Jan's open problems

Parity preserving branching and annihilation

Statement of the problem

In systems with *parity preserving branching and annihilation*, particles always give birth to an even number of offspring, and two particles on the same position can annihilate, leading to a decrease in the number of particles of two. As a result, parity is preserved, meaning that if the initial number of particles is even or odd, then it will stay even or odd for all time. The most important open problem is whether in one dimension, for low branching rates, the process started with an even number of particles dies out almost surely.

For concreteness, let us look at some systems where each site can contain at most one particle, although this is not essential. (See [BEM07] for systems where more particles can share the same site.) Let $\mathcal{S} = \{0,1\}^{\mathbb{Z}}$ be the space of all configurations $x = (x(i))_{i\in\mathbb{Z}}$ of zeros and ones on the integer lattice. We interpret a site $i \in \mathbb{Z}$ with $x(i) = 1$ (resp. $= 0$) as being occupied by a particle (resp. empty). We let $\mathcal{S}_{\rm fin} := \{x \in \mathcal{S} : |x| < \infty\}$ with $|x| := \sum_{i\in\mathbb{Z}} x(i)$ denote the space of configurations with finitely many particles and set $\mathcal{S}_{\rm even} := \{x \in \mathcal{S}_{\rm fin} : |x| \text{ is even}\}$ and $\mathcal{S}_{\rm odd} := \{x \in \mathcal{S}_{\rm fin} : |x| \text{ is odd}\}$.

We define $\delta_i \in \mathcal{S}_{\rm fin}$ by $\delta_i(j) := 1$ if $i = j$ and $:= 0$ otherwise and let $(x \oplus y)(i) := x(i) + y(i) \bmod(2)$ denote the pointwise sum modulo 2 of two configurations x and y. For each $i, j, j' \in \mathbb{Z}$, we define an *annihilating branching map* $\mathtt{abr}_{ijj'} : \mathcal{S} \to \mathcal{S}$ by

$$\mathtt{abr}_{ijj'}(x) := \begin{cases} x \oplus \delta_j \oplus \delta_{j'} & \text{if } x(i) = 1, \\ x & \text{if } x(i) = 0. \end{cases} \tag{1}$$

This has the interpretation that if there is a particle at i, then this particle produces two new particles at j and j', which immediately annihilate with any particles that may already be present on these sites. We now fix two probability laws p and q on $\mathbb{Z}$ with finite support and consider the continuous-time Markov chain with generator given by

$$\begin{aligned} Gf(x) := &\sum_{i,j\in\mathbb{Z}} q(j-i)\big\{f\big(\mathtt{abr}_{iij}(x)\big) - f\big(x\big)\big\} \\ &+\beta \sum_{i,j,j'\in\mathbb{Z}} p(j-i)p(j'-i)\big\{f\big(\mathtt{abr}_{ijj'}(x)\big) - f\big(x\big)\big\}. \end{aligned} \tag{2}$$

Note that when we apply the map $\mathtt{abr}_{iij}$, the new particle at i annihilates its parent, so we can alternatively interpret this map as a jump of a particle from i to j. As a result, the continuous-time Markov chain $(X_t)_{t\geq 0}$ with generator G has the following description.

- Each particle jumps with rate one from its present position i to a new position $i + j$, where j is chosen according to the probability law q. If the particle lands on an occupied site, then it annihilates with the particle that was present there.
- Each particle branches with rate β, which means that if i is its position, then it places two new particles at positions $i+j$ and $i+j'$, where j and j' are independent and chosen according to the probability law p. The new particles immediately annihilate with any particles that may already be present on these sites.

The open problem is now:

- Show that if β is small, then the process started in any configuration with two particles dies out almost surely, i.e., $|X_0| = 2$ implies that almost surely there exists a time $\tau < \infty$ such that $|X_\tau| = 0$.

What is known

In the special case that $p = q$ is the uniform distribution on $\{-1, 1\}$, it is known that we have almost sure extinction for all branching rates $\beta \geq 0$ (no matter how large). This was first proved in [Sud90]; see also [NP99] and [SS08], where this is called the DBARW. However, this case is special and does not seem to shed much light on the general problem. For a variety of models, it has been proved that if β is large enough, then the process started with two particles survives with positive probability, see [NP99, SS08].
It is known that there is a close relation with the behaviour of the process started in an odd initial state. Let us call two configurations $x, y \in \mathcal{S}$ equivalent if one is a translation of the other, i.e., there exists a $j \in \mathbb{Z}$ such that $x(i) = y(i + j)$ for all $i \in \mathbb{Z}$. Let $\tilde{x}$ denote the equivalence class containing x and let $\tilde{\mathcal{S}}_{\text{odd}} := \{\tilde{x} : x \in \mathcal{S}_{\text{odd}}\}$. It is easy to see that if we start the process with an odd number of particles, then $(\tilde{X}_t)_{t \geq 0}$ is a continuous-time Markov chain with countable state space $\tilde{\mathcal{S}}_{\text{odd}}$. By definition, the process is *stable* if $\tilde{\delta}_0$ is a positively recurrent state for this Markov chain. If the process is stable, then the continuous-time Markov chain with initial state $\tilde{\delta}_0$ satisfies

