Noninvadability implies noncoexistence for a class of cancellative systems

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Abstract

There exist a number of results proving that for certain classes of interacting particle systems in population genetics, mutual invadability of types implies coexistence. In this paper we prove a sort of converse statement for a class of one-dimensional cancellative systems that are used to model balancing selection. We say that a model exhibits strong interface tightness if started from a configuration where to the left of the origin all sites are of one type and to the right of the origin all sites are of the other type, the configuration as seen from the interface has an invariant law in which the number of sites where both types meet has finite expectation. We prove that this implies noncoexistence, i.e., all invariant laws of the process are concentrated on the constant configurations. The proof is based on special relations between dual and interface models that hold for a large class of one-dimensional cancellative systems and that are proved here for the first time.

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1 Introduction and main result

In spatial population genetics, one often considers interacting particle systems where each site in the lattice can be occupied by one of two different types, respresenting different genetic types of the same species or even different species. It is natural to conjecture that if each type is able to invade an area that is so far occupied by the other type only, then coexistence should be possible, i.e., there should exist invariant laws that are concentrated on configurations in which both types are present. There exist a number of rigorous results of this nature. In particular, Durrett [Dur02] has proved a general result of this sort for systems with fast stirring; see also, e.g., [DN97] for similar results. In a more restricted context, the same idea (mutual invadability implies coexistence) is also behind the proofs of, e.g., [NP99, Thm 1 (b)] or [CP07, Thm 4].

In this paper, we will prove a converse claim. We will show that for a class of one-dimensional cancellative systems that treat the two types symmetrically, mutual non-invadability implies non-coexistence. In particular, this applies to several generalizations of the standard, one-dimensional voter model that are used to model *balancing selection* (sometimes also called heterozygosity selection or negative frequency dependent selection), which is the effect, observed in many natural populations, that types that are locally in the minority have a selective advantage, since they are able to use resources not available to the other type.

Since our general theorem needs quite a bit of preparation to formulate, as a warm-up and motivation for what will follow, we first describe three particular models that our result applies to. These models also occur in [SS08]. We refer to that paper for a more detailed motivation and a proof that they are indeed cancellative systems.

Restricting ourselves to the one-dimensional case, as we will throughout the paper, let $\{0,1\}^{\mathbb{Z}}$ be the space of configurations $x = (x(i))_{i\in\mathbb{Z}}$ of zeros and ones on $\mathbb{Z}$. We sometimes identify sets with indicator functions and write $|x| = |\{i : x(i) = 1\}|$ for the number of ones in a configuration x. We recall that an interacting particle system (with two types), in one dimension, is a Markov process $X = (X_t)_{t\geq 0}$ with state space $\{0,1\}^{\mathbb{Z}}$ that is defined by its local transition rates [Lig85]. Let us say that such an interacting particle system is *type-symmetric* if its dynamics are symmetric under a simultaneous interchanging of all types, that is, the transition $x \mapsto x'$ happens at the same rate as the transition $(1-x) \mapsto (1-x')$.

The first model that our result applies to is the neutral Neuhauser-Pacala model, which is a special case of the model introduced in [NP99]. Fix $R \geq 2$ and for each $x \in \{0,1\}^{\mathbb{Z}}$, let us write

$$f_\tau(x,i) := \frac{1}{2R} \sum_{\substack{j\in\mathbb{Z} \\ 0<|i-j|\leq R}} 1_{\{x(j)=\tau\}} \qquad \big(\tau = 0,1,\ x \in \{0,1\}^{\mathbb{Z}},\ i \in \mathbb{Z}\big) \tag{1.1}$$

for the local frequency of type τ near i. Then the *neutral Neuhauser-Pacala model* with *competition parameter* $0 \leq \alpha \leq 1$ is the type-symmetric interacting particle system such that

$$x(i) \text{ flips } 0 \mapsto 1 \text{ with rate } f_1(x,i)\big(f_0(x,i) + \alpha f_1(x,i)\big), \tag{1.2}$$

and similarly for flips $1 \mapsto 0$, by type-symmetry. Similarly, the *affine voter model* with competition parameter $0 \leq \alpha \leq 1$ is the type-symmetric interacting particle system such that

$$x(i) \text{ flips } 0 \mapsto 1 \text{ with rate } \alpha f_1(x,i) + (1-\alpha)1_{\{f_1(x,i)\,>\,0\}}. \tag{1.3}$$

The affine voter model interpolates between the threshold voter model (corresponding to $\alpha = 0$) studied in, e.g., [CD91, Han99, Lig94] and the usual range R voter model (for $\alpha = 1$). The neutral Neuhauser-Pacala model likewise reduces to a range R voter model for $\alpha = 1$. Finally, the *rebellious voter model*, introduced in[SS08], with competition parameter $0 \leq \alpha \leq 1$ is the type-symmetric interacting particle system such that

$$\begin{aligned} x(i) \text{ flips } 0 \leftrightarrow 1 \text{ with rate } \ &\tfrac{1}{2}\alpha\big(1_{\{x(i-1)\,\neq\, x(i)\}} + 1_{\{x(i)\,\neq\, x(i+1)\}}\big) \\ &+\tfrac{1}{2}(1-\alpha)\big(1_{\{x(i-2)\,\neq\, x(i-1)\}} + 1_{\{x(i+1)\,\neq\, x(i+2)\}}\big). \end{aligned} \tag{1.4}$$

For $\alpha = 1$, this model reduces to the standard nearest-neighbour voter model.

