Subcritical contact processes seen from a typical infected site

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Abstract

What is the long-time behavior of the law of a contact process started with a single infected site, distributed according to counting measure on the lattice? This question is related to the configuration as seen from a typical infected site and gives rise to the definition of so-called eigenmeasures, which are possibly infinite measures on the set of nonempty configurations that are preserved under the dynamics up to a multiplicative constant. In this paper, we study eigenmeasures of contact processes on general countable groups in the subcritical regime. We prove that in this regime, the process has a unique spatially homogeneous eigenmeasure. As an application, we show that the exponential growth rate is continuously differentiable and strictly decreasing as a function of the recovery rate, and we give a formula for the derivative in terms of the eigenmeasures of the contact process and its dual.

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1 Introduction and main results

1.1 Introduction

It is known that contact processes on regular trees behave quite differently from contact processes on the d-dimensional integer lattice $\mathbb{Z}^d$. Indeed, if λ_c and λ'_c denote the critical infection rates associated with global and local survival, respectively, then one has $\lambda_c < \lambda'_c$ on trees while $\lambda_c = \lambda'_c$ on $\mathbb{Z}^d$. For $\lambda > \lambda'_c$, the process exhibits complete convergence and the upper invariant law is the only nontrivial invariant law, while on trees, in the intermediate regime $\lambda_c < \lambda \leq \lambda'_c$, there is a multitude of (not spatially homogeneous) invariant laws. The situation is reminiscent of what is known about unoriented percolation on transitive graphs, where one has uniqueness of the infinite cluster if the graph is amenable, while it is conjectured, and proved in some cases, that on nonamenable graphs there is an intermediate paremeter regime with infinitely many infinite clusters. We refer to [Lig99] as a general reference to contact processes on $\mathbb{Z}^d$ and trees and [Hag11] for percolation beyond $\mathbb{Z}^d$.

In general, it is not hard (but also not very interesting) to determine the limit behavior of contact processes started from a spatially homogeneous (i.e., translation invariant) initial law. On the other hand, it seems much more difficult to study the process started with a finite number of infected sites. For example, it seems quite difficult to prove that $\lambda_c = \lambda'_c$ on any amenable transitive graph. As an intermediate problem, in [Swa09, Problem 1 from Section 1.5], it has been proposed to study the process started with a single infected site, chosen uniformly from the lattice. For infinite lattices, the resulting 'law' at time t will be an infinite measure. However, as shown in [Swa09, Lemma 4.2], conditioning such a measure on the origin being infected yields a probability law, which can be interpreted as the process seen from a typical infected site.

There is a close connection between the law of the process seen from a typical infected site and the exponential growth rate r of a contact process. This can be understood by realizing that the number of healthy sites surrounding a typical infected site determines the number of infections that can be made and hence the speed at which the infection grows. In the context of infinite laws, which cannot be normalized, it is natural to generalize the concept of an invariant measure to an 'eigenmeasure', which is a measure on the set of nonempty configurations that is preserved under time evolution up to a multiplicative constant. In particular, if the suitably rescaled law at time t of the process started with a single, uniformly distributed site has a nontrivial long-time limit, then it follows from results in [Swa09] that such a limit law must be an eigenmeasure whose eigenvalue is the exponential growth rate r of the process.

In the present paper, we study eigenmeasures of subcritical contact processes on general countable groups. Our set-up includes translation-invariant contact processes on $\mathbb{Z}^d$ and on regular trees, as well as long-range processes and asymmetric processes. We will show that such processes have a unique homogeneous eigenmeasure which is the vague limit of the rescaled law at time t of the process started in any homogeneous, possibly infinite, initial law. As an application of our results, we give an expression for the derivative of the exponential growth rate as a function of the recovery rate in terms of the eigenmeasures of the process and its dual, and we use this to show that this derivative is strictly negative and continuous.

1.2 Contact processes on groups

Let Λ be a finite or countably infinite group with group action $(i, j) \mapsto ij$, inverse operation $i \mapsto i^{-1}$, and unit element 0 (also refered to as the origin). Let $a : \Lambda \times \Lambda \to [0, \infty)$ be a function such that $a(i,i)=0$ $(i\in\Lambda)$ and

$$\begin{array}{lll}\text{(i)} & a(i,j)=a(ki,kj) & (i,j,k\in\Lambda),\\ \text{(ii)} & |a|:=\sum_{i\in\Lambda}a(0,i)<\infty, & \end{array}\tag{1.1}$$

and let $\delta\geq 0$. By definition, the (Λ,a,δ)-*contact process* is the Markov process $\eta=(\eta_t)_{t\geq 0}$, taking values in the space $\mathcal{P}=\mathcal{P}(\Lambda):=\{A:A\subset\Lambda\}$ consisting of all subsets of Λ, with the formal generator

$$\begin{aligned}Gf(A):=&\sum_{i,j\in\Lambda}a(i,j)1_{\{i\in A\}}1_{\{j\notin A\}}\{f(A\cup\{j\})-f(A)\}\\ &+\delta\sum_{i\in\Lambda}1_{\{i\in A\}}\{f(A\backslash\{i\})-f(A)\}.\end{aligned}\tag{1.2}$$

If $i\in\eta_t$, then we say that the site i is infected at time t; otherwise it is healthy. Then (1.2) says that an infected site i infects another site j with *infection rate* $a(i,j)\geq 0$, and infected sites become healthy with *recovery rate* $\delta\geq 0$.

We will usually assume that the infection rates are irreducible in some sense or another. To make this precise, let us write $i\overset{a}{\rightsquigarrow}j$ if the site j can be infected through a chain of infections starting from i. Then we say that a is *irreducible* if $i\overset{a}{\rightsquigarrow}j$ for all $i,j\in\Lambda$. We say that a is *weakly irreducible* if for all $i,j\in\Lambda$, either $i\overset{a}{\rightsquigarrow}j$ or $j\overset{a}{\rightsquigarrow}i$. Finally, we will sometimes need the intermediate condition

$$\forall i,j\in\Lambda:\ \exists k,l\in\Lambda:\ k\overset{a}{\rightsquigarrow}i,\ k\overset{a}{\rightsquigarrow}j,\ i\overset{a}{\rightsquigarrow}l,\ j\overset{a}{\rightsquigarrow}l.\tag{1.3}$$

In words, this says that for any two sites i,j there exists a site k from which both i and j can be infected, and a site l that can be infected both from i and from j. If the rates a are symmetric, or more generally if one has $a(i,j)>0$ iff $a(j,i)>0$, then all three conditions are equivalent. In general, irreducibility implies (1.3) which implies weak irreducibility, but none of the converse implications holds.

Define *reversed infection rates* $a^\dagger(i,j):=a(j,i)$. It is known (and easy to see from the graphical representation) that the (Λ,a,δ)-contact process and $(\Lambda,a^\dagger,\delta)$-contact process are dual in the following sense. Let $(\eta^A_t)_{t\geq 0}$ and $(\eta^{\dagger\,B}_t)_{t\geq 0}$ denote the respective processes started in $\eta^A_0=A$ and $\eta^{\dagger\,B}_0=B$. Then

$$\mathbb{P}[\eta^A_t\cap B\neq\emptyset]=\mathbb{P}[A\cap\eta^{\dagger\,B}_t\neq\emptyset]\qquad(A,B\in\mathcal{P}(\Lambda),\ t\geq 0).\tag{1.4}$$

We note that unless $a=a^\dagger$ or the group Λ is abelian, the (Λ,a,δ)- and $(\Lambda,a^\dagger,\delta)$-contact processes have in general different dynamics and need to be distinguished. (If Λ is abelian, then the (Λ,a,δ)- and $(\Lambda,a^\dagger,\delta)$-contact processes can be mapped into each other by the transformation $i\mapsto i^{-1}$.) We say that the (Λ,a,δ)-contact process *survives* if

$$\rho(A):=\mathbb{P}\big[\eta^A_t\neq\emptyset\ \forall t\geq 0\big]>0\tag{1.5}$$

for some, and hence for all nonempty A of finite cardinality $|A|$. We set $\theta=\theta(\Lambda,a,\delta):=\rho(\{0\})$ and call

$$\delta_{\rm c}=\delta_{\rm c}(\Lambda,a):=\sup\{\delta\geq 0:\theta(\Lambda,a,\delta)>0\}\tag{1.6}$$

the *critical recovery rate*. It is known that $\delta_c < \infty$. If Λ is finitely generated, then moreover $\delta_c > 0$ provided a is weakly irreducible [Swa07, Lemma 4.18], but for non-finitely generated groups weak irreducibility is in general not enough [AS10]. It is well-known that

$$\mathbb{P}\big[\eta_t^\Lambda \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}, \tag{1.7}$$

where $\overline{\nu}$ is an invariant law of the (Λ, a, δ)-contact process, known as the *upper invariant law*. Using duality, it is not hard to prove that $\overline{\nu} = \delta_\emptyset$ if the $(\Lambda, a^\dagger, \delta)$-contact process dies out, while $\overline{\nu}$ is concentrated on the nonempty subsets of Λ if the process survives.

1.3 Eigenmeasures

It follows from subadditivity (see [Swa09, Lemma 1.1]) that any (Λ, a, δ)-contact process has a well-defined exponential growth rate, i.e., there exists a constant $r = r(\Lambda, a, \delta)$ with $-\delta \leq r \leq |a| - \delta$ such that

$$r = \lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}\big[|\eta_t^A|\big] \qquad (0 < |A| < \infty). \tag{1.8}$$

The following theorem lists some properties of the function $r(\Lambda, a, \delta)$.

Theorem 0 (Properties of the exponential growth rate)
For any (Λ, a, δ)-contact process:

(a) $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$.

(b) *The function $\delta \to r(\Lambda, a, \delta)$ is nonincreasing and Lipschitz continuous on $[0, \infty)$, with Lipschitz constant 1.*

(c) *If $r(\Lambda, a, \delta) > 0$, then the (Λ, a, δ)-contact process survives.*

(d) $\{\delta \geq 0 : r(\Lambda, a, \delta) < 0\} = (\delta_c, \infty)$.

The (easy) proofs of parts (a)–(c) can be found in [Swa09, Theorem 1.2]. The analogue of part (d) for unoriented percolation on $\mathbb{Z}^d$ was first proved by Menshikov [Men86] and Aizenman and Barsky [AB87]. Using the approach of the latter paper, Bezuidenhout and Grimmett [BG91, formula (1.13)] proved the statement in part (d) for contact processes on $\mathbb{Z}^d$. This has been generalized to processes on general transitive graphs in [AJ07]. As we point out in Appendix A, their arguments are not restricted to graphs but apply in the generality we need here. We note that it follows from parts (a) and (d) that $\delta_c(\Lambda, a) = \delta_c(\Lambda, a^\dagger)$. In general, it is not known if survival of a (Λ, a, δ)-contact process implies survival of the dual $(\Lambda, a^\dagger, \delta)$-contact process but any counterexample would have to be at $\delta = \delta_c$, while by [Swa09, Corollary 1.3], Λ would have to be amenable. If Λ is a finitely generated group of subexponential growth and the infection rates satisfy an exponential moment condition (for example, if $\Lambda = \mathbb{Z}^d$ and a is nearest-neighbor), then $r \leq 0$ [Swa09, Thm 1.2 (e)], but in general (e.g. on trees), it is possible that $r > 0$. Indeed, one of the main results of [Swa09] says that if Λ is nonamenable, the (Λ, a, δ)-contact process survives, and the infection rates satisfy the irreducibility condition (1.3), then $r > 0$ [Swa09, Thm. 1.2 (f)].

We next turn our attention to eigenmeasures. Recall that $\mathcal{P} = \mathcal{P}(\Lambda)$ denotes the space of all subsets of Λ. We let $\mathcal{P}_+ := \{A : |A| > 0\}$ and $\mathcal{P}_{\rm fin} := \{A : |A| < \infty\}$ denote the subspaces consisting of all nonempty, respectively finite subsets of Λ. We observe that $\mathcal{P} \cong \{0,1\}^\Lambda$ and equip it with the product topology and Borel-σ-field $\mathcal{B}(\mathcal{P})$. Note that since $\mathcal{P}$ is compact,

$\mathcal{P}_+ = \mathcal{P}\backslash\{\emptyset\}$ is a locally compact space. For $A \subset \Lambda$ and $i \in \Lambda$, we write $iA := \{ij : j \in A\}$, and for any $\mathcal{A} \subset \mathcal{P}$ we write $i\mathcal{A} := \{iA : A \in \mathcal{A}\}$. It follows from (1.1) (i) that $i\eta_t^A$ and η_t^{iA} are equally distributed. We say that a measure μ on $\mathcal{P}$ is (spatially) *homogeneous* if it is invariant under the left action of the group, i.e., if $\mu(\mathcal{A}) = \mu(i\mathcal{A})$ for each $i \in \Lambda$ and $\mathcal{A} \in \mathcal{B}(\mathcal{P})$. We say that μ is *nontrivial* if μ is concentrated on $\mathcal{P}_+$.

Following [Swa09], we say that a measure μ on $\mathcal{P}_+$ is an *eigenmeasure* of the (Λ, a, δ)-contact process if μ is nonzero, locally finite, and there exists a constant $\lambda \in \mathbb{R}$ such that

$$\int \mu(\mathrm{d}A)\mathbb{P}[\eta_t^A \in \cdot\,]\big|_{\mathcal{P}_+} = e^{\lambda t}\mu \qquad (t \geq 0), \tag{1.9}$$

where $|_{\mathcal{P}_+}$ denotes restriction (of a measure) to $\mathcal{P}_+$. We call λ the associated *eigenvalue*.

It follows from [Swa09, Prop. 1.4] that each (Λ, a, δ)-contact process has a (spatially) homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue $r = r(\Lambda, a, \delta)$. Since $\mathring{\nu}$ may be an infinite measure, its normalization is somewhat arbitrary. We will adopt the convention that $\int \mathring{\nu}(\mathrm{d}A)1_{\{0\in A\}} = 1$. In general, it is not known if $\mathring{\nu}$ is unique. Under the irreducibility condition (1.3), it has been shown in [Swa09, Thm. 1.5] that if the upper invariant measure $\overline{\nu}$ of a (Λ, a, δ)-contact process is nontrivial and $r(\Lambda, a, \delta) = 0$, then $\mathring{\nu}$ is unique and in fact $\mathring{\nu} = c\,\overline{\nu}$ for some $c > 0$. The main aim of the present paper is to investigate eigenmeasures in the subcritical case $r < 0$. Here is our main result.

Theorem 1 (Eigenmeasures in the subcritical case) *Assume that the infection rates satisfy the irreducibility condition (1.3) and that the exponential growth rate from (1.8) satisfies $r < 0$. Then there exists a unique homogeneous eigenmeasure $\mathring{\nu}$ of the (Λ, a, δ)-contact process such that $\int \mathring{\nu}(\mathrm{d}A)1_{\{0\in A\}} = 1$. This eigenmeasure has eigenvalue r and is concentrated on $\mathcal{P}_{\rm fin}(\Lambda)$. If μ is any nonzero, homogeneous, locally finite measure on $\mathcal{P}_+(\Lambda)$, then*

$$e^{-rt}\int \mu(\mathrm{d}A)\mathbb{P}[\eta_t^A \in \cdot\,]\big|_{\mathcal{P}_+(\Lambda)} \underset{t\to\infty}{\Longrightarrow} c\,\mathring{\nu}, \tag{1.10}$$

where $\Rightarrow$ denotes vague convergence of locally finite measures on $\mathcal{P}_+(\Lambda)$ and $c > 0$ is a constant, given by

$$c = \frac{\int \mu(\mathrm{d}A)\int \mathring{\nu}^\dagger(\mathrm{d}B)|A\cap B|^{-1}1_{\{0\in A\cap B\}}}{\int \mathring{\nu}(\mathrm{d}A)\int \mathring{\nu}^\dagger(\mathrm{d}B)|A\cap B|^{-1}1_{\{0\in A\cap B\}}}, \tag{1.11}$$

where $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ denote the homogeneous eigenmeasures of the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact processes, respectively, normalized such that $\int \mathring{\nu}(\mathrm{d}A)1_{\{0\in A\}} = 1 = \int \mathring{\nu}^\dagger(\mathrm{d}A)1_{\{0\in A\}}$.

