \tag{4}$$

where $\Rightarrow$ denotes weak convergence of probability measures on $\mathcal{P}(\Lambda)$ and $\overline{\nu}$ is an invariant law of the (Λ, a, δ)-contact process, known as the *upper invariant law*. For any finite $A \subset \Lambda$, let

$$\rho(A) := \mathbb{P}\big[\eta_t^A \neq \emptyset \ \forall t \geq 0\big] \tag{5}$$

denote the survival probability started from the initial state A. By definition, a (Λ, a, δ)-contact process exhibits *complete convergence* if

$$\mathbb{P}\big[\eta_t^A \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \rho(A)\overline{\nu} + \big(1 - \rho(A)\big)\delta_\emptyset \tag{6}$$

for each finite $A \subset \Lambda$. By a simple subadditivity argument, it is easy to see that for each $i \in \Lambda$ and $t > 0$, the following limit exists:

$$R(\Lambda, a, \delta, i, t) = R(i, t) := \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}\big[i^n \in \eta_{nt}^{\{0\}}\big] \tag{7}$$

For any $i \in \Lambda$ and $A \subset \Lambda$, we adopt the notation $iA := \{ij : j \in A\}$. The following conjecture is a precise version of the earlier stated open problem. In particular, applying this to $i = 0$ it should not be hard to prove that there exists a critical death rate $\delta'_{\rm c}$ such that the process survives and complete convergence holds for $\delta < \delta'_{\rm c}$ while the probability that the origin is infected decays exponentially fast in the whole regime $\delta > \delta'_{\rm c}$.

Conjecture 1 (Sharpness of the phase transition for local survival) *Assume that a (Λ, a, δ)-contact process satisfies $R(\Lambda, a, \delta, i, t) = 0$ for some $i \in \Lambda$ and $t > 0$. Then for each $\delta' < \delta$, the (Λ, a, δ')-contact process started from any finite $A \subset \Lambda$ satisfies*

$$\mathbb{P}\big[i^{-n}\eta_{nt}^A \in \cdot\,\big] \underset{n\to\infty}{\Longrightarrow} \rho(A)\overline{\nu} + \big(1 - \rho(A)\big)\delta_\emptyset. \tag{8}$$

What is known

We say that the (Λ, a, δ)-contact process *survives* if $\mathbb{P}[\eta_t^A \neq \emptyset \ \forall t \geq 0] > 0$ for some, and hence for all finite nonempty A. We let

$$\theta(\Lambda, a, \delta) := \rho(\{0\}) = \mathbb{P}\big[\eta_t^{\{0\}} \neq \emptyset \ \forall t \geq 0\big] \tag{9}$$

denote the survival probability started from a single infected site, and call

$$\delta_{\rm c} = \delta_{\rm c}(\Lambda, a) := \sup\big\{\delta \geq 0 : \theta(\Lambda, a, \delta) > 0\big\} \tag{10}$$

the *critical death rate*. A simple subadditivity argument (see [Swa09, Lemma 1.1]) shows that for any (Λ, a, δ)-contact process, there exists a constant $r = r(\Lambda, a, \delta)$ such that

$$r = \lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}\big[|\eta_t^A|\big] \qquad \text{for all finite nonempty } A \subset \Lambda. \tag{11}$$

The following theorem is known. We can think of this as a "global" version of Conjecture 1.

Theorem 2 (Sharpness of the phase transition for global survival) *For any (Λ, a, δ)-contact process, one has $r(\Lambda, a, \delta) < 0$ if and only if $\delta > \delta_{\rm c}$.*

A basic tool in the study of the contact process is Harris' graphical representation [Har78], which shows that the contact process is essentially a continuous-time version of oriented percolation. Sharpness of the phase transition type results were first proved for unoriented percolation in [Men86] and [AB87]. The method of [AB87] was adapted to the contact process on $\mathbb{Z}^d$ in [BG91] and to contact processes on general transitive graphs in [AJ07]. The arguments in the latter paper carry over to general (Λ, a, δ)-contact processes, as spelled out in the appendix of [SS14], leading to Theorem 2.

The method of [AB87] for proving sharpness of the phase transition for unoriented percolation is quite complicated. A much simpler argument has been described in [DT16a, DT16b]. This argument also works for oriented percolation. With a bit of extra work, it seems likely that it can be adapted to also prove Theorem 2, although nobody has taken the trouble to write down the details. Yet another, completely independent and surprisingly simple proof of Theorem 2 can be found in [Swa18].

The contact process is one of the most studied interacting particle systems for which a lot of techniques are available and a lot is known. A good general reference is [Lig99].

What is conjectured

We have already formulated the precise Conjecture 1. This conjecture is related to the open problem of determining on which lattices the critical death rates for global and local survival coincide, respectively, on which lattices the inequality $\delta'_{\rm c} \leq \delta_{\rm c}$ is strict. One conjecture is that this is an equality if and only if the group Λ is amenable. There is a parallel conjecture in unoriented percolation, which says that uniqueness of the infinite cluster holds if and only if the graph is amenable. The "if" part of the latter conjecture follows from the argument in [BK89], while there are a number of results proving the "only if" part under weak additional assumptions. By comparison, much less is known in the oriented setting.

How to attack the problem

There are basically two possible strategies for proving sharpness of the phase transition results in percolation:

I. Assume that for some value of the percolation parameter (in our case: the death rate), there is no percolation (in our case: the proces dies out). Prove that if you change the parameter a bit (in our case: increase the death rate), then correlations decay exponentially fast.

II. Assume that for some value of the percolation parameter (in our case: the death rate), correlations decay slower than exponentially. Prove that if you change the parameter a bit (in our case: decrease the death rate), then there is percolation (in our case: the proces survives).

All known proofs of sharpness of the phase transition with the exception of [Men86] follow Strategy II. The reason seems to be that it is very difficult to give explicit bounds on the exponential decay rate, but relatively easier to give lower bounds on the survival probability. In fact, all known proofs yield as a side result a bound on the (conjectured) critical exponent associated with the survival probability θ. All proofs except [Swa18] also rely heavily on Russo's formula. Given that Conjecture 1 is still open, some new proof ingredients may be needed. There may be a relation between the exponential rate $R(i,t)$ and large deviations of Markov chains, but it remains to be seen how explicit this can be made and how useful it is.

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