Let X be a type-symmetric interacting particle system. Then, by type-symmetry, it is easy to see that the process Y defined by

$$Y_t(i) := 1_{\{X_t(i-\frac{1}{2})\,\neq\, X_t(i+\frac{1}{2})\}} \qquad (t \geq 0,\ i \in \mathbb{Z} + \tfrac{1}{2}) \tag{1.5}$$

(where $\mathbb{Z} + \frac{1}{2} := \{i + \frac{1}{2} : i \in \mathbb{Z}\}$) is a Markov process. We call Y the *interface model* of X. We will often say that a site i is occupied by a particle if $Y_t(i) = 1$; otherwise the site is empty. Under mild assumptions on the flip rates of X (e.g. finite range), Y is itself an interacting particle system, where always an even number of sites flip at the same time. Let $\underline{0}, \underline{1} \in \{0,1\}^{\mathbb{Z}}$ denote the configurations that are constantly zero or one, respectively. If $\underline{0}$ (and hence by type-symmetry also $\underline{1}$) is a trap for the process X, then, under mild assumptions on the flip rates (e.g. finite range), we have that $|Y_0| < \infty$ implies $|Y_t| < \infty$ a.s. for all $t \geq 0$. In particular, this applies to all models introduced above. It is easy to see that Y *preserves parity*, i.e., $|Y_0| \bmod(2) = |Y_t| \bmod(2)$ a.s. for all $t \geq 0$. If $|Y_0|$ is finite and odd, then we let $l_t := \inf\{i \in \mathbb{Z} + \frac{1}{2} : Y_t(i) = 1\}$ denote the position of the left-most particle and we let

$$\hat{Y}_t(i) := Y(l_t + i) \qquad (t \geq 0,\ i \in \mathbb{N}) \tag{1.6}$$

denote the process Y *viewed from the left-most particle.* Note that $\hat{Y}$ takes values in the countable state space $\hat{S}$ of all functions $\hat{y} : \mathbb{N} \to \{0,1\}$ such that $|\hat{y}|$ is finite and odd and $\hat{y}(0) = 1$. Let δ_0 denote the unique state in $\hat{S}$ that contains a single particle. We let $\hat{S}_{\delta_0}$ denote the set of states in $\hat{S}$ that can be reached with positive probability from the state δ_0.

Following terminology first introduced in [CD95], we say that a type-symmetric interacting particle system X exhibits *interface tightness* if its corresponding interface model $\hat{Y}$ viewed from the left-most particle is positive recurrent on $\hat{S}_{\delta_0}$. In particular, this implies that the process $\hat{Y}$ started from $\hat{Y}_0 = \delta_0$ spends a positive fraction of its time in δ_0 and is ergodic with a unique invariant law on $\hat{S}_{\delta_0}$. Let $\hat{Y}_\infty$ be distributed according to this invariant law. Then, by definition, we will say that X exhibits *strong interface tightness* if $\mathbb{E}[|\hat{Y}_\infty|] < \infty$. Strong interface tightness will be our way of rigorously formulating the idea of 'noninvadability', i.e., that neither type is able to penetrate the area occupied by the other type.

We say that X exhibits *coexistence* if there exists an invariant law μ such that $\mu(\{\underline{0}, \underline{1}\}) = 0$, i.e., μ is concentrated on configurations in which both types are present, and we say that X *survives* if the process X started with a single one (and all other sites of type zero) satisfies $\mathbb{P}[X_t \neq \underline{0}\ \forall t \geq 0] > 0$. We will prove the following theorem.

Theorem 1 (Strong interface tightness implies noncoexistence) *Let X be either a neutral Neuhauser-Pacala model, or an affine voter model, or a rebellious voter model, with competition parameter $0 < \alpha \leq 1$. Assume that X exhibits strong interface tightness. Then X exhibits noncoexistence.*

To put this into context, let us look at what is known, both rigorously and nonrigorously, about these models. Numerical simulations for the rebellious voter model, reported in [SV10], give the following picture. There exists a critical parameter $\alpha_c \approx 0.510 \pm 0.002$ such that the process survives and coexistence holds if and only if $\alpha < \alpha_c$, while interface tightness holds if and only if $\alpha > \alpha_c$ (in particular, at $\alpha = \alpha_c$ one has neither survival, coexistence, nor interface tightness). Moreover, it seems that whenever interface tightness holds, one has strong interface tightness and in fact the probability $\mathbb{P}[|\hat{Y}_\infty| = (2n+1)]$ decays exponentially fast in n.[1] The behaviour of the neutral Neuhauer-Pacala model and affine voter model is supposed to be similar.

Most of these numerical 'facts' are unproven but for the rebellious voter model it has been rigorously shown that for α sufficiently close to zero one has coexistence and no interface tightness [SS08, Thm 4]. It is moreover known that coexistence is equivalent to survival [SS08, Lemma 2]. It is also known rigorously that the Neuhauser-Pacala model exhibits coexistence for α sufficiently close to zero [NP99, Thm 1 (b)] and that the affine voter model exhibits coexistence at $\alpha = 0$ [Lig94]. It is likely this latter result can be extended to α sufficiently small.

It is an open problem to prove either noncoexistence or interface tightness for any of these models for any $\alpha < 1$.[2] (For $\alpha = 1$, which corresponds to a one-dimensional pure voter model, noncoexistence and strong interface tightness are known.) The present result, therefore, unfortunately does not prove anything new for these models, except that it shows that *if* by some means one is able to prove strong interface tightness, then this implies noncoexistence (and of course also interface tightness).

The rest of the paper is organized as follows. In the next section, we formulate our general result. We introduce a class of one-dimensional cancellative systems that will be our general framework and point out some interesting relations between their interface models and their dual models in the sense of cancellative systems duality. In particular, we show that each one-dimensional, type-symmetric, cancellative system X has a rather peculiar dual X' that is also type-symmetric and cancellative. This sort of duality was sort of implicit in [SS08] but is for the first time formally written down here. We then observe that strong interface tightness for X implies the existence of a harmonic function for X' that allows us to prove that this process dies out and hence, by duality, that noncoexistence holds for X. The final section of the paper contains proofs.

[1] By contrast, for pure voter models of range $R \geq 2$, where strong interface tightness has been rigorously proved, it is known that the *length* of the interface $\sup\{i \in \mathbb{N} : \hat{Y}(i) = 1\}$ has a heavy-tailed distribution with infinite first moment [B&06, Thm 1.4].