1.4 The process seen from a typical infected site

Let $(\eta_t^{\{0\}})_{t\geq 0}$ be a (Λ, a, δ)-contact process, started with a single infected site at the origin, where $\eta_t^{\{0\}} = \eta_t^{\{0\}}(\omega)$ is defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, for each $t \geq 0$, we can define a new probability law $\hat{\mathbb{P}}_t$ on a suitably enriched probability space $\hat{\Omega}$ that also contains a Λ-valued random variable ι, by setting

$$\hat{\mathbb{P}}_t\big[\omega \in \mathcal{A},\ \iota = i\big] := \frac{\mathbb{P}[\omega \in \mathcal{A},\ i \in \eta_t^{\{0\}}(\omega)]}{\mathbb{E}[|\eta_t^{\{0\}}|]} \qquad (\mathcal{A} \in \mathcal{F},\ i \in \Lambda). \tag{1.12}$$

The law $\hat{\mathbb{P}}_t$ is a Campbell law (closely related to the more well-known Palm laws). In words, $\hat{\mathbb{P}}_t$ is obtained from the original law $\mathbb{P}$ by size-biasing on the number $|\eta_t^{\{0\}}|$ of infected sites at time t and then choosing one site ι from $\eta_t^{\{0\}}$ with equal probabilities.

Let $\mu_t := \sum_{i\in\Lambda} \mathbb{P}[\eta_t^{\{i\}} \in \cdot\,]|_{\mathcal{P}_+}$ be the infinite 'law' of the process started with a single infection at a uniformly chosen site in the lattice. Then it has been shown in [Swa09, Lemma 4.2] that

$$\mu_t\big(\,\cdot\,\big|\,\{A : 0 \in A\}\big) = \hat{\mathbb{P}}_t\big[\iota^{-1}\eta_t^{\{0\}} \in \cdot\,\big], \tag{1.13}$$

i.e., μ_t conditioned on the origin being infected describes the distribution of $\eta_t^{\{0\}}$ under the Campbell law $\hat{\mathbb{P}}_t$ with the 'typical infected site' ι shifted to the origin.

In view of this, Theorem 1 gives information about the long-time limit law of the process seen from a typical infected site. We will apply this to study the derivative of the exponential growth rate $r(\Lambda, a, \delta)$ with respect to the recovery rate δ. Let $\eta_t^{\delta,\{0\}}$ denote the process with a given recovery rate δ (and (Λ, a) fixed), constructed with the graphical representation of the contact process (see Section 2.1). A version of Russo's formula (compare [Gri99, Thm 2.25]) tells us that

$$\frac{\partial}{\partial\delta}\frac{1}{t}\log\mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big] = \frac{1}{t}\int_0^t \hat{\mathbb{P}}_t\big[\exists j \in \Lambda \text{ s.t. } (0,0) \rightsquigarrow_{(j,s)} (\iota, t)\big]\mathrm{d}s, \tag{1.14}$$

where $(0,0) \rightsquigarrow_{(j,s)} (\iota, t)$ denotes the event that in the graphical representation, all open paths from $(0,0)$ to (ι, t) lead through (j, s). In other words, the right-hand side of (1.14) is the fraction of time that there is a *pivotal* site on the way from $(0,0)$ to the typical site (ι, t).

By grace of Theorem 1, we are able to control the long-time limit of formula (1.14), leading to the following result.

Theorem 2 (Derivative of the exponential growth rate) *Assume that the infection rates satisfy the irreducibility condition (1.3). Then the function $\delta \mapsto r(\Lambda, a, \delta)$ is continuously differentiable on $(\delta_{\mathrm{c}}, \infty)$ and satisfies $\frac{\partial}{\partial\delta}r(\Lambda, a, \delta) < 0$ on $(\delta_{\mathrm{c}}, \infty)$. Moreover, one has*

$$-\frac{\partial}{\partial\delta}r(\Lambda, a, \delta) = \frac{\int \mathring{\nu}(\mathrm{d}A)\int \mathring{\nu}^{\dagger}(\mathrm{d}B)1_{\{A\cap B=\{0\}\}}}{\int \mathring{\nu}(\mathrm{d}A)\int \mathring{\nu}^{\dagger}(\mathrm{d}B)|A\cap B|^{-1}1_{\{0\in A\cap B\}}}, \tag{1.15}$$

where $\mathring{\nu}$ and $\mathring{\nu}^{\dagger}$ denote the homogeneous eigenmeasures of the (Λ, a, δ)- and $(\Lambda, a^{\dagger}, \delta)$-contact processes, respectively, defined in Theorem 1.

The differentiability of the exponential growth rate in the subcritical regime is expected. Indeed, for normal (unoriented) percolation in the subcritical regime, it is even known that the number of open clusters per vertex and the mean size of the cluster at the origin depend analytically on the percolation parameter. This result is due to Kesten [Kes81]; see also [Gri99, Section 6.4]. For oriented percolation in one plus one dimension in the *supercritical* regime, Durrett [Dur84, Section 14] has shown that the percolation probability is infinitely differentiable as a function of the percolation parameter. It is not immediately clear, however, if the methods in these papers can be adapted to cover the exponential growth rate. At any rate, they would not give very explicit information about the derivative such as positivity.

In principle, if for a given lattice one can control the right-hand side of (1.15) uniformly as $\delta \downarrow \delta_{\mathrm{c}}$, then this would imply trivility of the critical exponent associated with the exponential growth rate. But this is probably difficult in the most interesting cases, such as $\mathbb{Z}^d$ above the critical dimension.

1.5 Discussion and outlook

This paper is part of a larger program, initiated in [Swa09], which aims to describe all homogeneous eigenmeasures of (Λ, a, δ)-contact processes and prove convergence for suitable starting measures. There are several regimes of interest: the subcritical regime $\delta > \delta_{\rm c}$, the critical regime $\delta = \delta_{\rm c}$, and the supercritical regime $\delta < \delta_{\rm c}$, which needs to be distinguised into processes for which $r = 0$ in the supercritical regime (such as processes on $\mathbb{Z}^d$) and processes for which $r > 0$ in the supercritical regime (such as processes on trees).

In [Swa09], rather weak results have been derived for processes with $r = 0$ in the supercritical regime. In particular, it was shown that for such processes, there exists a unique homogeneous eigenmeasure with eigenvalue zero [Swa09, Thm. 1.5], but it has not been proved whether there are homogeneous eigenmeasures with other eigenvalues, while convergence has only been shown for one special initial measure and Laplace-transformed times [Swa09, Corollary 3.4].

Our present paper treats the subcritical case fairly conclusively. Arguably, this should be the easiest regime. Indeed, our analysis is made easier by the fact that the homogeneous eigenmeasure is concentrated on finite sets, which allows us to use a 'compensated' h-transform to translate problems related to long-time behavior into positive recurrence of a continuous-time Markov chain (see Proposition 15 below). In contrast, in the critical and supercritical regimes, we expect homogeneous eigenmeasures to be concentrated on infinite sets, hence these techniques are not available.

Nevertheless, our methods give some hints on what to do in some of the other regimes as well. Formula (1.15), which we expect to hold more generally, says, roughly speaking, that $-\frac{\partial}{\partial \delta} r(\Lambda, a, \delta)$ is the probability that two independent sets, which are distributed according to the forward and dual eigenmeasures $\mathring{\nu}$ and $\mathring{\nu}^\dagger$, intersect in a single point. In view of this, it is tempting to try to replace the fact that $\mathring{\nu}$ is concentrated on finite sets, which much helped our present analysis but holds only in the subcritical regime, by the weaker assumption that the 'intersection measure' of $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ (formally defined in Section 2.6) is concentrated on finite sets. In particular, one wonders if this always holds in the regime $r > 0$.

A simpler problem, which we have not pursued in the present paper, is to investigate higher-order derivatives of $r(\Lambda, a, \delta)$ with respect to δ or derivatives with respect to the infection rates $a(i, j)$. It seems likely that the latter are strictly positive in the subcritical regime and given by a formula similar to (1.15). Controlling higher-order derivatives of $r(\Lambda, a, \delta)$ might be more difficult; in particular, we do not know if the function $\delta \mapsto r(\Lambda, a, \delta)$ is concave, or (which would be a stronger statement), if the conditional laws $\mathring{\nu}_\delta(\,\cdot\,|\{A : 0 \in A\})$ are decreasing in the stochastic order, as a function of δ. These latter questions seem quite hard.

2 Eigenmeasures

2.1 Preliminaries

Since this will be needed on several occasions, we start by recalling the graphical representation of a contact process. Let $\omega = (\omega^{\rm r}, \omega^{\rm i})$ be a pair of independent, locally finite random subsets of $\Lambda \times \mathbb{R}$ and $\Lambda \times \Lambda \times \mathbb{R}$, respectively, produced by Poisson point processes with intensity δ and $a(i, j)$, respectively. We visualize this by plotting Λ horizontally and $\mathbb{R}$ vertically, marking points $(i, s) \in \omega^{\rm r}$ with a recovery symbol $*$, and drawing an infection arrow from (i, t) to (j, t) for each $(i, j, t) \in \omega^{\rm i}$. For any $(i, s), (j, u) \in \Lambda \times \mathbb{R}$ with $s \le u$, by definition, an *open path* from

(i,s) to (j,u) is a cadlag function $\pi : [s,u] \to \Lambda$ such that $\{(\pi(t),t) : t \in [s,u]\} \cap \omega^{\rm r} = \emptyset$ and $(\pi(t-),\pi(t),t) \in \omega^{\rm i}$ whenever $\pi(t-) \neq \pi(t)$. Thus, open paths must avoid recovery symbols and may follow infection arrows. We write $(i,s) \rightsquigarrow (j,u)$ to indicate the presence of an open path from (i,s) to (j,u). Then, for any $s \in \mathbb{R}$, we can construct a (Λ,a,δ)-contact process started in an initial state $A \in \mathcal{P}$ by setting

$$\eta^{A,s}_t := \{j \in \Lambda : (i,s) \rightsquigarrow (j,s+t) \text{ for some } i \in A\} \qquad (A \in \mathcal{P},\ s \in \mathbb{R},\ t \geq 0), \tag{2.1}$$

and likewise, we can construct a dual $(\Lambda,a^\dagger,\delta)$-contact process started in A by

$$\eta^{\dagger\,A,s}_t := \{j \in \Lambda : (j,s-t) \rightsquigarrow (i,s) \text{ for some } i \in A\} \qquad (A \in \mathcal{P},\ s \in \mathbb{R},\ t \geq 0). \tag{2.2}$$

In particular, we write $\eta^A_t := \eta^{A,0}_t$ and $\eta^{\dagger\,A}_t := \eta^{\dagger\,A,0}_t$.

For certain technical arguments that are needed in the proof of Theorem 2, we need to equip the group Λ with a metric d such that $d(0,i)$ tends to infinity as $i \to \infty$ sufficiently slowly in order for exponential moments of a (Λ,a,δ)-contact process to be finite. The next lemma guarantees the existence of the sort of metric we need.

Lemma 3 (Slowly growing metric) *Let Λ be a countable group and let $a : \Lambda \times \Lambda \to [0,\infty)$ satisfy (1.1). Then there exists a metric d on Λ such that*

$$\begin{array}{rll} \text{(i)} & d(i,j) = d(ki,kj) & (i,j,k \in \Lambda), \\ \text{(ii)} & \big|\{i \in \Lambda : d(0,i) \leq M\}\big| < \infty & (0 \leq M < \infty), \\ \text{(iii)} & \displaystyle\sum_i a(0,i) e^{\gamma d(0,i)} < \infty & (0 \leq \gamma < \infty). \end{array} \tag{2.3}$$

Proof We can find finite $\{0\} = \Delta_1 \subset \Delta_2 \subset \cdots$ such that $\sum_{i \in \Lambda \backslash \Delta_n} a(0,i) \leq |a| e^{-(n-1)}$. Making the sets Δ_n for $n \geq 2$ larger if necessary, we can moreover choose these sets such that they are symmetric, i.e., $\{i^{-1} : i \in \Delta_n\} = \Delta_n$ and such that $\Delta_\infty := \bigcup_{n \geq 1} \Delta_n$ generates Λ. (In particular, we can always choose $\Delta_\infty = \Lambda$, but for nearest-neighbor processes on graphs this leads to a somewhat unnatural metric d, which is why we only assume here that Δ_∞ generates Λ.) We set $\Delta_0 := \emptyset$ and define

$$\phi(i) := \begin{cases} n & (i \in \Delta_n \backslash \Delta_{n-1},\ n \geq 1) \\ \infty & (i \in \Lambda \backslash \Delta_\infty). \end{cases} \tag{2.4}$$

Since $a(0,i) = 0$ for $i \notin \Delta_\infty$,

$$\sum_{i \in \Lambda} a(0,i) \phi(i)^\gamma = \sum_{n \geq 1} n^\gamma \sum_{i \in \Delta_n \backslash \Delta_{n-1}} a(0,i) \leq |a| \sum_{n \geq 1} n^\gamma e^{-(n-2)} < \infty \tag{2.5}$$

for each $0 \leq \gamma < \infty$. Set

$$d'(i,j) = d'(0,i^{-1}j) := \log(\phi(i^{-1}j)) \qquad (i,j \in \Lambda). \tag{2.6}$$

Then d' satisfies properties (2.3) (i)–(iii), $d'(i,j) = 0$ if and only if $i = j$, and $d'(i,j) = d'(j,i)$ (by the symmetry of the sets Δ_n). Since d' need not yet be a metric, we define

$$d(i,j) := \inf \Big\{ \sum_{k=1}^n d'(i_{k-1},i_k) : n \geq 1,\ i_0,\ldots,i_n \in \Lambda,\ i_0 = i,\ i_n = j \Big\}, \tag{2.7}$$

i.e., $d(i,j)$ is a graph-style distance between i and j, defined as the shortest path from i to j where an edge from i_{k-1} to i_k has length $d'(i_{k-1},i_k)$. Note that $d(i,j)<\infty$ for each $i,j\in\Lambda$ since Δ_∞ generates Λ and $d(i,j)>0$ for each $i\neq j$ since $d'(i,j)\geq\log(2)$ for each $i\neq j$. It is now straightforward to check that d is a metric on Λ and that $d(i,j)=d(ki,kj)$ for all $i,j,k\in\Lambda$. Since $d(i,j)\leq d'(i,j)$, the metric d also enjoys property (2.3) (iii). Property (2.3) (ii), finally, follows from the fact that

$$\{i\in\Lambda:d(0,i)\leq M\}\subset\{j_1\cdots j_n:1\leq n\leq M/\log(2),\ d'(0,j_k)\leq M\ \forall k=1,\ldots,n\},\tag{2.8}$$

where we use that $d'(i,j)\geq\log(2)$ for all $i\neq j$, and we observe that if $d(0,i)\leq M$ $(i\neq 0)$, then there must be some $n\geq 1$ and $0=i_0,\ldots,i_n=i$ with $\sum_{k=1}^n d'(i_{k-1},i_k)\leq M$. Setting $j_k:=i_{k-1}^{-1}i_k$ we see that i must be of the form $i=j_1\cdots j_n$ with $\sum_{k=1}^n d'(0,j_k)\leq M$. ∎

For each $0\leq\gamma<\infty$, let us define a function $e_\gamma:\mathcal{P}_{\rm fin}\to[0,\infty)$ by

$$e_\gamma(A):=\sum_{i\in A}e^{\gamma d(0,i)}\qquad(\gamma\geq 0,\ A\in\mathcal{P}_{\rm fin}).\tag{2.9}$$

We note that a similar (but not entirely identical) function has proved useful in the study of contact processes on trees, see [Lig99, formula (I.4.3)].