$$\mathbb{P}^{\tilde{\delta}_0}\big[\tilde{X}_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu \tag{3}$$

where ν is an invariant law that is concentrated on $\tilde{\mathcal{S}}_{\text{odd}}$. Let $\tilde{X}_\infty$ be a random variable with this law. If $\mathbb{E}[|\tilde{X}_\infty|] < \infty$, then we say that the process is *strongly stable*. It has been proved in [Swa13] that strong stability implies almost sure extinction started from even initial states. It seems intuitively clear that for (strong) stability to be possible, it should be true for the process with $\beta = 0$ started with three particles that the time we have to wait till two of these particles annihilate each other has finite expectation. This is indeed known to be true. If q is the uniform distribution on $\{-1, 1\}$, then the precise asymptotics of this time are known, see [SS13, Lemma 9].

What is conjectured

Numerical simulations strongly suggest the existence of a critical value $0 < \beta_* < \infty$ such that the process started with an even number of particles dies out a.s. and is strongly stable for $\beta < \beta_*$ and survives with positive probability and is unstable for $\beta > \beta_*$. For $\beta = \beta_*$ it seems the process dies out but is unstable. Proving a result in this strong form seems an almost hopeless undertaking in view of the lack of monotonicity of the problem, which means that the usual coupling arguments fail and we cannot exclude that the process survives for some value of β but dies out for some *larger* value of β, unnatural as that may seem. This is why the open problem above has been formulated in a weaker form: just show that there exists some $\beta_+ > 0$ (which may be ridiculously small) such that the process started with an even number of particles dies out a.s. for $\beta < \beta_+$. The numerical evidence seems to say that strong stability holds in the whole subcritical regime $\beta < \beta_*$ and that the random variable $|\tilde{X}_\infty|$ has exponentially light tails. By contrast, the *width* of $\tilde{X}_\infty$, which is the distance from the left-most particle to the right-most particle, is believed to have infinite expectation.

How to attack the problem

I have probably spent more time on this problem than on any other open problem I have worked on. If I had a really good idea how to attack the problem, then I would probably have solved it. The only thing I can say is that I am convinced that the best hope is trying to prove strong stability and then use the result in [Swa13] to conclude that the process dies out. For proving stability, I am convinced some form of multiscale argument is needed. For the rest, I can just say what does not work. It is easy to bound the process from above by branching and coalescing particles, but this does not bring anything since such systems are not stable. A standard way for proving positive recurrence is to find a Lyapunov function and apply Foster's theorem, but all my efforts at finding a suitable Lyapunov function have failed and this approach seems very hard. Systems with parity preserving branching and annihilation can often be viewed as describing the interface between two populations. This point of view is explained in detail in [SS08] and also exploited in [Swa13]. In this context, stability translates into *interface tightness*. Interface tightness can sometimes be proved with Lyapunov function techniques; see [SSY19].

References

[BEM07] J. Blath, A.M. Etheridge, and M.E. Meredith. Coexistence in locally regulated competing populations and survival of BARW. *Ann. Appl. Probab.* 17 (2007), 1474–1507.

[NP99] C. Neuhauser and S.W. Pacala. An explicitly spatial version of the Lotka–Volterra model with interspecific competition. *Ann. Appl. Probab.* 9 (1999), 1226–1259.

[SS08] A. Sturm and J.M. Swart. Voter models with heterozygosity selection. *Ann. Appl. Probab.* 18(1) (2008), 59–99.

[SS13] A. Sturm and J.M. Swart. A particle system with cooperative branching and coalescence. *Ann. Appl. Probab.* 25(3) (2015), 1616–1649.

[SSY19] R. Sun, J.M. Swart, and J. Yu. Equilibrium interfaces of biased voter models. *Ann. Appl. Probab.* 29(4) (2019), 2556–2593.

[Sud90] A. Sudbury. The branching annihilating process: An interacting particle system. *Ann. Probab.* 18 (1990), 581–601.

[Swa13] J.M. Swart. Noninvadability implies noncoexistence for a class of cancellative systems. *Electron. Commun. Probab.* 18 (2013) paper No. 38, 1–12.