[2] Setting $R = 1$ in either the neutral Neuhauer-Pacala model or affine voter model yields, up to a trivial rescaling of time, the *disagreement voter model* (using terminology from [SS08]), which is known to exhibit noncoexistence and interface tightness for all $0 \leq \alpha < 1$. For the special case $\alpha = \frac{1}{2}$, moreover strong interface tightness has been proved in [ALM92, Corollary of Thm 2]. But, as explained in [SS08], this model has special properties that give few clues on how to prove noncoexistence for any of the other models.

2 Methods and further results

2.1 Cancellative systems

Cancellative systems are a special class of interacting particle systems that are linear with respect to addition modulo 2. It will be convenient to allow the lattice to be $\mathbb{I} = \mathbb{Z}$ or $\mathbb{I} = \mathbb{Z} + \frac{1}{2}$. It is well-known (though for probabilists perhaps not always at the front of their minds) that linear spaces can be defined over any field. In particular, we may view the space $\{0,1\}^{\mathbb{I}}$ of all functions $x : \mathbb{I} \to \{0,1\}$ as a linear space over the finite field $\{0,1\}$, where the latter is equipped with addition modulo 2 (and the usual product). To distinguish this from the usual addition in $\mathbb{R}$ (which we will sometimes also need), we will use the symbol $\oplus$ for (componentwise) addition modulo 2.

We equip $\{0,1\}^{\mathbb{I}}$ with the product topology and let $\mathcal{L}(\mathbb{I})$ denote the space of all continuous linear maps $A : \{0,1\}^{\mathbb{I}} \to \{0,1\}^{\mathbb{I}}$. The matrix $(A(i,j))_{i,j\in\mathbb{I}}$ of such a linear operator is defined as

$$A(i,j) := (A\delta_j)(i) \quad \text{where} \quad \delta_j(i) := 1_{\{i=j\}} \qquad (i,j \in \mathbb{I}). \tag{2.1}$$

It is not hard to see that the continuity of A is equivalent to the requirement that $|\{j \in \mathbb{I} : A(i,j) = 1\}| < \infty$ for all $i \in \mathbb{I}$ and that

$$Ax(i) = \bigoplus_{j\in\mathbb{I}} A(i,j)x(j) \qquad (i \in \mathbb{I}), \tag{2.2}$$

where the infinite sum reduces to a finite sum and hence is well-defined. Identifying sets with indicator functions as we sometimes do, we associate A with the set $\{(i,j) : A(i,j) = 1\} \subset \mathbb{I}^2$. We call $\mathcal{L}_{\rm loc}(\mathbb{I}) := \{A \in \mathcal{L}(\mathbb{I}) : |A| < \infty\}$ (where $|A|$ denotes the cardinality of $A \subset \mathbb{I}^2$) the set of *local* operators on $\{0,1\}^{\mathbb{I}}$

Slightly specializing[3] from the set-up in [Gri79], we will say that an interacting particle system on $\mathbb{I}$ is *cancellative* if for each $A \in \mathcal{L}_{\rm loc}(\mathbb{I})$, there exists a rate $r(A) \geq 0$ (possibly zero), such that X makes the transition

$$x \mapsto x \oplus Ax \qquad \text{with rate } r(A). \tag{2.3}$$

We will always assume that the rates are translation invariant, i.e.,

$$r(A) = r(T_k(A)) \quad (k \in \mathbb{Z}) \quad \text{where} \quad T_k(A) := \{(i+k, j+k) : (i,j) \in A\}, \tag{2.4}$$

For technical convenience, we will also assume that our models are finite range, i.e., there exists an $R < \infty$ such that

$$r(A) = 0 \quad \text{whenever} \quad \exists (i,j) \in A \text{ with } |i-j| > R. \tag{2.5}$$

It follows from standard results [Lig85, Thm I.3.9] that any collection of rates $(r(A))_{A\in\mathcal{L}_{\rm loc}(\mathbb{I})}$ satisfying (2.4) and (2.5) corresponds to a well-defined $\{0,1\}^{\mathbb{I}}$-valued Markov process X. We refer to [SS08] for the not immediately obvious fact that the neutral Neuhauser-Pacala model, the affine voter model, and the rebellious voter model are cancellative systems.

It is not hard to see that a cancellative system is type-symmetric if and only if its rates satisfy $r(A) = 0$ unless

$$|\{j \in \mathbb{I} : (i,j) \in A\}| \text{ is even for all } i \in \mathbb{I}. \tag{2.6}$$

Similarly, a cancellative system is parity preserving if and only if its rates satisfy $r(A) = 0$ unless

$$|\{i \in \mathbb{I} : (i,j) \in A\}| \text{ is even for all } j \in \mathbb{I}. \tag{2.7}$$

We let $\mathcal{L}_{\rm ts}(\mathbb{I})$ and $\mathcal{L}_{\rm pp}(\mathbb{I})$ denote the sets of all $A \in \mathcal{L}_{\rm loc}(\mathbb{I})$ satisfying (2.6) and (2.7), respectively (where the subscript ts and pp stand for type-symmetry and parity preservation, respectively).

[3] In [Gri79], a cancellative system is defined by a percolation substructure $\mathcal{P}(\lambda; V, W)$ where $\lambda_{i,x}$ is the rate of the i-th alarm clock at a site $x \in \mathbb{Z}^d$, $V_{i,x}$ is the set of sites where particles are spontaneously born if this alarm clock rings, and $W_{i,x}$ is a collection of percolation arrows. We will restrict ourselves to the case that $V_{i,x} = \emptyset$ for all i, x. The matrix $1 \oplus A$ (where 1 is the identity matrix) corresponds to the set of arrows $W_{i,x}$ in [Gri79].