Lemma 4 (Existence of exponential moments) *Let $(\eta^A_t)_{t\geq 0}$ be a (Λ,a,δ)-contact process started in a finite initial state $\eta^A_0=A\in\mathcal{P}_{\rm fin}$ and let d be a metric on Λ as in Lemma 3. Then*

$$\mathbb{E}\big[e_\gamma(\eta^A_t)\big]\leq e^{K_\gamma t}e_\gamma(A)\quad(t\geq 0)\quad\textit{where}\quad K_\gamma:=\sum_{i\in\Lambda}a(0,i)e^{\gamma d(0,i)}.\tag{2.10}$$

Proof For $\gamma=0$ this follows from [Swa09, Prop. 2.1]. To prove the statement for $\gamma>0$, let G be the generator of the (Λ,a,δ)-contact process as defined in (1.2). Then

$$\begin{aligned}Ge_\gamma(A)&=\sum_{i\in A}\sum_{j\notin A}a(i,j)e^{\gamma d(0,j)}-\delta\sum_{i\in A}e^{-\gamma d(0,i)}\\&\leq\sum_{i\in A}\sum_{j\in\Lambda}a(i,j)e^{\gamma(d(0,i)+d(i,j))}=K_\gamma e_\gamma(A),\end{aligned}\tag{2.11}$$

where we have used that $\sum_{j\in\Lambda}a(i,j)e^{\gamma d(i,j)}=\sum_{j\in\Lambda}a(0,i^{-1}j)e^{\gamma d(0,i^{-1}j)}=K_\gamma$ $(i\in\Lambda)$.

Set $\tau_N:=\inf\{t\geq 0:e_\gamma(\eta^A_t)\geq N\}$. Since the stopped process is a Markov process with finite state space, it follows by standard arguments from (2.11) that

$$\mathbb{E}\big[e_\gamma(\eta^A_{t\wedge\tau_N})\big]\leq e^{K_\gamma t}e_\gamma(A)\quad(t\geq 0,\ N\geq 1),\tag{2.12}$$

which in turn implies that $\mathbb{P}[e_\gamma(\eta^A_{t\wedge\tau_N})\geq N]\to 0$ as $N\to\infty$ and hence $\tau_N\to\infty$ a.s. Therefore, letting $N\to\infty$ in (2.12), we arrive at (2.10). ∎

Lemma 5 (Exponential growth rates) *Let $(\eta^{\{0\}}_t)_{t\geq 0}$ be the (Λ,a,δ)-contact process started in $\eta^{\{0\}}_0=\{0\}$, let d be a metric on Λ as in Lemma 3, and let e_γ be the function defined in (2.9). Then, for each $0\leq\gamma<\infty$, the limit*

$$r_\gamma=r_\gamma(\Lambda,a,\delta):=\lim_{t\to\infty}\tfrac{1}{t}\log\mathbb{E}\big[e_\gamma(\eta^{\{0\}}_t)\big]=\inf_{t>0}\tfrac{1}{t}\log\mathbb{E}\big[e_\gamma(\eta^{\{0\}}_t)\big]\tag{2.13}$$

exists. The function $\gamma\mapsto r_\gamma$ is nondecreasing and satisfies $-\delta\leq r_\gamma(\Lambda,a,\delta)\leq K_\gamma(\Lambda,a)$ $(\gamma\geq 0)$ where $K_\gamma=K_\gamma(\Lambda,a)$ is defined in (2.10).

Proof Note that $r_0(\Lambda, a, \delta) = r(\Lambda, a, \delta)$ is the exponential growth rate from (1.8). The statement for $\gamma = 0$ has been proved in [Swa09, Lemma 1.1 and formula (3.5)]. To prove the general statement, set $\pi_t^\gamma := \mathbb{E}\big[e_\gamma(\eta_t^{\{0\}})\big]$. Formula (2.13) will follow from standard facts [Lig99, Thm B.22] if we show that $t \mapsto \log \pi_t^\gamma$ is subadditive. Recalling the graphical representation of the (Λ, a, δ)-contact process, we observe that indeed

$$\begin{aligned} \pi_{s+t}^\gamma &= \sum_i \mathbb{P}[(0,0) \rightsquigarrow (i, s+t)] e^{\gamma d(0,i)} \\ &\leq \sum_{ij} \mathbb{P}[(0,0) \rightsquigarrow (j,s) \rightsquigarrow (i, s+t)] e^{\gamma(d(0,j)+d(j,i))} = \pi_s^\gamma \pi_t^\gamma, \end{aligned} \tag{2.14}$$

which implies the subadditivity of $t \mapsto \log \pi_t^\gamma$ and hence formula (2.13). Since $e_\gamma(A) \leq e_{\gamma'}(A)$ for all $\gamma \leq \gamma'$, it is clear that $\gamma \mapsto r_\gamma$ is nondecreasing. The fact that $-\delta \leq r_0$ has been proved in [Swa09, Lemma 1.1] while the estimate $r_\gamma \leq K_\gamma$ is immediate from Lemma 4. ∎

Lemma 6 (Right-continuity in γ) *The function $[0, \infty) \ni \gamma \mapsto r_\gamma$ defined in Lemma 5 is right-continuous.*

Proof It follows from (2.13) that for any $t_n \uparrow \infty$,

$$r_\gamma = \lim_{n\to\infty} \inf_{1 \leq k \leq n} \frac{1}{t_k} \log \mathbb{E}\big[e_\gamma(\eta_{t_k}^{\{0\}})\big]. \tag{2.15}$$

By dominated convergence and the finiteness of exponential moments (Lemma 4) we have that for each fixed $t > 0$, the function $\gamma \mapsto \frac{1}{t} \log \mathbb{E}[e_\gamma(\eta_t^{\{0\}})]$ is continuous. Therefore, being the decreasing limit of continuous functions, $\gamma \mapsto r_\gamma$ must be upper semi-continuous. Since $\gamma \mapsto r_\gamma$ is nondecreasing, this is equivalent to continuity from the right. ∎

2.2 Existence

In this section, we prove the existence part of Theorem 1. We start by recalling how homogeneous eigenmeasures with eigenvalue r are constructed in [Swa09]. For any (Λ, a, δ)-contact process, we can define homogeneous, locally finite measures μ_t on $\mathcal{P}_+ = \mathcal{P}_+(\Lambda)$ by

$$\mu_t := \sum_{i \in \Lambda} \mathbb{P}[\eta_t^{\{i\}} \in \cdot\,]\big|_{\mathcal{P}_+} \qquad (t \geq 0). \tag{2.16}$$

We can think of μ_t as the law of a contact process started with one infected site, distributed according to the counting measure on Λ. It is not hard to show (see [Swa09, formulas (3.8) and (3.20)]) that

$$\mu_t(\{A : 0 \in A\}) = \mathbb{E}\big[|\eta_t^{\{0\}}|\big] =: \pi_t. \tag{2.17}$$

Let $\hat{\mu}_\lambda$ be the Laplace transform of $(\mu_t)_{t \geq 0}$, i.e.,

$$\hat{\mu}_\lambda := \int_0^\infty \mu_t \, e^{-\lambda t} \mathrm{d}t \qquad (\lambda > r). \tag{2.18}$$

Then

$$\hat{\mu}_\lambda(\{A : 0 \in A\}) = \int_0^\infty \pi_t \, e^{-\lambda t} \mathrm{d}t =: \hat{\pi}_\lambda \qquad (\lambda > r), \tag{2.19}$$

which is finite for $\lambda > r$ by the definition of the exponential growth rate (see (1.8)). We cite the following result from [Swa09, Corollary 3.4].

Proposition 7 (Convergence to eigenmeasure) *The measures $\frac{1}{\hat\pi_\lambda}\hat\mu_\lambda$ $(\lambda > r)$ are relatively compact in the topology of vague convergence of locally finite measures on $\mathcal{P}_+(\Lambda)$, and each subsequential limit as $\lambda \downarrow r$ is a homogeneous eigenmeasure of the (Λ, a, δ)-contact process, with eigenvalue $r(\Lambda, a, \delta)$.*

The next lemma gives a uniform estimate on expections of the functions $e_\gamma(A)$ defined in (2.9) under the measures $\frac{1}{\hat\pi_\lambda}\hat\mu_\lambda$. We note that although the bound in (2.20) holds regardless of the values of γ and $r = r(\Lambda, a, \delta)$, the right-hand side will usually be infinite, unless $r < 0$ and γ is small enough (see the proofs of Lemma 9 and Proposition 24).

Lemma 8 (Uniform exponential moment bound) *Let $\hat\mu_\lambda$ and $\hat\pi_\lambda$ be defined as in (2.18)–(2.19) and for $\gamma \geq 0$, let e_γ be the function defined in (2.9) in terms of a metric d satisfying (2.3). Then, for any (Λ, a, δ)-contact process with exponential growth rate $r = r(\Lambda, a, \delta)$,*

$$\limsup_{\lambda\downarrow r} \frac{1}{\hat\pi_\lambda} \int \hat\mu_\lambda(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A) \leq (|a| + \delta) \int_0^\infty e^{-rt}\mathrm{d}t\, \mathbb{E}\big[e_\gamma(\eta_t^{\{0\}})\big]^2. \tag{2.20}$$

Proof Fix $\gamma \geq 0$ and, to ease notation, set $\psi_\gamma(i,j) := e^{\gamma d(i,j)}$ $(i, j, k \in \Lambda)$. We observe that

$$\begin{aligned}
&\int \hat\mu_\lambda(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A) = \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \mathbb{E}\big[1_{\{0\in\eta_t^{\{i\}}\}}1_{\{j\in\eta_t^{\{i\}}\}}\psi_\gamma(0,j)\big] \\
&= \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \mathbb{E}\big[1_{\{i^{-1}\in\eta_t^{\{0\}}\}}1_{\{i^{-1}j\in\eta_t^{\{0\}}\}}\psi_\gamma(i^{-1}, i^{-1}j)\big] \\
&= \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_t^{\{0\}},\ j \in \eta_t^{\{0\}}\big].
\end{aligned} \tag{2.21}$$

Set $f_i(A) := 1_{\{i\in A\}}$. Then

$$\mathbb{P}\big[i \in \eta_t^{\{0\}},\ j \in \eta_t^{\{0\}}\big] = \mathbb{E}\big[f_i(\eta_t^{\{0\}})\big]\,\mathbb{E}\big[f_j(\eta_t^{\{0\}})\big] + \mathrm{Cov}\big(f_i(\eta_t^{\{0\}}), f_j(\eta_t^{\{0\}})\big). \tag{2.22}$$

By a standard covariance formula (see [Swa09, Prop. 2.2]), for any functions f, g of polynomial growth (as in (2.49) below), one has

$$\mathrm{Cov}\big(f(\eta_t^{\{0\}}), g(\eta_t^{\{0\}})\big) = 2\int_0^t \mathbb{E}\big[\Gamma(P_sf, P_sg)(\eta_{t-s}^{\{0\}})\big]\mathrm{d}s \qquad (t \geq 0), \tag{2.23}$$

where $(P_t)_{t\geq0}$ denotes the semigroup of the (Λ, a, δ)-contact process and $\Gamma(f, g) = \frac{1}{2}(G(fg) - fGg - gGf)$, with G as in (1.2). A little calculation (see [Swa09, formula (4.6)]) shows that

$$\begin{aligned}
2\Gamma(P_sf, P_sg)(A) = &\sum_{k\in A}\sum_{l\notin A} a(k,l)\big(P_sf(A\cup\{l\}) - P_sf(A)\big)\big(P_sg(A\cup\{l\}) - P_sg(A)\big) \\
&+\delta\sum_{k\in A}\big(P_sf(A\backslash\{k\}) - P_sf(A)\big)\big(P_sg(A\backslash\{k\}) - P_sg(A)\big).
\end{aligned} \tag{2.24}$$

Applying (2.24) to the functions $f = f_i$, $g = f_j$, using the fact that, by the graphical representation,

$$\big|P_sf_i(A\cup\{l\}) - P_sf_i(A)\big| = \big|\mathbb{P}\big[i \in \eta_s^{A\cup\{l\}}\big] - \mathbb{P}\big[i \in \eta_s^A\big]\big| \leq \mathbb{P}\big[i \in \eta_s^{\{l\}}\big], \tag{2.25}$$

we find that

$$2\big|\Gamma(P_sf_i,P_sf_j)(A)\big| \le \sum_{k\in A}\sum_{l\notin A} a(k,l)\mathbb{P}\big[i\in\eta_s^{\{l\}}\big]\mathbb{P}\big[j\in\eta_s^{\{l\}}\big] + \delta\sum_{k\in A}\mathbb{P}\big[i\in\eta_s^{\{k\}}\big]\mathbb{P}\big[j\in\eta_s^{\{k\}}\big], \tag{2.26}$$

which by (2.23) implies that

$$\begin{aligned} &\big|\mathrm{Cov}\big(f_i(\eta_t^{\{0\}}), f_j(\eta_t^{\{0\}})\big)\big| \\ &\quad\le \int_0^t \sum_{k,l} a(k,l)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}},\ l\notin\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{l\}}\big]\mathbb{P}\big[j\in\eta_s^{\{l\}}\big]\mathrm{d}s \\ &\qquad+\delta\int_0^t\sum_k \mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{k\}}\big]\mathbb{P}\big[j\in\eta_s^{\{k\}}\big]\mathrm{d}s. \end{aligned} \tag{2.27}$$

Inserting this into (2.22), we obtain for the quantity in (2.21) the estimate

$$\begin{aligned} &\int_0^\infty e^{-\lambda t}\mathrm{d}t\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}},\ j\in\eta_t^{\{0\}}\big] \\ &\quad\le \int_0^\infty e^{-\lambda t}\mathrm{d}t\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big] \\ &\qquad+\int_0^\infty e^{-\lambda t}\mathrm{d}t\int_0^t\mathrm{d}s\sum_{i,j,k,l}\psi_\gamma(i,j)a(k,l)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}},\ l\notin\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{l\}}\big]\mathbb{P}\big[j\in\eta_s^{\{l\}}\big] \\ &\qquad+\delta\int_0^\infty e^{-\lambda t}\mathrm{d}t\int_0^t\mathrm{d}s\sum_{i,j,k}\psi_\gamma(i,j)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{k\}}\big]\mathbb{P}\big[j\in\eta_s^{\{k\}}\big]. \end{aligned} \tag{2.28}$$

Here

$$\begin{aligned} &\sum_{i,j,k}\psi_\gamma(i,j)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{k\}}\big]\mathbb{P}\big[j\in\eta_s^{\{k\}}\big] \\ &\quad=\sum_{i,j,k}\psi_\gamma(k^{-1}i,k^{-1}j)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[k^{-1}i\in\eta_s^{\{0\}}\big]\mathbb{P}\big[k^{-1}j\in\eta_s^{\{0\}}\big] \\ &\quad=\Big(\sum_k\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\Big)\Big(\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_s^{\{0\}}\big]\mathbb{P}\big[j\in\eta_s^{\{0\}}\big]\Big) \\ &\quad=\mathbb{E}\big[|\eta_{t-s}^{\{0\}}|\big]\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_s^{\{0\}}\big]\mathbb{P}\big[j\in\eta_s^{\{0\}}\big] \end{aligned} \tag{2.29}$$

and similarly

$$\begin{aligned} &\sum_{i,j,k,l}\psi_\gamma(i,j)a(k,l)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}},\ l\notin\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{l\}}\big]\mathbb{P}\big[j\in\eta_s^{\{l\}}\big] \\ &\quad\le\sum_{i,j,k,l}\psi_\gamma(l^{-1}i,l^{-1}j)a(k,l)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[l^{-1}i\in\eta_s^{\{0\}}\big]\mathbb{P}\big[l^{-1}j\in\eta_s^{\{0\}}\big] \\ &\quad=\Big(\sum_{k,l}a(k,l)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\Big)\Big(\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_s^{\{0\}}\big]\mathbb{P}\big[j\in\eta_s^{\{0\}}\big]\Big) \\ &\quad=|a|\,\mathbb{E}\big[|\eta_{t-s}^{\{0\}}|\big]\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_s^{\{0\}}\big]\mathbb{P}\big[j\in\eta_s^{\{0\}}\big]. \end{aligned} \tag{2.30}$$

Inserting this into (2.28) yields

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}},\ j\in\eta_t^{\{0\}}\big] \\
&\quad\le \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big] \\
&\quad\quad +(|a|+\delta)\int_0^\infty e^{-\lambda t}\mathrm{d}t\int_0^t \mathrm{d}s\,\mathbb{E}\big[|\eta_{t-s}^{\{0\}}|\big]\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_s^{\{0\}}\big]\mathbb{P}\big[j\in\eta_s^{\{0\}}\big] \\
&\quad = \Big(1+(|a|+\delta)\int_0^\infty e^{-\lambda t}\mathrm{d}t\,\mathbb{E}\big[|\eta_t^{\{0\}}|\big]\Big)\Big(\int_0^\infty e^{-\lambda t}\mathrm{d}t\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big]\Big),
\end{aligned} \tag{2.31}$$

where in the last step we have changed the integration order on the set $\{(s,t) : 0\le s\le t\}$. Using the fact that $\psi_\gamma(i,j)=e^{\gamma d(i,j)}$ where d is a metric, we may further estimate the sum in the second factor on the right-hand side of (2.31) as

$$\begin{aligned}
&\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big] = \sum_{i,j}e^{\gamma d(i,j)}\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big] \\
&\le \quad \sum_{i,j}e^{\gamma(d(0,i)+d(0,j))}\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big] \\
&= \quad \big(\sum_i e^{\gamma d(0,i)}\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\big)^2 = \mathbb{E}\big[\sum_{i\in\eta_t^{\{0\}}}e^{\gamma d(0,i)}\big]^2.
\end{aligned} \tag{2.32}$$

Inserting this into (2.31) yields

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}},\ j\in\eta_t^{\{0\}}\big] \\
&\quad\le \Big(1+(|a|+\delta)\int_0^\infty e^{-\lambda t}\mathrm{d}t\,\mathbb{E}\big[|\eta_t^{\{0\}}|\big]\Big)\int_0^\infty e^{-\lambda t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta_t^{\{0\}})\big]^2.
\end{aligned} \tag{2.33}$$

We note that setting $\gamma=0$ in (2.13) shows that

$$e^{rt}\le\mathbb{E}\big[|\eta_t^{\{0\}}|\big]\qquad(t\ge 0), \tag{2.34}$$

and therefore

$$\lim_{\lambda\downarrow r}\hat\pi_\lambda=\lim_{\lambda\downarrow r}\int_0^\infty e^{-\lambda t}\mathrm{d}t\,\mathbb{E}\big[|\eta_t^{\{0\}}|\big]=\infty. \tag{2.35}$$

Using this and (2.33), we arrive at (2.20). ∎

As a simple application of Lemma 8, we can prove the following result.

Lemma 9 (Existence of an eigenmeasure on finite configurations) *Assume that the exponential growth rate $r=r(\Lambda,a,\delta)$ of the (Λ,a,δ)-contact process satisfies $r<0$. Then there exists a homogeneous eigenmeasure $\mathring\nu$ with eigenvalue r of the (Λ,a,δ)-contact process such that*

$$\int\mathring\nu(\mathrm{d}A)|A|1_{\{0\in A\}}<\infty. \tag{2.36}$$

Proof By Proposition 7, we can choose $\lambda_n \downarrow r$ such that the measures $\frac{1}{\hat\pi_{\lambda_n}}\hat\mu_{\lambda_n}$ converge vaguely to a homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue r. Setting $\gamma = 0$ in (2.20), we obtain the formula

$$\limsup_{\lambda\downarrow r}\frac{1}{\hat\pi_\lambda}\int\hat\mu_\lambda(\mathrm{d}A)1_{\{0\in A\}}|A| \le (|a|+\delta)\int_0^\infty e^{-rt}\mathrm{d}t\,\mathbb{E}\big[|\eta^{\{0\}}_t|\big]^2. \tag{2.37}$$

It follows from (1.8) that $\mathbb{E}[|\eta^{\{0\}}_t|] = e^{rt+o(t)}$ where $t\mapsto o(t)$ is a continuous function such that $o(t)/t\to 0$ as $t\to\infty$, hence provided $r<0$,

$$\int_0^\infty e^{-rt}\mathrm{d}t\,\mathbb{E}\big[|\eta^{\{0\}}_t|\big]^2 = \int_0^\infty e^{2rt-rt+o(t)}\mathrm{d}t < \infty \qquad (r<0). \tag{2.38}$$

Let Λ_k be finite sets such that $0\in\Lambda_k\subset\Lambda$ and $\Lambda_k\uparrow\Lambda$. It is easy to check that $A\mapsto f_k(A) := |A\cap\Lambda_k|1_{\{0\in A\}}$ is a continuous, compactly supported real function on $\mathcal{P}_+(\Lambda)$. Therefore, by the vague convergence of the $\frac{1}{\hat\pi_{\lambda_n}}\hat\mu_{\lambda_n}$ to $\mathring{\nu}$,

$$\int\mathring{\nu}(\mathrm{d}A)f_k(A) = \lim_{n\to\infty}\frac{1}{\hat\pi_{\lambda_n}}\int\hat\mu_{\lambda_n}(\mathrm{d}A)f_k(A) \le \liminf_{n\to\infty}\frac{1}{\hat\pi_{\lambda_n}}\int\hat\mu_{\lambda_n}(\mathrm{d}A)|A|1_{\{0\in A\}}. \tag{2.39}$$

Letting $k\uparrow\infty$, using the fact that the right-hand side of (2.37) is finite by (2.38), we arrive at (2.36). ∎

2.3 Homogeneous laws on finite sets

Lemma 9 shows that if $r(\Lambda,a,\delta)<0$, then the (Λ,a,δ)-contact process has a homogeneous eigenmeasure that is concentrated on the set $\mathcal{P}_{\rm fin}=\mathcal{P}_{\rm fin}(\Lambda)$ of all finite subsets of Λ. In this section, we give a simple, but useful representation for any locally finite, homogeneous measure on $\mathcal{P}_+$ that is concentrated on $\mathcal{P}_{\rm fin}$.

We will need to consider finite sets 'modulo shifts'. To this aim, we define an equivalence relation on $\mathcal{P}_{\rm fin}$ by

$$A\sim B \quad\text{iff}\quad A=iB \quad\text{for some } i\in\Lambda, \tag{2.40}$$

and we let $\tilde{\mathcal{P}}_{\rm fin} := \{\tilde A : A\in\mathcal{P}_{\rm fin}\}$ with $\tilde A := \{iA : i\in\Lambda\}$ denote the set of equivalence classes. Below, we set $\mathcal{P}_{{\rm fin},+} := \mathcal{P}_+\cap\mathcal{P}_{\rm fin}$.

Lemma 10 (Homogeneous measures on the finite sets) *Let Δ be a $\mathcal{P}_{{\rm fin},+}(\Lambda)$-valued random variable such that $\mathbb{E}[|\Delta|]<\infty$ and let $c\ge 0$. Then*

$$\mu := c\sum_{i\in\Lambda}\mathbb{P}\big[i\Delta\in\cdot\,\big] \tag{2.41}$$

defines a locally finite homogeneous measure on $\mathcal{P}_+(\Lambda)$ that is concentrated on $\mathcal{P}_{\rm fin}(\Lambda)$. Conversely, if μ is a locally finite homogeneous measure on $\mathcal{P}_+(\Lambda)$ that is concentrated on $\mathcal{P}_{\rm fin}(\Lambda)$, then there exists a $\mathcal{P}_{{\rm fin},+}(\Lambda)$-valued random variable Δ such that $\mathbb{E}[|\Delta|]<\infty$ and μ is given by (2.41), where

$$c=c(\mu) := \int\mu(\mathrm{d}A)|A|^{-1}1_{\{0\in A\}}. \tag{2.42}$$

Moreover, if μ is nonzero, then the law of $\tilde\Delta$ is uniquely determined by μ.

Proof Formula (2.41) obviously defines a homogeneous measure on $\mathcal{P}_{\text{fin},+} = \mathcal{P}_{\text{fin},+}(\Lambda)$. Since

$$\mu(\{A : 0 \in A\}) = c\sum_i \mathbb{P}[0 \in i\Delta] = c\sum_i \mathbb{P}[i^{-1} \in \Delta] = cE\big[|\Delta|\big] < \infty, \tag{2.43}$$

it follows from [Swa09, Lemma 3.1] that μ is locally finite. If μ is given by (2.41), then

$$\begin{aligned} \int \mu(\mathrm{d}A)|A|^{-1}1_{\{0\in A\}} &= c\sum_{i\in\Lambda} \mathbb{E}\big[|i\Delta|^{-1}1_{\{0\in i\Delta\}}\big] \\ &= c\mathbb{E}\big[|\Delta|^{-1}\big(\sum_{i\in\Lambda} 1_{\{i^{-1}\in\Delta\}}\big)\big] = c\mathbb{E}\big[|\Delta|^{-1}|\Delta|\big] = c, \end{aligned} \tag{2.44}$$

which shows that c must be given by (2.42). To see that every locally finite homogeneous measure μ on $\mathcal{P}_{\text{fin},+}$ can be written in the form (2.41), assume without loss of generality that μ is nonzero and define a probability law ρ on $\mathcal{P}_0 := \{A \in \mathcal{P}_{\text{fin}} : 0 \in A\}$ by

$$\rho(\{A\}) := c^{-1}\mu(\{A\})|A|^{-1}1_{\{0\in A\}}, \tag{2.45}$$

where $c = c(\mu)$ is given by (2.42). Let Δ be a random variable with law ρ. Then

$$\mathbb{E}\big[|\Delta|\big] = \sum_{A\in\mathcal{P}_0} \rho(\{A\})|A| = c^{-1}\sum_{A\in\mathcal{P}_0} \mu(\{A\}) = c^{-1}\mu(\{A : 0 \in A\}) < \infty \tag{2.46}$$

by the local finiteness of μ. We claim that μ is given by (2.41). To check this, we calculate, for $A \in \mathcal{P}_{\text{fin},+}$:

$$\begin{aligned} &c\sum_{i\in\Lambda} \mathbb{P}\big[i\Delta = A\big] = c\sum_{i\in\Lambda} \mathbb{P}\big[\Delta = i^{-1}A\big] = c\sum_{i\in\Lambda} \rho(\{i^{-1}A\}) \\ &\quad= \sum_{i\in\Lambda} \mu(\{i^{-1}A\})|i^{-1}A|^{-1}1_{\{0\in i^{-1}A\}} = \mu(\{A\})|A|^{-1}\sum_{i\in\Lambda} 1_{\{i\in A\}} = \mu(\{A\}), \end{aligned} \tag{2.47}$$

where we have used the homogeneity of μ. Since

$$\begin{aligned} \mu(\{A\}) &= c\sum_{i\in\Lambda} \mathbb{P}[i\Delta = A] = c\,m(A)\mathbb{P}[\tilde\Delta = \tilde A] \\ \text{where}\quad m(A) &:= |\{i \in \Lambda : iA = A\}| \qquad \big(A \in \mathcal{P}_{\text{fin},+}\big), \end{aligned} \tag{2.48}$$

the law of $\tilde\Delta$ is uniquely determined by μ. Note that $m(A) = 1$ for all $A \in \mathcal{P}_{\text{fin},+}$ if Λ is infinite, which is usually our main case of interest. ∎

2.4 A transformed Markov process

Let $(\eta_t)_{t\geq 0}$ be a (Λ, a, δ)-contact process and let μ be a homogeneous eigenmeasure of the dual $(\Lambda, a^\dagger, \delta)$-contact process, with eigenvalue λ. In this section, we show how such an eigenmeasure μ can be used to define a Markov process $(\xi_t)_{t\geq 0}$ taking values in the space $\mathcal{P}_{\text{fin},+}(\Lambda)$ of finite, nonempty subsets of Λ, that is a 'compensated' h-transform of $(\eta_t)_{t\geq 0}$. If the $(\Lambda, a^\dagger, \delta)$-contact process has a nontrivial upper invariant law $\overline{\nu}^\dagger$, then setting $\mu = \overline{\nu}^\dagger$ one has that $(\xi_t)_{t\geq 0}$ is the (Λ, a, δ)-contact process conditioned on survival. As we will see, a similar interpretation is possible if $r(\Lambda, a, \delta) < 0$ and μ is the (unique) homogeneous eigenmeasure of the $(\Lambda, a^\dagger, \delta)$-contact process.

Let

$$\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda)) := \{f : \mathcal{P}_{\rm fin}(\Lambda) \to \mathbb{R} : |f(A)| \le K|A|^k + M \text{ for some } K, M, k \ge 0\}. \tag{2.49}$$

denote the class of real functions on $\mathcal{P}_{\rm fin} = \mathcal{P}_{\rm fin}(\Lambda)$ of polynomial growth. It has been shown in [Swa09, Prop. 2.1] that the operator G maps the space $\mathcal{S}(\mathcal{P}_{\rm fin})$ into itself and for each $f \in \mathcal{S}(\mathcal{P}_{\rm fin})$ and $A \in \mathcal{P}_{\rm fin}$, the process

$$M_t := f(\eta^A_t) - \int_0^t Gf(\eta^A_s)\mathrm{d}s \qquad (t \ge 0) \tag{2.50}$$

is a martingale with respect to the filtration generated by η^A.