2.2 Dual and interface models

We set

$$\begin{aligned} S_\pm(\mathbb{I}) &:= \big\{x \in \{0,1\}^{\mathbb{I}} : \lim_{i\to\pm\infty} x(i) = 0\big\}, \\ S_{\rm fin}(\mathbb{I}) &:= \big\{x \in \{0,1\}^{\mathbb{I}} : |x| < \infty\big\} = S_-(\mathbb{I}) \cap S_+(\mathbb{I}). \end{aligned} \tag{2.8}$$

If X is a cancellative system, then it is not hard to check that

$$\left.\begin{array}{lllll} \text{(i)} & \mathbb{E}\big[\inf X_0\big] > -\infty & \text{implies} & \mathbb{E}\big[\inf X_t\big] > -\infty, \\ \text{(ii)} & \mathbb{E}\big[|X_0|\big] < \infty & \text{implies} & \mathbb{E}[|X_t|] < \infty, \end{array}\right\} \qquad (t \geq 0), \tag{2.9}$$

where we notationally identify sets and indicator functions as before, i.e., $\inf x = \inf\{i \in \mathbb{I} : x(i) = 1\}$. It follows that $X_0 \in S_-(\mathbb{I})$ a.s. implies $X_t \in S_-(\mathbb{I})$ a.s. for all $t \geq 0$ and by symmetry analogue statements hold for $S_+(\mathbb{I})$ and $S_{\rm fin}(\mathbb{I})$.

We let xy denote the pointwise product of $x, y \in \{0,1\}^{\mathbb{I}}$ and write

$$\|x\| := \bigoplus_{i\in\mathbb{I}} x(i) = |x| \bmod(2) \qquad \big(x \in S_{\rm fin}(\mathbb{I})\big). \tag{2.10}$$

Let $\mathcal{G}(\mathbb{I},\mathbb{I})$ be the set of all pairs (x,y) satisfying any of the following conditions: 1. $x \in S_-(\mathbb{I})$ and $y \in S_+(\mathbb{I})$, or 2. $x \in S_+(\mathbb{I})$ and $y \in S_-(\mathbb{I})$, or 3. $x \in S_{\rm fin}(\mathbb{I})$, or 4. $y \in S_{\rm fin}(\mathbb{I})$. We observe that the bilinear form

$$\mathcal{G}(\mathbb{I},\mathbb{I}) \ni (x,y) \mapsto \|xy\| \tag{2.11}$$

is very much like an inner product. In particular, $\|xy\| = 0$ for all $y \in S_{\rm fin}(\mathbb{I})$ implies $x = 0$.

Let $A^\dagger(i,j) := A(j,i)$ denote the adjoint of a matrix A. It follows from general theory (see [Gri79, Thm III.1.5]) that the cancellative system X defined by rates $(r_X(A))_{A\in\mathcal{L}_{\rm loc}(\mathbb{I})}$ is dual to the cancellative system Y' defined by the rates

$$r_{Y'}(A) := r_X(A^\dagger) \qquad \big(A \in \mathcal{L}_{\rm loc}(\mathbb{I})\big), \tag{2.12}$$

in the sense that

$$\mathbb{E}\big[\|X_0 Y'_t\|\big] = \mathbb{E}\big[\|X_t Y'_0\|\big] \qquad (t \geq 0) \tag{2.13}$$

whenever X and Y' are independent (with arbitrary initial laws) and $(X_0, Y_0) \in \mathcal{G}(\mathbb{I},\mathbb{I})$ a.s. Note that $\mathbb{E}[\|X_0 Y'_t\|] = \mathbb{P}[|X_t Y'_0| \text{ is odd}]$. By (2.6) and (2.7), Y' is parity preserving if and only if X is type-symmetric. The duality in (2.13) is the analogue of linear systems duality (see [Lig85, Thm IX.1.25]) with normal addition replaced by addition modulo 2.

We next consider interface models. Let us define an 'interface operator' or 'discrete differential operator' $\nabla : \{0,1\}^{\mathbb{I}} \to \{0,1\}^{\mathbb{I}+\frac{1}{2}}$ by

$$(\nabla x)(i) = x(i - \tfrac{1}{2}) \oplus x(i + \tfrac{1}{2}) \qquad (i \in \mathbb{I} + \tfrac{1}{2}). \tag{2.14}$$

Note that if $X = (X_t)_{t\geq 0}$ is a type-symmetric cancellative system on $\mathbb{I}$ then $Y := (\nabla(X_t))_{t\geq 0}$ is its interface model as in (1.5). Recall the definitions of $\mathcal{L}_{\rm ts}(\mathbb{I})$ and $\mathcal{L}_{\rm pp}(\mathbb{I})$ from (2.6) and (2.7). The next lemma says that the interface model of each type-symmetric cancellative system is a parity preserving cancellative system, and conversely, each parity preserving cancellative system is the interface model of a unique type-symmetric cancellative system.

Lemma 2 (Interface model) *There exists a unique bijection* $\Psi : \mathcal{L}_{\rm ts}(\mathbb{I}) \to \mathcal{L}_{\rm pp}(\mathbb{I} + \frac{1}{2})$ *such that*

$$\nabla A x = \Psi(A) \nabla x \qquad \big(x \in \{0,1\}^{\mathbb{I}}\big). \tag{2.15}$$

Moreover, if X is a type-symmetric cancellative system on $\mathbb{I}$ defined by rates $r_X(A)$ with $A \in \mathcal{L}_{\rm ts}(\mathbb{I})$, then its interface model is the parity preserving cancellative system on $\mathbb{I} + \frac{1}{2}$ with rates defined by

$$r_Y(A) := r_X\big(\Psi^{-1}(A)\big) \qquad \big(A \in \mathcal{L}_{\rm pp}(\mathbb{I} + \tfrac{1}{2})\big). \tag{2.16}$$

An explicit formula for $\Psi(A)$ is given in (3.6) below.

We have just seen that every type-symmetric cancellative system X gives in a natural way rise to two (in most cases different) parity preserving cancellative systems: its dual Y' in the sense of (2.13) and its interface model Y as in (1.5). Now, by Lemma 2, Y' is itself the interface model of some type-symmetric cancellative system X' and Y is the dual of some type-symmetric cancellative system X'', so it seems as if continuing in this way, one could in principle generate infinitely many different models. It turns out that this is not the case, however. As the next lemma shows, we have $X' = X''$ and the process stops here.