We say that a function $f : \mathcal{P}_{\rm fin} \to \mathbb{R}$ is *shift-invariant* if $f(iA) = f(A)$ for all $i \in \Lambda$, *monotone* if $A \subset B$ implies $f(A) \le f(B)$, and *subadditive* if $f(A \cup B) \le f(A) + f(B)$, for all $A, B \in \mathcal{P}_{\rm fin}$. Recall the definition of the generator G of the (Λ, a, δ)-contact process from (1.2). We cite the following lemma from [Swa09, Lemma 3.5].

Lemma 11 (Eigenmeasures and harmonic functions) *If μ is a homogeneous eigenmeasure with eigenvalue λ of the $(\Lambda, a^\dagger, \delta)$-contact process, then*

$$h(A) := \int \mu(\mathrm{d}B)1_{\{A\cap B\neq\emptyset\}} \qquad (A \in \mathcal{P}_{\rm fin}) \tag{2.51}$$

defines a shift-invariant, monotone, subadditive function such that $h(\emptyset) = 0$, $h(A) > 0$ for any $\emptyset \neq A \in \mathcal{P}_{\rm fin}$, $h \in \mathcal{S}(\mathcal{P}_{\rm fin})$, and $Gh = \lambda h$.

The 'compensated h-transform' of a Markov generator G is defined in [FS02, Lemma 3] as $G^h f := (G(hf) - (Gh)f)/h$. We want to apply this to the generator G from (1.2) and the function h from Lemma 11. Our next result shows that this indeed yields a well-defined Markov generator. Below, $\mathcal{S}(\mathcal{P}_{\rm fin,+})$ denotes the space of real functions on $\mathcal{P}_{\rm fin,+}$ of polynomial growth, defined analogously to (2.49).

Proposition 12 (Transformed generator) *Let G be the generator of a (Λ, a, δ)-contact process as in (1.2) and let h be given by (2.51). Then there exists a unique generator G^h of a Markov process $(\xi_t)_{t\ge0}$ in $\mathcal{P}_{\rm fin,+}$ such that*

$$G^h f\big|_{\mathcal{P}_{\rm fin,+}}(A) = G(hf)(A)/h(A) - \lambda f(A) \qquad \big(A \in \mathcal{P}_{\rm fin,+},\ f \in \mathcal{S}(\mathcal{P}_{\rm fin})\big), \tag{2.52}$$

where $f|_{\mathcal{P}_{\rm fin,+}}$ denotes the restriction of f to $\mathcal{P}_{\rm fin,+}$. One has

$$\begin{aligned} G^h f(A) :=& \sum_{ij} \frac{h(A\cup\{j\})}{h(A)} a(i,j)1_{\{i\in A\}}1_{\{j\notin A\}}\{f(A\cup\{j\}) - f(A)\} \\ &+\delta \sum_i \frac{h(A\backslash\{i\})}{h(A)} 1_{\{i\in A\}}\{f(A\backslash\{i\}) - f(A)\} \end{aligned} \tag{2.53}$$

$(A \in \mathcal{P}_{\rm fin,+},\ f \in \mathcal{S}(\mathcal{P}_{\rm fin,+}))$. If $(\xi^A_t)_{t\ge0}$ denotes the process with generator G^h started in $\xi^A_0 = A$, then

$$\mathbb{P}\big[(\xi^A_s)_{0\le s\le t} \in \mathrm{d}w\big] = e^{-\lambda t}\,\frac{h(w_t)}{h(A)}\,\mathbb{P}\big[(\eta^A_s)_{0\le s\le t} \in \mathrm{d}w\big] \qquad (t \ge 0). \tag{2.54}$$

Proof Let

$$P_t(A,B) := \mathbb{P}[\eta_t^A = B] \qquad \big(t \geq 0,\ A,B \in \mathcal{P}_{\rm fin}\big) \tag{2.55}$$

denote the transition probabilities of the (Λ, a, δ)-contact process and let $(P_t)_{t\geq 0}$ denote the associated semigroup defined as $P_t f(A) = \sum_B P_t(A,B) f(B)$. It follows from [Swa09, Prop. 2.1] that P_t maps the space $\mathcal{S}(\mathcal{P}_{\rm fin})$ into itself. Set

$$P_t^h(A,B) := e^{-\lambda t}\,\frac{h(B)}{h(A)}\,P_t(A,B) \qquad \big(t \geq 0,\ A,B \in \mathcal{P}_{\rm fin,+}\big), \tag{2.56}$$

which is well-defined since $h(A) > 0$ for $A \neq \emptyset$. Using (2.51) and contact process duality, it is easy to show (see the proof of [Swa09, Lemma 3.5]) that $P_t h = e^{\lambda t} h$ $(t \geq 0)$ hence P_t^h is a transition probability on $\mathcal{P}_{\rm fin,+}$ and its associated semigroup satisfies

$$h|_{\mathcal{P}_{\rm fin,+}} P_t^h(f|_{\mathcal{P}_{\rm fin,+}}) = \big(e^{-\lambda t} P_t(hf)\big)\big|_{\mathcal{P}_{\rm fin,+}} \qquad \big(t \geq 0,\ f \in \mathcal{S}(\mathcal{P}_{\rm fin})\big). \tag{2.57}$$

It is now straightforward to check that for each $A \in \mathcal{P}_{\rm fin,+}$, formula (2.54) consistently defines a probability law on the space of cadlag paths $w : [0,\infty) \to \mathcal{P}_{\rm fin,+}$ and that this is the law of the Markov process with semigroup $(P_t^h)_{t\geq 0}$ and initial state A. We need to show that this Markov process has a generator G^h given by (2.52) and (2.53).

It has been proved in [Swa09, formula (2.25)] that for each $g \in \mathcal{S}(\mathcal{P}_{\rm fin})$, one has

$$\lim_{t\to 0} t^{-1}\big(P_t g - g\big)(A) = Gg(A) \qquad \big(A \in \mathcal{P}_{\rm fin}\big). \tag{2.58}$$

Applying this to $g = hf$ in (2.57) we see that

$$G^h f|_{\mathcal{P}_{\rm fin,+}}(A) := \lim_{t\to 0} t^{-1}\big(P_t^h f|_{\mathcal{P}_{\rm fin,+}} - f|_{\mathcal{P}_{\rm fin,+}}\big)(A) = G(hf)(A)/h(A) - \lambda f(A) \tag{2.59}$$

$(A \in \mathcal{P}_{\rm fin,+},\ f \in \mathcal{S}(\mathcal{P}_{\rm fin}))$. We may write the operator G in the form

$$Gf(A) = \sum_{B\in\mathcal{P}_{\rm fin}} r(A,B)\big(f(B) - f(A)\big) \qquad \big(A \in \mathcal{P}_{\rm fin}\big), \tag{2.60}$$

where $r(A,B)$ denotes the rate of jumps from A to B. Since

$$\begin{aligned} G(hf)(A) &= \sum_B r(A,B)\big(h(B)f(B) - h(A)f(A)\big) \\ &= \sum_B r(A,B)h(B)\big(f(B) - f(A)\big) + \sum_B r(A,B)\big(h(B) - h(A)\big)f(A), \end{aligned} \tag{2.61}$$

we see from (2.59) and the fact that $Gh = \lambda h$ that

$$\begin{aligned} G^h f|_{\mathcal{P}_{\rm fin,+}}(A) &= \big(G(hf)(A) - (Gh)f(A)\big)/h(A) \\ &= \sum_{B\in\mathcal{P}_{\rm fin,+}} r^h(A,B)\big(f(B) - f(A)\big) \end{aligned} \tag{2.62}$$

$(A \in \mathcal{P}_{\rm fin,+},\ f \in \mathcal{S}(\mathcal{P}_{\rm fin}))$, where we have defined

$$r^h(A,B) = \frac{h(B)}{h(A)} r(A,B) \qquad \big(A,B \in \mathcal{P}_{\rm fin,+}\big). \tag{2.63}$$

This proves that the operator G^h, defined in (2.59), is given by (2.53). In particular, this shows that $(\xi_t)_{t\geq 0}$ jumps from A to B with rate $r^h(A,B)$. ∎

Lemma 13 (Martingale problem for transformed process) *The operator G^h from Proposition 12 maps the space $\mathcal{S}(\mathcal{P}_{\rm fin})$ into itself. If $(\xi^A_t)_{t\geq 0}$ is the process with generator G^h started in $\xi^A_0 = A \in \mathcal{P}_{\rm fin,+}$, then for each $f \in \mathcal{S}(\mathcal{P}_{\rm fin,+})$, the process*

$$M_t := f(\xi^A_t) - \int_0^t G^h f(\xi^A_s)\mathrm{d}s \qquad (t \geq 0) \tag{2.64}$$

is a martingale with respect to the filtration generated by $(\xi^A_t)_{t\geq 0}$. Moreover, setting $z^{\langle k\rangle} := \prod_{i=0}^{k-1}(z+i)$, one has

$$\mathbb{E}\big[|\xi^A_t|^{\langle k\rangle}\big] \leq |A|^{\langle k\rangle} e^{2k|a|t} \qquad (A \in \mathcal{P}_{\rm fin,+},\ k \geq 1,\ t \geq 0). \tag{2.65}$$

Proof Recall that the jump rates of the process with generator G^h are given by (2.63). Since by Lemma 11, the function h is monotone and subadditive, we have

$$\frac{h(A\backslash\{i\})}{h(A)} \leq 1 \quad \text{and} \quad \frac{h(A\cup\{j\})}{h(A)} \leq 1 + \frac{h(\{j\})}{h(A)} \leq 2, \tag{2.66}$$

hence $r^h(A,B) \leq 2r(A,B)$ for all $A \in \mathcal{P}_{\rm fin,+}$. Therefore, the proof of [Swa09, Prop. 2.1] carries over except for the constant in the exponent in (2.65) which follows from a somewhat more rough estimate than [Swa09, formula (2.14)]. ∎

2.5 The process conditioned on survival

We continue to consider (Λ,a,δ)-contact processes with exponential growth rate $r = r(\Lambda,a,\delta)$. In this section, we assume that $r < 0$. Theorem 0 (a) says that for any (Λ,a,δ)-contact process, $r(\Lambda,a,\delta) = r(\Lambda,a^\dagger,\delta)$. Therefore, by Lemma 9, there exist homogeneous eigenmeasure $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ of the (Λ,a,δ)- and $(\Lambda,a^\dagger,\delta)$-contact process, respectively, both with eigenvalue r, such that

$$\int \mathring{\nu}(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty \quad \text{and} \quad \int \mathring{\nu}^\dagger(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty. \tag{2.67}$$

We normalize $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ such that $\int \mathring{\nu}(\mathrm{d}A)1_{\{0\in A\}} = 1 = \int \mathring{\nu}^\dagger(\mathrm{d}A)1_{\{0\in A\}}$. For the moment, we do not know yet if $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ are unique. We fix any two such measures and define, in analogy with (2.51),

$$\left.\begin{aligned} h(A) &:= \int \mathring{\nu}^\dagger(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}, \\ h^\dagger(A) &:= \int \mathring{\nu}(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}, \end{aligned}\right\} \quad (B \in \mathcal{P}_{\rm fin}). \tag{2.68}$$

Then Lemma 11 tells us that $Gh = rh$ and $G^\dagger h^\dagger = rh^\dagger$, where G and $G^\dagger$ denote the generators of the (Λ,a,δ)- and $(\Lambda,a^\dagger,\delta)$-contact process, respectively. We will be interested in the (compensated) h-transformed (Λ,a,δ)-contact process with generator G^h defined in Proposition 12. Likewise, we will sometimes need the $h^\dagger$-transformed $(\Lambda,a^\dagger,\delta)$-contact process with generator $G^{\dagger\, h^\dagger}$.

We note that if μ is a homogeneous, locally finite measure on $\mathcal{P}_+$ such that (compare (2.67))

$$\int \mu(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty, \tag{2.69}$$

then μ is concentrated on $\mathcal{P}_{\rm fin}$ and the weighted measure $h\mu(\mathrm{d}A) := h(A)\mu(\mathrm{d}A)$ is a locally finite measure on $\mathcal{P}_+$. Indeed, since h is shift-invariant and subadditive by Lemma 11, it follows that $h(A) \leq h(\{0\})|A|$ and therefore $\int h(A)\mu(\mathrm{d}A)1_{\{0\in A\}} \leq h(\{0\})\int \mu(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty$, which by [Swa09, Lemma 3.1] implies that μ is locally finite.

The next lemma shows that h-transformation maps eigenmeasures into invariant measures.

Lemma 14 (Invariant laws of the transformed process) *Let μ be a homogeneous, locally finite measure on $\mathcal{P}_+(\Lambda)$ that satifies (2.69). Then μ is an eigenmeasure of the (Λ, a, δ)-contact process with eigenvalue $r = r(\Lambda, a, \delta)$ if and only if $h\mu$ is an invariant law of the h-transformed (Λ, a, δ)-contact process with generator G^h, where h is defined in (2.68).*

Proof The measure μ is an eigenmeasure of the (Λ, a, δ)-contact process with eigenvalue r if and only if

$$\sum_{A\in\mathcal{P}_{\rm fin,+}} \mu(\{A\})P_t(A,B) = e^{rt}\mu(\{B\}), \tag{2.70}$$

which by (2.56) (which follows from (2.54)) and the fact that $h(A) > 0$ for $A \in \mathcal{P}_{\rm fin,+}$ is equivalent to

$$\sum_{A\in\mathcal{P}_{\rm fin,+}} \mu(\{A\})h(A)P^h_t(A,B) = h(B)\mu(\{B\}), \tag{2.71}$$

i.e., $h\mu$ is an invariant law of the h-transformed (Λ, a, δ)-contact process. ∎

Recall from Section 2.3 that for any $A \in \mathcal{P}_{\rm fin}$, we let $\tilde{A} \in \tilde{\mathcal{P}}_{\rm fin}$ denote the set A 'modulo shifts'. It follows from the shift-invariance of our infection rates (formula (1.1) (i)) and the spatial homogeneity of μ that if $(\xi_t)_{t\geq 0}$ is the Markov process with generator G^h, then the $\tilde{\mathcal{P}}_{\rm fin,+}(\Lambda)$-valued process $(\tilde{\xi}_t)_{t\geq 0}$ is also a Markov process. We call this the *h-transformed (Λ, a, δ)-contact process modulo shifts.*

By the remarks below (2.69), the weighted measures $h\mathring{\nu}$ and $h^\dagger\mathring{\nu}^\dagger$ are locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\rm fin}$. Therefore, by Lemma 10, there exist $\mathcal{P}_{\rm fin,+}$-valued random variables ξ_∞ and $\xi^\dagger_\infty$ such that

$$h\mathring{\nu} = c(h\mathring{\nu})\sum_{i\in\Lambda}\mathbb{P}\big[i\xi_\infty \in \cdot\,\big] \quad \text{and} \quad h^\dagger\mathring{\nu}^\dagger = c(h^\dagger\mathring{\nu}^\dagger)\sum_{i\in\Lambda}\mathbb{P}\big[i\xi^\dagger_\infty \in \cdot\,\big]. \tag{2.72}$$

The following simple observation will be very useful. Below, we use the word 'irreducible' in the sense as defined in Section 1.2, i.e., for each two states in the state space there is a positive probability of going from one to the other.

Proposition 15 (Positive recurrence) *Assume that $r(\Lambda, a, \delta) < 0$ and let h be as defined above. Assume that a satisfies the irreducibility condition (1.3). Then the h-transformed (Λ, a, δ)-contact process modulo shifts is a positively recurrent, irreducible Markov process with countable state space $\tilde{\mathcal{P}}_{\rm fin,+}(\Lambda)$, and $\mathbb{P}[\tilde{\xi}_\infty \in \cdot\,]$ is its unique invariant law.*

Proof By Lemma 14, for any $c > 0$, the measure

$$\mu := c\sum_{i\in\Lambda}\mathbb{P}\big[i\xi_\infty \in \cdot\,\big] \tag{2.73}$$

is an invariant law for the h-transformed (Λ, a, δ)-contact process. It is sort of clear that this implies that $\mathbb{P}[\tilde{\xi}_\infty \in \cdot\,]$ is an invariant law for the h-transformed (Λ, a, δ)-contact process modulo shifts, but for completeness, we prove this formally.

If Λ is finite, then we can without loss of generality assume that μ is a probability measure and that the law of ξ_∞ is shift-invariant, hence $\mu = \mathbb{P}[\xi_\infty \in \cdot\,]$. Now if $(\xi_t)_{t\geq 0}$ is the stationary h-transformed (Λ, a, δ)-contact process started in $\mathbb{P}[\xi_0 \in \cdot\,] = \mu$, then $(\tilde{\xi}_t)_{t\geq 0}$ is a stationary process with law $\mathbb{P}[\tilde{\xi}_\infty \in \cdot\,]$, hence the latter is an invariant law for the h-transformed (Λ, a, δ)-contact process modulo shifts.

If Λ is infinite, then we can without loss of generality assume that $c = 1$. The transition probabilities of the h-transformed (Λ, a, δ)-contact process modulo shifts are given by

$$\tilde{P}^h_t(\tilde{A}, \tilde{B}) = \sum_{i\in\Lambda} P^h_t(A, iB) \qquad \big(t \geq 0,\ A, B \in \mathcal{P}_{\text{fin},+}\big). \tag{2.74}$$

Therefore, by (2.48),

$$\begin{aligned}\sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\text{fin},+}} \mathbb{P}[\tilde{\xi}_\infty = \tilde{A}]\tilde{P}^h_t(\tilde{A}, \tilde{B}) &= \sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\text{fin},+}} \sum_{i\in\Lambda} \mu(\{A\}) P^h_t(A, iB)\\ = \sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\text{fin},+}} \sum_{i\in\Lambda} \mu(\{i^{-1}A\}) P^h_t(i^{-1}A, B) &= \sum_{A\in\mathcal{P}_{\text{fin},+}} \mu(\{A\}) P^h_t(A, B) = \mu(\{B\}) = \mathbb{P}[\tilde{\xi}_\infty = \tilde{B}],\end{aligned} \tag{2.75}$$

which shows that $\mathbb{P}[\tilde{\xi}_\infty \in \cdot\,]$ is an invariant law for the h-transformed (Λ, a, δ)-contact process modulo shifts.

Since the h-transformed (Λ, a, δ)-contact process modulo shifts has an invariant law, positive recurrence and the other statements of the proposition will follow once we prove irreducibility. It follows from (2.56) and the fact that $h(A) > 0$ for all $A \neq \emptyset$ that $P^h_t(A, B) > 0$ if and only if $P_t(A, B) > 0$ $(A, B \in \mathcal{P}_{\text{fin},+}(\Lambda))$. Our assumption that $r < 0$ entails that $\delta > 0$. Therefore, since it may happen that all sites except one recover, for each finite set A and $i \in A$ we have $P^h_t(A, \{i\}) > 0$. On the other hand, by (1.3), for each finite set A there exists an $i \in \Lambda$ such that all sites in A can be infected from i, hence $\tilde{P}^h_t(\{i\}, A) > 0$. This proves the irreducibility of the h-transformed (Λ, a, δ)-contact process modulo shifts. ∎

2.6 Uniqueness and convergence

In this section we prove Theorem 1. To prepare for this, for any measures μ, ν on $\mathcal{P}_+$, we let $\psi(\mu, \nu)$ denote the restriction to $\mathcal{P}_+$ of the image of the product measure $\mu \otimes \nu$ under the map $(A, B) \mapsto A \cap B$. Note that

$$\int \psi(\mu, \nu)(\mathrm{d}C) f(C) := \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B) f\big(A \cap B\big) \tag{2.76}$$

for any bounded measurable $f : \mathcal{P}_+ \to \mathbb{R}$. We claim that $\psi(\mu, \nu)$ is locally finite if μ and ν are. Indeed, this follows from [Swa09, Lemma 3.1] and the fact that

$$\begin{aligned}\int \psi(\mu, \nu)(\mathrm{d}C) 1_{\{i\in C\}} &= \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B) 1_{\{i\in A\cap B\}}\\ &= \Big(\int \mu(\mathrm{d}A) 1_{\{i\in A\}}\Big)\Big(\int \nu(\mathrm{d}B) 1_{\{i\in B\}}\Big) < \infty \quad (i \in \Lambda).\end{aligned} \tag{2.77}$$

We call $\psi(\mu, \nu)$ the *intersection measure* associated with μ and ν. Note that if μ, ν are probability measures, then $\psi(\mu, \nu)$ is the law of the intersection of two independent random sets with laws μ and ν, respectively. The following simple lemma will be useful later on.

Lemma 16 (Continuity of intersection measure) *Let μ_n, μ, ν_n, ν be locally finite measures on $\mathcal{P}_+$ and let $\Rightarrow$ denote vague convergence of locally finite measures on $\mathcal{P}_+$. Then $\mu_n \Rightarrow \mu$ and $\nu_n \Rightarrow \nu$ imply that $\psi(\mu_n, \nu_n) \Rightarrow \psi(\mu, \nu)$.*

Proof By [Swa09, Lemma 3.2], the vague convergence $\psi(\mu_n, \nu_n) \Rightarrow \psi(\mu, \nu)$ is equivalent to

$$\int \psi(\mu_n, \nu_n)(\mathrm{d}C) 1_{\{C\cap D\neq\emptyset\}} \underset{n\to\infty}{\longrightarrow} \int \psi(\mu, \nu)(\mathrm{d}C) 1_{\{C\cap D\neq\emptyset\}} \qquad (D \in \mathcal{P}_{\mathrm{fin},+}). \tag{2.78}$$

Since

$$1_{\{C\cap D\neq\emptyset\}} = 1 - \prod_{i\in D} 1_{\{i\notin C\}} = 1 - \prod_{i\in D}(1 - 1_{\{i\in C\}}) = \sum_{\substack{D'\subset D\\ D'\neq\emptyset}} (-1)^{|D'|+1} \prod_{i\in D'} 1_{\{i\in C\}}, \tag{2.79}$$

formula (2.78) is equivalent to

$$\int \psi(\mu_n, \nu_n)(\mathrm{d}C) 1_{\{D\subset C\}} \underset{n\to\infty}{\longrightarrow} \int \psi(\mu, \nu)(\mathrm{d}C) 1_{\{D\subset C\}} \qquad (D \in \mathcal{P}_{\mathrm{fin},+}). \tag{2.80}$$

Now

$$\begin{aligned}\int \psi(\mu_n, \nu_n)(\mathrm{d}C) 1_{\{D\subset C\}} &= \int \mu_n(\mathrm{d}A) \int \nu_n(\mathrm{d}B) 1_{\{D\subset(A\cap B)\}} \\ &= \Big(\int \mu_n(\mathrm{d}A) 1_{\{D\subset A\}}\Big)\Big(\int \nu_n(\mathrm{d}B) 1_{\{D\subset B\}}\Big),\end{aligned} \tag{2.81}$$

which, by our assumptions that $\mu_n \Rightarrow \mu$ and $\nu_n \Rightarrow \nu$, converges to the analogue formula with μ_n, ν_n replaced by μ, ν. ∎

Lemma 17 (Intersection and weighted measure) *Let μ, ν be homogeneous locally finite measures on $\mathcal{P}_+$, assume that $\int \mu(\mathrm{d}A)|A| 1_{\{0\in A\}} < \infty$, and define $h : \mathcal{P}_{\mathrm{fin}} \to \mathbb{R}$ by $h(A) := \int \nu(\mathrm{d}B) 1_{\{A\cap B\neq\emptyset\}}$. Then both $h\mu$ and $\psi(\mu, \nu)$ are homogeneous locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\mathrm{fin}}$, and one has*

$$c\big(\psi(\mu, \nu)\big) = c(h\mu), \tag{2.82}$$

where $c(\psi(\mu, \nu))$ and $c(h\mu)$ are defined in (2.42).

Proof Since $\int \mu(\mathrm{d}A)|A| 1_{\{0\in A\}} < \infty$, the measure μ and therefore also $\psi(\mu, \nu)$ are concentrated on $\mathcal{P}_{\mathrm{fin}}$. It follows from the way h is defined that h is shift-invariant and subadditive, hence $h(A) \leq h(\{0\})|A|$, from which in the same way as below (2.69) we see that $h\mu$ is locally finite. By Lemma 10, there exists a $\mathcal{P}_{\mathrm{fin},+}$-valued random variable Δ with $\mathbb{E}[|\Delta|] < \infty$ such that

$\mu = c(\mu)\sum_{i\in\Lambda}\mathbb{P}[i\Delta \in \cdot\,]$. Now

$$\begin{aligned}
& c\big(\psi(\mu,\nu)\big) = \int \psi(\mu,\nu)(\mathrm{d}C)|C|^{-1}1_{\{0\in C\}} = \int \mu(\mathrm{d}A)\int \nu(\mathrm{d}B)\,|A\cap B|^{-1}1_{\{0\in A\cap B\}} \\
& = c(\mu)\sum_{i\in\Lambda}\int \nu(\mathrm{d}B)\,\mathbb{E}\big[|i\Delta\cap B|^{-1}1_{\{0\in i\Delta\cap B\}}\big] = c(\mu)\sum_{i\in\Lambda}\int \nu(\mathrm{d}B)\,\mathbb{E}\big[|i\Delta\cap iB|^{-1}1_{\{0\in i\Delta\cap iB\}}\big] \\
& = c(\mu)\int \nu(\mathrm{d}B)\,\mathbb{E}\big[|\Delta\cap B|^{-1}\sum_{i\in\Lambda}1_{\{i^{-1}\in\Delta\cap B\}}\big] = c(\mu)\int \nu(\mathrm{d}B)\,\mathbb{P}[\Delta\cap B\neq\emptyset] \\
& = c(\mu)\int \nu(\mathrm{d}B)\,\mathbb{E}\big[1_{\{\Delta\cap B\neq\emptyset\}}|\Delta|^{-1}\sum_{i\in\Lambda}1_{\{i^{-1}\in\Delta\}}\big] \\
& = c(\mu)\sum_{i\in\Lambda}\int \nu(\mathrm{d}B)\,\mathbb{E}\big[1_{\{i\Delta\cap iB\neq\emptyset\}}|i\Delta|^{-1}1_{\{0\in i\Delta\}}\big] \\
& = c(\mu)\sum_{i\in\Lambda}\int \nu(\mathrm{d}B)\,\mathbb{E}\big[1_{\{i\Delta\cap B\neq\emptyset\}}|i\Delta|^{-1}1_{\{0\in i\Delta\}}\big] = \int \mu(\mathrm{d}A)\int \nu(\mathrm{d}B)\,1_{\{A\cap B\neq\emptyset\}}|A|^{-1}1_{\{0\in A\}} \\
& = \int h(A)\mu(\mathrm{d}A)\,|A|^{-1}1_{\{0\in A\}} = c(h\mu),
\end{aligned} \tag{2.83}$$

where we have used the homogeneity of ν. ∎

Proof of Theorem 1 The existence of $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ has already been proved in Lemma 9, so uniqueness will follow once we prove the convergence in (1.10). By duality (1.4) and formula (2.56), we observe that for any $B\in\mathcal{P}_{\mathrm{fin},+}$,

$$\begin{aligned}
& \int e^{-rt}\mu(\mathrm{d}A)\mathbb{P}\big[\eta^A_t\cap B\neq\emptyset\big] = \int e^{-rt}\mu(\mathrm{d}A)\mathbb{P}\big[A\cap \eta^{\dagger\,B}_t\neq\emptyset\big] \\
& \quad = \int e^{-rt}\mu(\mathrm{d}A)\sum_{B'}P^\dagger_t(B,B')1_{\{A\cap B'\neq\emptyset\}} = e^{-rt}\sum_{B'}P^\dagger_t(B,B')h'(B') \\
& \quad = h^\dagger(B)\sum_{B'}P^{\dagger\,h^\dagger}_t(B,B')h^\dagger(B')^{-1}h'(B'),
\end{aligned} \tag{2.84}$$

where $P^\dagger_t$ and $P^{\dagger\,h^\dagger}_t$ denote the transition probabilities of the $(\Lambda,a^\dagger,\delta)$-contact process and the $h^\dagger$-transformed $(\Lambda,a^\dagger,\delta)$-contact process, respectively, and we have defined

$$h'(B) := \int \mu(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} \qquad (B\in\mathcal{P}_{\mathrm{fin}}). \tag{2.85}$$

We claim that $h'/h^\dagger$ is a bounded function. To see this, note that by the fact that $\mathring{\nu}$ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and Lemma 10, there exists a $\mathcal{P}_{\mathrm{fin},+}$-valued random variable Δ such that $\mathring{\nu}$ can be written as in (2.41). Let κ be a Λ-valued random variable such that $\kappa\in\Delta$ a.s. Then

$$\begin{aligned}
h^\dagger(A) &= \int \mathring{\nu}(\mathrm{d}B)1_{\{A\cap B\neq\emptyset\}} = c(\mathring{\nu})\sum_{i\in\Lambda}\mathbb{P}[A\cap i\Delta\neq\emptyset] \\
&\geq c(\mathring{\nu})\sum_{i\in\Lambda}\mathbb{P}[A\cap\{i\kappa\}\neq\emptyset] = c(\mathring{\nu})\,\mathbb{E}\big[\sum_{i\in\Lambda}1_{\{i\kappa\in A\}}\big] = c(\mathring{\nu})|A|.
\end{aligned} \tag{2.86}$$

On the other hand, it is easy to see from (2.85) that h' is subadditive and shift-invariant, hence $h'(A)\leq h'(\{0\})|A|$ and therefore $h'(A)/h^\dagger(A)\leq h'(\{0\})/c(\mathring{\nu})$.