Lemma 3 (Duals and interface models) *Let $\Psi : \mathcal{L}_{\rm ts}(\mathbb{I}) \to \mathcal{L}_{\rm pp}(\mathbb{I} + \frac{1}{2})$ be as in Lemma 2. Then*

$$\Psi(A)^\dagger = \Psi^{-1}(A^\dagger) \qquad \big(A \in \mathcal{L}_{\rm ts}(\mathbb{I})\big). \tag{2.17}$$

Lemma 3, together with formulas (2.12) and (2.16), shows that for any type-symmetric cancellative system X, there exists another type-symmetric cancellative system X' as well as parity preserving cancellative systems Y and Y' such that the following commutative diagram holds:

$$\begin{array}{ccc} X & \xrightarrow{\text{interface}} & Y \\ {\scriptstyle\text{dual}}\updownarrow & & \updownarrow{\scriptstyle\text{dual}} \\ Y' & \xleftarrow[\text{interface}]{} & X' \end{array} \tag{2.18}$$

An example of such a commutative diagram was given in [SS08], but as far as we know, the general case is proved for the first time here. If X and X' are as in (2.18), then X and X' are in fact themselves dual in the following sense.

Lemma 4 (Duality of type-symmetric cancellative systems) *Let X be a type-symmetric cancellative system as in (2.3)–(2.5) and let X' be the dual of the interface model of X, or equivalently (by Lemma 3), let X be the dual of the interface model of X'. Then X and X' are dual in the sense that*

$$\mathbb{E}\big[H(X_t, X'_0)\big] = \mathbb{E}\big[H(X_0, X'_t)\big] \qquad (t \geq 0) \tag{2.19}$$

whenever X and X' are independent and satisfy $(X_0, X'_0) \in \mathcal{G}(\mathbb{I}, \mathbb{I} + \frac{1}{2})$ a.s. (with $\mathcal{G}(\mathbb{I}, \mathbb{I} + \frac{1}{2})$ defined analogously to $\mathcal{G}(\mathbb{I}, \mathbb{I})$), and $H(x, x')$ is the duality function

$$H(x, x') := \|(\nabla x)x'\| = \|x(\nabla x')\|. \tag{2.20}$$

Using the graphical representation of cancellative systems [Gri79], the duality in (2.19) can be made into a strong pathwise duality. (For this concept, and more general theory of Markov process duality, see [JK12].)

Remark 1 A special property of the rebellious voter model, that in fact motivated its introduction in [SS08], is that it is self-dual with respect to the duality in (2.19).

Remark 2 It is possible for a cancellative system to be both type-symmetric and parity preserving. In particular, this applies to the symmetric exclusion process Y, which is part of a commutative diagram of the form:[4]

$$\begin{array}{ccccc} X & \xrightarrow{\text{interface}} & Y & \xrightarrow{\text{interface}} & Z \\ {\scriptstyle\text{dual}}\updownarrow & & {\scriptstyle\text{dual}}\updownarrow & & {\scriptstyle\text{dual}}\updownarrow \\ Z & \xleftarrow[\text{interface}]{} & Y & \xleftarrow[\text{interface}]{} & X \end{array} \tag{2.21}$$

[4]Here X has pure 'disagreement' dynamics (see footnote 2). Its dual Z is an annihilating particle model, known as the double branching annihilating process, where particles with a certain rate give birth to two new particles, situated on their neighbouring positions.

Remark 3 It is interesting to speculate how much of the above goes through if $\{0,1\}$ is replaced by a more general finite field. It seems that at least the duality formula (2.13) holds more generally.

Remark 4 If X, X', Y and Y' are as in (2.18), then also Y and Y' are dual to each other, in a sense. If X and X' are voter models and Y and Y' are systems of annihilating random walks, then this is a form of non-crossing duality similar to the duality of the Brownian web.

2.3 A harmonic function

If X and X' are type-symmetric cancellative systems that are dual in the sense of (2.19), then it is not hard to show that coexistence of X is equivalent to survival of X'. In fact, this is just [SS08, Lemma 1(a)], translated into our present notation (compare also formula (3.15) below). Our strategy for proving Theorem 1 will be to show that strong interface tightness for X implies extinction of X'.

It is well-known that a duality between two Markov processes translates invariant measures of one process into harmonic functions of the other process. Mimicking a trick used in [SS11], we will apply this to the infinite, translation-invariant measure

$$\mu := \sum_{i\in\mathbb{I}+\frac{1}{2}} \mathbb{P}\big[(\hat{Y}_\infty + i) \in \cdot\,\big], \tag{2.22}$$

where $\hat{Y}_\infty$ is distributed according to the invariant law of the interface model of X viewed from the left-most particle and $\hat{Y}_\infty + i$ denotes the configuration obtained from $\hat{Y}_\infty$ by shifting all particles by i. (In set-notation, $\hat{Y}_\infty + i = \{j+i : j \in \hat{Y}_\infty\}$.) It is not hard to see that μ is indeed an invariant measure of the interface model Y of X. We will not directly use this fact, but it provides the idea for the following lemma.

Lemma 5 (Harmonic function) *Let X and X' be type-symmetric cancellative systems defined by rates as in (2.3)–(2.5), on $\mathbb{I}$ and $\mathbb{I}+\frac{1}{2}$ respectively, that are dual in the sense of (2.19). Assume that strong interface tightness holds for X and let $\hat{Y}_\infty$ be distributed according to the invariant law of the interface model of X viewed from the left-most particle. Then*

$$h(x) := \sum_{i\in\mathbb{I}+\frac{1}{2}} \mathbb{E}\big[\|(\hat{Y}_\infty + i)x\|\big] \qquad \big(x \in S_{\rm fin}(\mathbb{I}+\tfrac{1}{2})\big) \tag{2.23}$$

defines a harmonic function $h : S_{\rm fin}(\mathbb{I}+\frac{1}{2}) \to [0,\infty)$ for the process X', i.e., for each deterministic initial state $X'_0 = x' \in S_{\rm fin}(\mathbb{I}+\frac{1}{2})$, the process $M = (M_t)_{t\geq 0}$ defined by

$$M_t := h(X'_t) \qquad (t \geq 0) \tag{2.24}$$

is a martingale with respect to the filtration generated by X'. Moreover, defining constants $0 < c \leq C < \infty$ by $c := \mathbb{P}[|\hat{Y}_\infty| = 1]$ and $C := \mathbb{E}[|\hat{Y}_\infty|]$, one has that

$$c|x| \leq h(x) \leq C|x| \qquad \big(x \in S_{\rm fin}(\mathbb{I}+\tfrac{1}{2})\big). \tag{2.25}$$

We note that if X is a nearest-neighbour voter model, then X' is also a nearest-neighbour voter model and $\hat{Y}_\infty = \delta_0$ a.s. Now the harmonic function h from Lemma 5 is just $h(x) = |x|$, which is a well-known harmonic function for X'. Numerical simulations in [SV10] suggest that for the rebellious voter model, as α is lowered from the pure voter case $\alpha = 1$, the function h defined in (2.23) changes smoothly as a function of α and can even be smoothly extended across the critical point.

Since the process $M_t = h(X'_t)$ in (2.24) is a nonnegative martingale, it converges a.s. We will show that this implies extinction for X' under the additional assumption that the dynamics of X (and hence also X') have a nearest-neighbour voter component. This latter assumption is made for technical convenience and can be relaxed; it seems however not easy to formulate simple, sufficient, yet general conditions on the dynamics of X that allow one to conclude from the convergence of $h(X'_t)$ that X' get extinct a.s. From the a.s. extinction of X' we obtain in fact a little more than just noncoexistence for X.

Theorem 6 (Strong interface tightness implies clustering) *Let X be a type-symmetric cancellative system on $\mathbb{Z}$ defined by translation invariant, finite range rates as in (2.3)–(2.5). Assume that the dynamics of X have a nearest-neighbour voter component, i.e.,*

$$r(\{(0,0),(0,1)\}) \vee r(\{(0,0),(-1,0)\}) > 0, \tag{2.26}$$

and that X exhibits strong interface tightness. Then, for the process started in an arbitrary initial law,

$$\mathbb{P}[X_t(i) = X_t(i+1)] \underset{t\to\infty}{\longrightarrow} 1 \qquad (i \in \mathbb{Z}). \tag{2.27}$$

The behaviour in (2.27) is called *clustering* and well-known for one-dimensional pure voter models. For pure voter models, if the initial law of X_0 is translation invariant, one has moreover that

$$\mathbb{P}[X_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} p\delta_{\underline{0}} + (1-p)\delta_{\underline{1}} \quad \text{with} \quad p := \mathbb{E}[X_0(0)], \tag{2.28}$$

where $\Rightarrow$ denotes weak convergence of probability laws on $\{0,1\}^{\mathbb{Z}}$. More generally, if X satisfies the assumptions of Theorem 6 and also X' exhibits interface tightness, then using duality it is not hard to check that (2.28) holds with

$$p := \mathbb{E}\big[\|X_0\hat{Y}'_\infty\|\big], \tag{2.29}$$

where $\hat{Y}'_\infty$ is independent of X_0 and distributed according to the invariant law of the dual of X viewed from its left-most particle. We note that in [ALM92], clustering is proved for a variation of the range R voter model that is not a cancellative system and for which no dual is known. The authors show that also for this model, the probability p in (2.28) depends in a nontrivial way on the initial law, and determine its asymptotics if X_0 is a product measure with low density.

3 Proofs

3.1 Duality and interface models

In this section we prove the lemmas from Section 2.2.

We equip $S_-(\mathbb{I})$ with the stronger topology such that $x_n \to x$ if and only if $x_n(i) \to x(i)$ for each $i \in \mathbb{I}$ and $\inf x_n \to \inf x$ (with notation as in (2.9)), and we let $\mathcal{L}_-(\mathbb{I})$ denote the space of all linear maps $A : S_-(\mathbb{I}) \to S_-(\mathbb{I})$ that are continuous with respect to this stronger topology. It is not hard to see that $A \in \mathcal{L}_-(\mathbb{I})$ if and only if its matrix, defined as in (2.1), satisfies

$$\begin{aligned} &\sup\{j \in I : A(i,j) = 1 \text{ for some } i \le k\} < \infty, \\ &\inf\{i \in I : A(i,j) = 1 \text{ for some } j \ge k\} > -\infty \end{aligned} \tag{3.1}$$

for all $k \in \mathbb{I}$. Note that for $A \in \mathcal{L}_-(\mathbb{I})$ and $x \in S_-(\mathbb{I})$, the infinite sum in (2.2) reduces to a finite sum and hence is well-defined. We define $\mathcal{L}_+(\mathbb{I})$ analogously. We observe that $\mathcal{L}_-(\mathbb{I}) \cap \mathcal{L}_+(\mathbb{I}) \subset \mathcal{L}(\mathbb{I})$ and that $A \in \mathcal{L}_-(\mathbb{I})$ if and only if $A^\dagger \in \mathcal{L}_+(\mathbb{I})$. One has

$$\|x(Ay)\| = \|(A^\dagger x)y\| \qquad \big(x \in \mathcal{S}_-(\mathbb{I}),\ y \in \mathcal{S}_+(\mathbb{I}),\ A \in \mathcal{L}_+(\mathbb{I})\big), \tag{3.2}$$

and the same holds if $A \in \mathcal{L}_-(\mathbb{I}) \cap \mathcal{L}_+(\mathbb{I})$ and $(x,y) \in \mathcal{G}(\mathbb{I},\mathbb{I})$.

We define the spaces $\mathcal{L}(\mathbb{I}, \mathbb{I}+\frac{1}{2})$ and $\mathcal{L}_\pm(\mathbb{I}, \mathbb{I}+\frac{1}{2})$ of continuous linear maps from $\{0,1\}^{\mathbb{I}}$ to $\{0,1\}^{\mathbb{I}+\frac{1}{2}}$ or from $S_\pm(\mathbb{I})$ to $S_\pm(\mathbb{I}+\frac{1}{2})$ analogous to $\mathcal{L}(\mathbb{I})$ and $\mathcal{L}_\pm(\mathbb{I})$, respectively. Recall the definition of the interface operator ∇ from (2.14). It is straightforward to check the following facts.