By Proposition 15, the $h^\dagger$-transformed $(\Lambda,a^\dagger,\delta)$-contact process modulo shifts is irreducible and positively recurrent. Using this and the fact that $h'/h^\dagger$ is a bounded shift-invariant function, we see from (2.84) and the definition of $h^\dagger$ in (2.68) that

$$\begin{aligned}\int e^{-rt}\mu(\mathrm{d}A)\mathbb{P}\big[\eta^A_t\cap B\neq\emptyset\big] &\underset{t\to\infty}{\longrightarrow} h^\dagger(B)\mathbb{E}\big[h'(\xi^\dagger_\infty)/h^\dagger(\xi^\dagger_\infty)\big]\\ &= \mathbb{E}\big[h'(\xi^\dagger_\infty)/h^\dagger(\xi^\dagger_\infty)\big]\int\mathring{\nu}(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}},\end{aligned}\tag{2.87}$$

where $\xi^\dagger_\infty$ is defined in (2.72). Since this holds for any $B\in\mathcal{P}_{\mathrm{fin},+}$, by [Swa09, Lemma 3.2] we conclude that

$$e^{-rt}\mu_t \underset{t\to\infty}{\Longrightarrow} c\mathring{\nu}\quad\text{where}\quad c:=\mathbb{E}\big[h'(\xi^\dagger_\infty)/h^\dagger(\xi^\dagger_\infty)\big]\tag{2.88}$$

and $\Rightarrow$ denotes vague convergence of locally finite measures on $\mathcal{P}_+$. To complete the proof, we must show that the constant c from (2.88) is given by formula (1.11). In light of (2.42) and (2.76), the right-hand side of (1.11) is equal to $c(\psi(\mu,\mathring{\nu}^\dagger))/c(\psi(\mathring{\nu},\mathring{\nu}^\dagger))$, which with the help of Lemma 17 can be further rewritten as

$$\begin{aligned}c(\psi(\mu,\mathring{\nu}^\dagger))/c(h^\dagger\mathring{\nu}^\dagger) &= c(h^\dagger\mathring{\nu}^\dagger)^{-1}\int\mu(\mathrm{d}A)\int\mathring{\nu}^\dagger(\mathrm{d}B)|A\cap B|^{-1}1_{\{0\in A\cap B\}}\\ &= c(h^\dagger\mathring{\nu}^\dagger)^{-1}\int\mu(\mathrm{d}A)\int\mathring{\nu}^\dagger(\mathrm{d}B)h^\dagger(B)h^\dagger(B)^{-1}|A\cap B|^{-1}1_{\{0\in A\cap B\}}\\ &= \sum_{i\in\Lambda}\int\mu(\mathrm{d}A)\mathbb{E}\big[h^\dagger(i\xi^\dagger_\infty)^{-1}|A\cap i\xi^\dagger_\infty|^{-1}1_{\{0\in A\cap i\xi^\dagger_\infty\}}\big]\\ &= \sum_{i\in\Lambda}\int\mu(\mathrm{d}A)\mathbb{E}\big[h^\dagger(i\xi^\dagger_\infty)^{-1}|iA\cap i\xi^\dagger_\infty|^{-1}1_{\{0\in iA\cap i\xi^\dagger_\infty\}}\big]\\ &= \int\mu(\mathrm{d}A)\mathbb{E}\big[h^\dagger(\xi^\dagger_\infty)^{-1}|A\cap\xi^\dagger_\infty|^{-1}\sum_{i\in\Lambda}1_{\{i^{-1}\in A\cap\xi^\dagger_\infty\}}\big]\\ &= \int\mu(\mathrm{d}A)\mathbb{E}\big[h^\dagger(\xi^\dagger_\infty)^{-1}1_{\{A\cap\xi^\dagger_\infty\neq\emptyset\}}\big] = \mathbb{E}\big[h^\dagger(\xi^\dagger_\infty)^{-1}h'(\xi^\dagger_\infty)\big],\end{aligned}\tag{2.89}$$

in agreement with (2.88). ∎

3 The derivative of the exponential growth rate

3.1 Continuity of the eigenmeasure

In this section, we prove Theorem 2. To prepare for this, in the present subsection, we show that the eigenmeasures $\mathring{\nu}$ from Theorem 1 depend continuously on the recovery rate δ. This will later be used to prove continuity of the right-hand side of (1.15) in δ.

Lemma 18 (Limits of eigenmeasures) *Let ν_n $(n\geq 0)$ be homogeneous eigenmeasures of (Λ,a,δ_n)-contact processes, with eigenvalues λ_n, normalized such that $\int\nu_n(\mathrm{d}A)1_{\{0\in A\}}=1$. Assume that $\lambda_n\to\lambda$, $\delta_n\to\delta$. Then the $(\nu_n)_{n\geq 0}$ are relatively compact in the topology of vague convergence, and each vague cluster point ν is a homogeneous eigenmeasure of the (Λ,a,δ)-contact processes, with eigenvalue λ.*

Proof By the homogeneity and normalization of the ν_n, one has

$$\int\nu_n(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}\leq\sum_{i\in B}\int\nu_n(\mathrm{d}A)1_{\{i\in A\}}=|B|.\tag{3.1}$$

Since this estimate is uniform in n, applying [Swa09, Lemma 3.2] we find that the $(\nu_n)_{n\geq 0}$ are relatively compact in the topology of vague convergence. By going to a subsequence if necessary, we may assume that the ν_n converge vaguely to a limit ν. Since the ν_n are eigenmeasures, denoting the (Λ,a,δ_n)-contact process started in A by $(\eta^{\delta_n,A}_t)_{t\geq 0}$, we have

$$\int \nu_n(\mathrm{d}A)\mathbb{P}[\eta^{\delta_n,A}_t\in\cdot\,]\big|_{\mathcal{P}_+}=e^{\lambda_n t}\nu_n\qquad(t\geq 0). \tag{3.2}$$

Since $\lambda_n\to\lambda$, the right-hand side of this equation converges vaguely to $e^{\lambda t}\nu$. To prove vague convergence of the left-hand side, by [Swa09, Lemma 3.2], it suffices to prove that for $B\in\mathcal{P}_{\rm fin}$,

$$\int \nu_n(\mathrm{d}A)\mathbb{P}[\eta^{\delta_n,A}_t\cap B\neq\emptyset]\to\int \nu(\mathrm{d}A)\mathbb{P}[\eta^{\delta,A}_t\cap B\neq\emptyset]. \tag{3.3}$$

We estimate

$$\begin{aligned}
&\Big|\int \nu_n(\mathrm{d}A)\mathbb{P}[\eta^{\delta_n,A}_t\cap B\neq\emptyset]-\int \nu(\mathrm{d}A)\mathbb{P}[\eta^{\delta,A}_t\cap B\neq\emptyset]\Big|\\
&\quad\leq\int \nu_n(\mathrm{d}A)\Big|\mathbb{P}[\eta^{\delta_n,A}_t\cap B\neq\emptyset]-\mathbb{P}[\eta^{\delta,A}_t\cap B\neq\emptyset]\Big| \qquad (3.4)\\
&\qquad+\Big|\int \nu_n(\mathrm{d}A)\mathbb{P}[\eta^{\delta,A}_t\cap B\neq\emptyset]-\int \nu(\mathrm{d}A)\mathbb{P}[\eta^{\delta,A}_t\cap B\neq\emptyset]\Big|. \qquad (3.5)
\end{aligned}$$

The term in (3.5) tends to zero as $n\to\infty$ by [Swa09, Lemmas 3.2 and 3.3]. By duality, we can rewrite the term in (3.4) as

$$\int \nu_n(\mathrm{d}A)\Big|\mathbb{P}[A\cap\eta^{\dagger\,\delta_n,B}_t\neq\emptyset]-\mathbb{P}[A\cap\eta^{\dagger\,\delta,B}_t\neq\emptyset]\Big|. \tag{3.6}$$

We couple the graphical representations for processes with different recovery rates in the natural way, by constructing a Poisson point process $\Omega^{\rm r}$ on $\Lambda\times\mathbb{R}_+\times\mathbb{R}_+$ with intensity one, and letting $\omega^{\rm r}_\delta:=\{(i,t):\exists 0\leq r\leq\delta\text{ s.t. }(i,t,r)\in\Omega^{\rm r}\}$ be the set of recovery symbols for the process with recovery rate δ. Then the quantity in (3.6) can be estimated from above by

$$\begin{aligned}
&\int \nu_n(\mathrm{d}A)\mathbb{P}\big[A\cap\eta^{\dagger\,0,B}_t\neq\emptyset,\ \eta^{\dagger\,\delta_n,B}_t\neq\eta^{\dagger\,\delta,B}_t\big]\\
&\quad=\int \mathbb{P}\big[\eta^{\dagger\,0,B}_t\in\mathrm{d}C,\ \eta^{\dagger\,\delta_n,B}_t\neq\eta^{\dagger\,\delta,B}_t\big]\int \nu_n(\mathrm{d}A)1_{\{A\cap C\neq\emptyset\}} \qquad (3.7)\\
&\quad\leq\int \mathbb{P}\big[\eta^{\dagger\,0,B}_t\in\mathrm{d}C,\ \eta^{\dagger\,\delta_n,B}_t\neq\eta^{\dagger\,\delta,B}_t\big]\,|C|=\mathbb{E}\big[|\eta^{\dagger\,0,B}_t|1_{\{\eta^{\dagger\,\delta_n,B}_t\neq\eta^{\dagger\,\delta,B}_t\}}\big],
\end{aligned}$$

where $\eta^{\dagger\,0,B}_t$ denotes the process with zero recovery rate and we have used (3.1). Since the right-hand side of (3.7) tends to zero by dominated convergence, this proves the lemma. ∎

Proposition 19 (Continuity of the eigenmeasure) *Assume that the infection rates satisfy the irreducibility condition (1.3). For $\delta\in(\delta_{\rm c},\infty)$, let $\mathring{\nu}_\delta$ denote the unique homogeneous eigenmeasure of the (Λ,a,δ)-contact process normalized such that $\int\mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}}=1$. Then the map $\delta\mapsto\mathring{\nu}_\delta$ is continuous on $(\delta_{\rm c},\infty)$ w.r.t. vague convergence of locally finite measures on $\mathcal{P}_+$.*

Proof Choose $\delta_n, \delta \in (\delta_c, \infty)$ such that $\delta_n \to \delta$. Since the eigenvalue $r(\Lambda, a, \delta)$ of the homogeneous eigenmeasure $\mathring{\nu}_\delta$ is continuous in δ by Theorem 0 (b), Lemma 18 implies that the measures $(\mathring{\nu}_{\delta_n})_{n\geq 0}$ are relatively compact in the topology of vague convergence, and each vague cluster point is a homogeneous eigenmeasure of the (Λ, a, δ)-contact processes with eigenvalue $r(\Lambda, a, \delta)$. By Theorem 1, this implies that $\mathring{\nu}_\delta$ is the only vague cluster point, hence the $\mathring{\nu}_{\delta_n}$ converge vaguely to $\mathring{\nu}_\delta$. ∎

3.2 Local convergence

In order to prove Theorem 2, we will need to strengthen the form of convergence in (1.10) for initial measures that are concentrated on finite sets, and likewise, we will need to strengthen the form of continuity of the map $\delta \mapsto \mathring{\nu}_\delta$ from Proposition 19.

For each $i \in \Lambda$, we define

$$\mathcal{P}_i := \{A \in \mathcal{P} : i \in A\} \quad \text{and} \quad \mathcal{P}_{\text{fin},\,i} := \mathcal{P}_{\text{fin}} \cap \mathcal{P}_i. \tag{3.8}$$

Note that since $\mathcal{P}_i$ is a compact subset of $\mathcal{P}_+$, the restriction $\mu|_{\mathcal{P}_i}$ of a locally finite measure μ on $\mathcal{P}_+$ to $\mathcal{P}_i$ is a finite measure. We note that there are two natural ways to equip the space $\mathcal{P}_{\text{fin},\,i}$ with a topology. On the one hand, $\mathcal{P}_{\text{fin},\,i}$ inherits the product topology from its embedding in $\mathcal{P} \cong \{0,1\}^\Lambda$. On the other hand, since $\mathcal{P}_{\text{fin},\,i}$ is a countable set, it is natural to equip it with the discrete topology. (I.e., $A_n \to A$ in the discrete topology if and only if there is an N such that $A_n = A$ for all $n \geq N$.) The following lemma will be proved below.

Lemma 20 (Vague and weak convergence) *Let μ_n, μ be locally finite measures on $\mathcal{P}_+$. Then the μ_n converge vaguely to μ if and only if for each $i \in \Lambda$, the $\mu_n|_{\mathcal{P}_i}$ converge weakly to $\mu|_{\mathcal{P}_i}$ with respect to the product topology.*

Motivated by this, if μ_n, μ are locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\text{fin},+}$, then we say that the μ_n converge to μ *locally on* $\mathcal{P}_{\text{fin},+}$, if for each $i \in \Lambda$, the $\mu_n|_{\mathcal{P}_{\text{fin},\,i}}$ converge weakly to $\mu|_{\mathcal{P}_{\text{fin},\,i}}$ with respect to the discrete topology on $\mathcal{P}_{\text{fin},\,i}$.

We need two more definitions. If μ_n, μ are locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\text{fin},+}$, then we say that the μ_n converge to μ *pointwise on* $\mathcal{P}_{\text{fin},+}$ if $\mu_n(\{A\}) \to \mu(\{A\})$ for all $A \in \mathcal{P}_{\text{fin},+}$. If $(\mu_n)_{n\geq 1}$ are locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\text{fin},+}$, then we say that the $(\mu_n)_{n\geq 1}$ are *locally tight* if for each $i \in \Lambda$ and $\varepsilon > 0$ there exists a finite $\mathcal{D} \subset \mathcal{P}_{\text{fin},\,i}$ such that $\sup_n \mu_n(\mathcal{P}_{\text{fin},\,i} \backslash \mathcal{D}) \leq \varepsilon$.

The next proposition connects all these definitions.

Proposition 21 (Local convergence) *Let μ_n, μ be locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\text{fin},+}$. Then the following statements are equivalent.*

(i) *$\mu_n \Rightarrow \mu$ locally on $\mathcal{P}_{\text{fin},+}$.*

(ii) *$\mu_n \to \mu$ pointwise on $\mathcal{P}_{\text{fin},+}$ and the $(\mu_n)_{n\geq 1}$ are locally tight.*

(iii) *$\mu_n \Rightarrow \mu$ vaguely on $\mathcal{P}_+$ and the $(\mu_n)_{n\geq 1}$ are locally tight.*

(iv) *$\mu_n \Rightarrow \mu$ vaguely on $\mathcal{P}_+$ and $\mu_n \to \mu$ pointwise on $\mathcal{P}_{\text{fin},+}$.*

(v) *$\mu_n \Rightarrow \mu$ vaguely on $\mathcal{P}_+$ and*

$$\liminf_{n\to\infty} \int \mu_n(\mathrm{d}A) 1_{\{i\in A\}} |A|^{-1} \geq \int \mu(\mathrm{d}A) 1_{\{i\in A\}} |A|^{-1} \qquad (i \in \Lambda).$$

Remark If μ_n, μ are homogeneous, then the condition on the limit inferior in (v) just says that $\liminf_{n\to\infty} c(\mu_n) \geq c(\mu)$, where $c(\mu)$ is defined in (2.42).

We start with a preliminary lemma.