Lemma 7 (Differential operator) *The map $\nabla : S_\pm(\mathbb{I}) \to S_\pm(\mathbb{I}+\frac{1}{2})$ is a bijection with inverse $\nabla_\pm^{-1} \in \mathcal{L}_\pm(\mathbb{I}+\frac{1}{2}, \mathbb{I})$ given by*

$$\nabla_-^{-1}(i,j) = 1_{\{i>j\}} \quad \text{and} \quad \nabla_+^{-1}(i,j) = 1_{\{i<j\}} \qquad (i \in \mathbb{I},\ j \in \mathbb{I}+\tfrac{1}{2}). \tag{3.3}$$

One has $\nabla^\dagger = \nabla$ and $(\nabla_-^{-1})^\dagger = \nabla_+^{-1}$.

Remark 1 If one defines right and left discrete derivatives as $\nabla_{\rm left}x(i) := x(i+1) \oplus x(i)$ and $\nabla_{\rm right}x(i) := x(i) \oplus x(i-1)$, then $(\nabla_{\rm right})^\dagger = \nabla_{\rm left}$. The main reason why we work with half-integers is that we want the operator ∇ to look as much as possible like a self-adjoint operator. (Note that since ∇ maps $S_\pm(\mathbb{I})$ into the different space $S_\pm(\mathbb{I}+\frac{1}{2})$, it is not strictly speaking self-adjoint.) Half-integers are also quite natural in view of the interpretation of ∇ as an interface operator.

Remark 2 We observe from (3.3) and (2.2) that

$$\nabla_-^{-1}x(i) = 1_{\{x(j)=1 \text{ for an odd number of sites } j<i\}},$$

and a similar formula holds for ∇_+^{-1}. Let

$$S_{\rm even}(\mathbb{I}) := \big\{x \in S_{\rm fin}(\mathbb{I}) : |x| \text{ is even}\big\} \quad \text{and} \quad S_{\rm odd}(\mathbb{I}) := \big\{x \in S_{\rm fin}(\mathbb{I}) : |x| \text{ is odd}\big\}. \tag{3.4}$$

Then $\nabla : S_{\rm fin}(\mathbb{I}) \to S_{\rm even}(\mathbb{I}+\frac{1}{2})$ is a bijection with inverse $\nabla_-^{-1} = \nabla_+^{-1}$ on $S_{\rm even}(\mathbb{I}+\frac{1}{2})$. On the other hand, $\nabla_-^{-1}x = \nabla_+^{-1}x \oplus \underline{1}$ for $x \in S_{\rm odd}(\mathbb{I}+\frac{1}{2})$.

Proof of Lemmas 2 and 3 For $A \in \mathcal{L}_{\rm pp}(\mathbb{I}+\frac{1}{2})$, we define $\Psi^{-1}(A)$ by

$$\Psi^{-1}(A) := \nabla_-^{-1}A\nabla = \nabla_+^{-1}A\nabla \qquad \big(A \in \mathcal{L}_{\rm pp}(\mathbb{I}+\tfrac{1}{2})\big). \tag{3.5}$$

Note that since $A(\,\cdot\,,j) \in S_{\rm even}(\mathbb{I})$ for each $j \in \mathbb{I}+\frac{1}{2}$, in view of Remark 2 below Lemma 7, we have $\nabla_-^{-1}A(i,j) = \nabla_+^{-1}A(i,j)$ for each i,j, so the two formulas for $\Psi^{-1}(A)$ coincide. Since $\nabla(i,\,\cdot\,) \in S_{\rm even}(\mathbb{I})$ for each $i \in \mathbb{I}+\frac{1}{2}$, we have that $\Psi^{-1}(A) \in \mathcal{L}_{\rm ts}(\mathbb{I})$. Next, for $A \in \mathcal{L}_{\rm ts}(\mathbb{I})$, we set

$$\Psi(A) := \big(\Psi^{-1}(A^\dagger)\big)^\dagger = (\nabla_\mp^{-1}A^\dagger\nabla)^\dagger = \nabla A \nabla_\pm^{-1}, \tag{3.6}$$

where we have used that by Lemma 7, $\nabla^\dagger = \nabla$ and $(\nabla_\mp^{-1})^\dagger = \nabla_\pm^{-1}$. Since $A \in \mathcal{L}_{\rm ts}(\mathbb{I})$ if and only if $A^\dagger \in \mathcal{L}_{\rm pp}(\mathbb{I})$, this clearly defines a map $\Psi : \mathcal{L}_{\rm ts}(\mathbb{I}) \to \mathcal{L}_{\rm pp}(\mathbb{I}+\frac{1}{2})$. Now

$$\Psi(\Psi^{-1}(A))x = \nabla\nabla_\pm^{-1}A\nabla\nabla_\pm^{-1}x = Ax = \nabla_\pm^{-1}\nabla A\nabla_\pm^{-1}\nabla x = \Psi^{-1}(\Psi(A))x \qquad \big(x \in S_\pm(\mathbb{I}+\tfrac{1}{2})\big), \tag{3.7}$$

which proves that Ψ and Ψ^{-1} are each other's inverses. Moreover,

$$\Psi(A)\nabla x = \nabla A\nabla_\mp^{-1}\nabla x = \nabla Ax \qquad \big(x \in S_\pm(\mathbb{I})\big). \tag{3.8}$$

Since each $x \in \{0,1\}^{\mathbb{I}}$ can be written as $x = x_- \oplus x_+$ with $x_\pm \in S_\pm(\mathbb{I})$, this proves (2.15). Since the map $\nabla : \{0,1\}^{\mathbb{I}} \to \{0,1\}^{\mathbb{I}+\frac{1}{2}}$ is surjective, $\Psi(A)$ is in fact uniquely characterized by (2.15).

The fact that the interface model of X is the parity-preserving cancellative system with rates as in (2.16) is immediate from (2.3) and (2.15). Lemma 3 follows from (3.6). ∎

Proof of Lemma 4 Immediate from Lemmas 2 and 3 and cancellative systems duality (2.13). Note that the two formulas for the duality function H coincide by Lemma 7. ∎

3.2 Noncoexistence

Proof of Lemma 5 We start by observing that

$$h(x) \leq \sum_{i\in\mathbb{I}+\frac{1}{2}} \mathbb{E}\big[|x(\hat{Y}_\infty+i)|\big] = \sum_{i\in\mathbb{I}+\frac{1}{2}} \mathbb{E}\Big[\sum_{j\in x}|\delta_j(\hat{Y}_\infty+i)|\Big] = \mathbb{E}\Big[\sum_{j\in x}|\hat{Y}_\infty|\Big] = |x|\,\mathbb{E}\big[|\hat{Y}_\infty|\big] \tag{3.9}$$

for all $x \in S_{\rm fin}(\mathbb{I}+\frac{1}{2})$, where, as we have done before, we notationally identify x with the set $\{i : x(i)=1\}$, and the second equality is obtained by moving the sum over i inside the expectation. Similarly

$$h(x) \geq \sum_{i\in\mathbb{I}+\frac{1}{2}} \mathbb{E}\big[\|x(\hat{Y}_\infty+i)\|1_{\{\hat{Y}_\infty=\delta_0\}}\big] = \mathbb{P}[\hat{Y}_\infty=\delta_0]\,|x|. \tag{3.10}$$

Since $\mathbb{E}\big[|\hat{Y}_\infty|\big] < \infty$ and $\mathbb{P}[\hat{Y}_\infty = \delta_0] > 0$ by the assumption of strong interface tightness, formulas (3.9) and (3.10) imply (2.25).

The upper bound of (2.25), together with (2.9), show that if $\mathbb{E}\big[|X'_0|\big] < \infty$, then $\mathbb{E}[h(X'_t)] < \infty$ for all $t \geq 0$. Let $Y = (Y_t)_{t\geq 0}$ be the interface model of X, started in the initial law $\mathbb{P}[Y_0 \in \cdot\,] := \mathbb{P}[\hat{Y}_\infty \in \cdot\,]$ if $\mathbb{I} + \frac{1}{2} = \mathbb{Z}$ and $\mathbb{P}[Y_0 \in \cdot\,] := \mathbb{P}[(\hat{Y}_\infty + \frac{1}{2}) \in \cdot\,]$ if $\mathbb{I} + \frac{1}{2} = \mathbb{Z} + \frac{1}{2}$, and independent of X'. Then by duality (2.13), letting l_t denote the position of the left-most particle of Y_t, we see that

$$\mathbb{E}\big[h(X'_t)\big] = \sum_{i\in\mathbb{Z}} \mathbb{E}\big[\|X'_t(Y_0+i)\|\big] = \sum_{i\in\mathbb{Z}} \mathbb{E}\big[\|X'_0(Y_t+i)\|\big] = \mathbb{E}\big[\sum_{i\in\mathbb{Z}} \|X'_0(Y_t+i-l_t)\|\big] = \mathbb{E}\big[h(X'_0)\big], \tag{3.11}$$

which proves (in combination with the Markov property of X') that $h(X'_t)$ is a martingale. ∎

Proof of Theorem 6 It is straightforward to check that the one-sided nearest neighbour voter model, in which sites with rate one copy the type on their left, is dual, in the sense of the duality in (2.19), to a one-sided nearest neighbour voter model in which sites with rate one copy the type on their right. Therefore, if the dynamics of X have a (left or right) nearest-neighbour voter component, then the the dynamics of X' have a (right or left) nearest-neighbour voter component. From this, it is easy to see that the probability that the process X' started in x gets extinct

$$q(x) := \mathbb{P}^x\big[\exists t \geq 0 \text{ s.t. } X'_t = \underline{0}\big] \tag{3.12}$$

can be uniformly bounded from below in the sense that

$$\inf\{q(x) : |x| \leq K\} > 0 \qquad \forall K < \infty. \tag{3.13}$$

Formula (3.13) is our sole reason for assuming that the dynamics of X has a (left or right) nearest-neighbour voter component; if this can be established by some other means then the conclusions of Theorem 6 remain valid.

Extinction of X' now follows from a standard argument: Letting $(\mathcal{F}_t)_{t\geq 0}$ denote the filtration generated by X', we have by the Markov property and the a.s. continuity of the conditional expectation with respect to increasing sequences of σ-fields that

$$q(X'_t) = \mathbb{P}\big[\exists s \geq 0 \text{ s.t. } X'_s = \underline{0}\,\big|\,\mathcal{F}_t\big] \underset{t\to\infty}{\longrightarrow} 1_{\{\exists s\geq 0 \text{ s.t. } X'_s=\underline{0}\}} \quad \text{a.s.} \tag{3.14}$$

In particular, $q(X'_t) \to 0$ a.s. on the event that X' does not get extinct, which by (3.13) implies that $|X'_t| \to \infty$ a.s. By the lower bound in (2.25), it follows that $h(X'_t) \to \infty$ a.s. on the event that X' does not get extinct. But Lemma 5 says that $h(X'_t)$ is a nonnegative martingale, so $h(X'_t) \to \infty$ has zero probability and hence the same must be true for the event that X' does not get extinct.

It follows that the interface model Y' of X' started in $Y'_0 = \delta_i + \delta_{i+1}$ also gets trapped in $\underline{0}$ a.s., so by the fact that Y' is dual to X in the sense of (2.13), we find that

$$\mathbb{P}\big[X_t(i) \neq X_t(i+1)\big] = \mathbb{E}\big[\|X_t(\delta_i + \delta_{i+1})\|\big] = \mathbb{E}\big[\|X_0 Y'_t\|\big] \leq \mathbb{P}[Y'_t \neq \underline{0}] \underset{t\to\infty}{\longrightarrow} 0. \tag{3.15}$$

∎

Proof of Theorem 1 Immediate from Theorem 6 and the fact that the neutral Neuhauser-Pacala model, the affine voter model, and the rebellious voter model are cancellative systems, which is proved in [SS08]. ∎

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