Lemma 22 (Compact classes) *If $\mathcal{C} \subset \mathcal{P}_+$ is compact, then there exists a finite $\Delta \subset \Lambda$ such that $\mathcal{C} \subset \bigcup_{i\in\Delta} \mathcal{P}_i$.*

Proof Choose $\Delta_n \uparrow \Lambda$. If $\mathcal{C} \not\subset \bigcup_{i\in\Delta_n} \mathcal{P}_i$ for each n, then we can find $A_n \in \mathcal{C}$ such that $A_n \cap \Delta_n = \emptyset$. It follows that $A_n \to \emptyset \notin \mathcal{C}$ (in the product topology), hence $\mathcal{C}$ is not a closed subset of $\mathcal{P}$ and therefore not compact. ∎

Proof of Lemma 20 Since $\mathcal{P}\backslash\mathcal{P}_i$ is a closed subset of $\mathcal{P}$, any continuous function $f : \mathcal{P}_i \to \mathbb{R}$ can be extended to a continuous, compactly supported function on $\mathcal{P}_+$ by putting $f(A) := 0$ for $A \in \mathcal{P}_+\backslash\mathcal{P}_i$. Therefore, if the μ_n converge vaguely to μ, it follows that the $\mu_n|_{\mathcal{P}_i}$ converge weakly to $\mu|_{\mathcal{P}_i}$. Conversely, if for each $i \in \Lambda$ the $\mu_n|_{\mathcal{P}_i}$ converge weakly to $\mu|_{\mathcal{P}_i}$, then for each $i, j \in \Lambda$ one has

$$\mu_n|_{\mathcal{P}_i\cap\mathcal{P}_j} \Rightarrow \mu|_{\mathcal{P}_i\cap\mathcal{P}_j}, \quad \mu_n|_{\mathcal{P}_i\backslash\mathcal{P}_j} \Rightarrow \mu|_{\mathcal{P}_i\backslash\mathcal{P}_j} \quad \text{and} \quad \mu_n|_{\mathcal{P}_j\backslash\mathcal{P}_i} \Rightarrow \mu|_{\mathcal{P}_j\backslash\mathcal{P}_i}, \tag{3.9}$$

where we have used that $\mathcal{P}_i \cap \mathcal{P}_j$, $\mathcal{P}_i\backslash\mathcal{P}_j$ and $\mathcal{P}_j\backslash\mathcal{P}_i$ are compact sets. Continuing this process, we see by induction that for each finite $\Delta \subset \Lambda$, the restrictions $\mu_n|_{\bigcup_{i\in\Delta}\mathcal{P}_i}$ converge weakly to $\mu|_{\bigcup_{i\in\Delta}\mathcal{P}_i}$. By Lemma 22, if $f : \mathcal{P}_+ \to \mathbb{R}$ is a compactly supported continuous function, then f is supported on $\bigcup_{i\in\Delta}\mathcal{P}_i$ for some finite $\Delta \subset \Lambda$. It follows that $\int \mu_n(\mathrm{d}A)f(A) \to \int \mu(\mathrm{d}A)f(A)$, proving that the μ_n converge vaguely to μ. ∎

Proof of Proposition 21 The equivalence of (i) and (ii) follows straightforwardly from Prohorov's theorem applied to the countable space $\mathcal{P}_{\mathrm{fin},\,i}$ with the discrete topology.

Since the discrete topology on $\mathcal{P}_{\mathrm{fin},\,i}$ is stronger than the product topology, weak convergence of the $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ with respect to the discrete topology implies weak convergence with respect to the product topology. By Lemma 20, this shows that local convergence on $\mathcal{P}_{\mathrm{fin},\,+}$ implies vague convergence on $\mathcal{P}_+$.

To prove (iii)⇒(i), note that by local tightness, for each $i \in \Lambda$ the measures $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ are relatively compact in the topology of weak convergence with respect to the discrete topology. Let μ^i_* be a subsequential limit. Since weak convergence with respect to the discrete topology implies weak convergence with respect to the product topology, by Lemma 20, we conclude that $\mu^i_* = \mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$. Since this is true for each cluster point, we conclude that the $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ converge weakly to $\mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$ with respect to the discrete topology.

We next claim that (iv)⇒(v). Choose finite $\mathcal{D}_k \uparrow \mathcal{P}_{\mathrm{fin},\,+}$. Then, by the pointwise convergence of the μ_n to μ on $\mathcal{P}_{\mathrm{fin},\,+}$, we see that

$$\begin{aligned} &\int \mu_n(\mathrm{d}A)|A|^{-1}1_{\{i\in A\}} \geq \sum_{A\in\mathcal{D}_k} \mu_n(\{A\})|A|^{-1}1_{\{i\in A\}} \\ &\underset{n\to\infty}{\longrightarrow} \sum_{A\in\mathcal{D}_k} \mu(\{A\})|A|^{-1}1_{\{i\in A\}} \quad \underset{k\to\infty}{\longrightarrow} \int \mu(\mathrm{d}A)|A|^{-1}1_{\{i\in A\}}, \end{aligned} \tag{3.10}$$

which shows that the limit inferior of the left-hand side is larger or equal than the right-hand side.

To prove (v)⇒(i), finally, let $\overline{\mathbb{N}} := \{0,1,\ldots\}\cup\{\infty\}$ be the one-point compactification of $\mathbb{N}$. Then $\mathcal{P}_i \times \overline{\mathbb{N}}$ is a compact set, where we equip $\mathcal{P}_i$ with the product topology. If $A_n \to A$ in the product topology, then $\liminf_{n\to\infty}|A_n| \geq |A|$. It follows that the set

$$\mathcal{Q}_i := \{(A,n)\in\mathcal{P}_i\times\overline{\mathbb{N}} : |A|\leq n\}, \tag{3.11}$$

is a closed subset of $\mathcal{P}_i\times\overline{\mathbb{N}}$, hence compact.

Let ρ_n be the image of the measure $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ under the map $A\mapsto(A,|A|)$. By the compactness of $\mathcal{Q}_i$, going to a subseqence if necessary, we may assume that the ρ_n converge weakly to a limit ρ with respect to the topology on $\mathcal{Q}_i$. By the vague convergence $\mu_n\Rightarrow\mu$ and Lemma 20, the first marginal of ρ is $\mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$. Now

$$\begin{aligned}\int\rho(\mathrm{d}(A,n))n^{-1} &= \lim_{n\to\infty}\int\rho_n(\mathrm{d}(A,n))n^{-1} = \lim_{n\to\infty}\int\mu_n(\mathrm{d}A)1_{\{i\in A\}}|A|^{-1}\\ &\geq \int\mu(\mathrm{d}A)1_{\{i\in A\}}|A|^{-1} = \int\rho(\mathrm{d}(A,n))|A|^{-1},\end{aligned} \tag{3.12}$$

which proves that ρ is concentrated on $\{(A,n)\in\mathcal{Q}_i : |A|=n\}$. We observe that if $A_n, A\in\mathcal{P}_{\mathrm{fin},\,i}$, then $A_n\to A$ in the discrete topology on $\mathcal{P}_{\mathrm{fin},\,i}$ if and only if $A_n\to A$ in the product topology on $\mathcal{P}_i$ and $|A_n|\to|A|$. It follows that the space $\{(A,n)\in\mathcal{Q}_i : A\in\mathcal{P}_{\mathrm{fin},\,i},\ |A|=n\}$ equipped with the induced topology from $\mathcal{Q}_i$ is isomorphic with the space $\mathcal{P}_{\mathrm{fin},\,i}$ equipped with the discrete topology. Therefore, since ρ_n,ρ are concentrated on this space and $\rho_n\Rightarrow\rho$ weakly with respect to the topology on $\mathcal{Q}_i$, we conclude that the $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ converge weakly to $\mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$ with respect to the discrete topology on $\mathcal{P}_{\mathrm{fin},\,i}$. ∎

The next result says that if the measure μ in Theorem 1 is concentrated on $\mathcal{P}_{\mathrm{fin}}$, then the convergence in (1.10) also happens locally on $\mathcal{P}_{\mathrm{fin},\,+}$.

Proposition 23 (Local convergence) *Let $r,\mu,c,\mathring{\nu}$ be as in Theorem 1, and assume that $\int\mu(\mathrm{d}A)|A|1_{\{0\in A\}}<\infty$. Then, in addition to (1.10), one has*

$$e^{-rt}\int\mu(\mathrm{d}A)\mathbb{P}[\eta^A_t\in\cdot\,]\big|_{\mathcal{P}_+}\underset{t\to\infty}{\Longrightarrow} c\,\mathring{\nu}\quad\textit{locally on } \mathcal{P}_{\mathrm{fin},\,+}. \tag{3.13}$$

Proof Let h be defined as in (2.68). By the remarks below (2.69), $h\mu$ is a locally finite measure on $\mathcal{P}_+$ that is concentrated on $\mathcal{P}_{\mathrm{fin}}$. Therefore, by Lemma 10 there exists some $\mathcal{P}_{\mathrm{fin},\,+}$-valued random variable Δ such that

$$h\mu = c(h\mu)\sum_{i\in\Lambda}\mathbb{P}\big[i\Delta\in\cdot\,\big]. \tag{3.14}$$

Let μ_t denote the left-hand side of (3.13). By Theorem 1 and Proposition 21 (i) and (iv), it suffices to show that μ_t converges pointwise on $\mathcal{P}_{\mathrm{fin},\,+}$ to the right-hand side of (3.13). By (2.56),

$$h(B)\mu_t(\{B\}) = \sum_{A\in\mathcal{P}_{\mathrm{fin},\,+}}\mu(\{A\})e^{-rt}P_t(A,B)h(B) = \sum_{A\in\mathcal{P}_{\mathrm{fin},\,+}}\mu(\{A\})h(A)P^h_t(A,B). \tag{3.15}$$

Let $(\xi_t)_{t\geq0}$ be a h-transformed (Λ,a,δ)-contact process started in the initial law $\mathbb{P}[\xi_0\in\cdot\,] = \mathbb{P}[\Delta\in\cdot\,]$. Then (3.14) and (3.15) show that

$$\begin{aligned}(h\mu_t)(\{B\}) &= \sum_{A\in\mathcal{P}_{\mathrm{fin},\,+}}(h\mu)(\{A\})P^h_t(A,B)\\ &= c(h\mu)\sum_{i\in\Lambda}\sum_{A\in\mathcal{P}_{\mathrm{fin},\,+}}\mathbb{P}\big[i\Delta=A\big]P^h_t(A,B) = c(h\mu)\sum_{i\in\Lambda}\mathbb{P}\big[i\xi_t=B\big]\qquad(t\geq0,\ B\in\mathcal{P}_{\mathrm{fin},\,+}).\end{aligned} \tag{3.16}$$

Since for finite Λ, vague convergence implies pointwise convergence, we can without loss of generality assume that Λ is infinite. Then (2.48) tells us that

$$(h\mu_t)(\{B\}) = c(h\mu)\mathbb{P}\big[\tilde{\xi}_t = \tilde{B}\big] \qquad (t \geq 0,\ B \in \mathcal{P}_{\mathrm{fin},+}). \tag{3.17}$$

By Proposition 15 and Lemma 14, it follows that

$$h(B)\mu_t(\{B\}) \underset{t\to\infty}{\longrightarrow} c(h\mu)\mathbb{P}\big[\tilde{\xi}_\infty = \tilde{B}\big] = \frac{c(h\mu)}{c(h\mathring{\nu})}h(B)\mathring{\nu}(\{B\}) \qquad (B \in \mathcal{P}_{\mathrm{fin},+}). \tag{3.18}$$

Here, by Lemma 17, $c(h\mu)/c(h\mathring{\nu}) = c$ is the constant from formula (1.11), so dividing both sides of (3.18) by $h(B)$ (which is strictly positive for each $B \in \mathcal{P}_{\mathrm{fin},+}$ by the fact that $\mathring{\nu}^\dagger$ is nonzero), we see that μ_t converges pointwise on $\mathcal{P}_{\mathrm{fin},+}$ to $(c(h\mu)/c(h\mathring{\nu}))\mathring{\nu}(\{B\})$, which by the equivalence (i)⇔(iv) of Proposition 21 implies the local convergence in (3.13). ∎

The following result says that the vague convergence in Proposition 19 can be strengthened to local convergence on $\mathcal{P}_{\mathrm{fin},+}$.

Proposition 24 (Local continuity of the eigenmeasure) *Assume that the infection rates satisfy the irreducibility condition (1.3). For $\delta \in (\delta_\mathrm{c}, \infty)$, let $\mathring{\nu}_\delta$ denote the unique homogeneous eigenmeasure of the (Λ, a, δ)-contact process normalized such that $\int \mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}} = 1$. Then the map $\delta \mapsto \mathring{\nu}_\delta$ is continuous on $(\delta_\mathrm{c}, \infty)$ in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$.*

Proof Vague continuity of the map $(\delta_\mathrm{c}, \infty) \ni \delta \mapsto \mathring{\nu}_\delta$ has been proved in Proposition 19, so by the equivalence (i)⇔(iii) in Proposition 21, it suffices to show that for any $\delta_* \in (\delta_\mathrm{c}, \infty)$ there exists an $\varepsilon > 0$ such that the measures $(\mathring{\nu}_\delta)_{\delta\in(\delta_*-\varepsilon,\delta_*+\varepsilon)}$ are locally tight. By Lemma 8 and an argument as in (2.39),

$$\int \mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A) \leq (|a| + \delta)\int_0^\infty e^{-r(\delta)t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta_t^{\delta,\{0\}})\big]^2, \tag{3.19}$$

where for $\gamma \geq 0$, the function e_γ is defined as in (2.9) in terms of a metric d satisfying (2.3), $(\eta_t^{\delta,\{0\}})_{t\geq 0}$ denotes the (Λ, a, δ)-contact process started in $\eta_0^{\delta,\{0\}} = \{0\}$, and $r = r(\delta)$ is its exponential growth rate. By property (2.3) (ii), for each $\gamma > 0$ and $K < \infty$, the set $\{A \in \mathcal{P}_{\mathrm{fin},0} : e_\gamma(A) \leq K\}$ is finite. Thus, to prove the required local tightness, it suffices to show that for each $\delta_* \in (\delta_\mathrm{c}, \infty)$ there exist a $\gamma > 0$ and $\varepsilon > 0$ such that

$$\sup_{\delta\in(\delta_*-\varepsilon,\delta_*+\varepsilon)}\int_0^\infty e^{-r(\delta)t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta_t^{\delta,\{0\}})\big]^2 < \infty. \tag{3.20}$$

By the continuity of $\delta \mapsto r(\delta)$ (Theorem 0 (b)), we can choose $\varepsilon > 0$ such that $\delta_\mathrm{c} < \delta_* - \varepsilon$ and

$$r(\delta_* - \varepsilon) \leq \frac{4}{5}r(\delta_* + \varepsilon). \tag{3.21}$$

Let $r_\gamma = r_\gamma(\delta)$ be the exponential growth rate associated with the function e_γ, as defined in Lemma 5. By Lemma 6, we can choose $\gamma > 0$ such that

$$r_\gamma(\delta_* - \varepsilon) \leq \frac{3}{4}r(\delta_* - \varepsilon). \tag{3.22}$$

By the fact that $r(\delta)$ is nonincreasing in δ and the law of $\eta_t^{\delta,\{0\}}$ is nonincreasing in δ with respect to the stochastic order, it follows that for all $\delta \in (\delta_* - \varepsilon, \delta_* + \varepsilon)$,

$$\begin{aligned}
&\int_0^\infty e^{-r(\delta)t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta_t^{\delta,\{0\}})\big]^2 \leq \int_0^\infty e^{-r(\delta_*+\varepsilon)t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta_t^{\delta_*-\varepsilon,\{0\}})\big]^2 \\
&\quad = \int_0^\infty \mathrm{d}t\, e^{\big(2r_\gamma(\delta_* - \varepsilon) - r(\delta_* + \varepsilon)\big)t + o(t)} \leq \int_0^\infty \mathrm{d}t\, e^{\frac{1}{5}r(\delta_* + \varepsilon)t + o(t)} < \infty,
\end{aligned} \tag{3.23}$$

where $t \mapsto o(t)$ is continuous, $o(t)/t \to 0$ for $t \to \infty$ by the definition of r_γ in Lemma 5, and we have used that $2r_\gamma(\delta_* - \varepsilon) \leq 2 \cdot \frac{3}{4} \cdot \frac{4}{5} r(\delta_* + \varepsilon) = \frac{6}{5} r(\delta_* + \varepsilon)$. This proves (3.20) and hence the required local tightness. ■

Lemma 25 (Local continuity of intersection measure) *Let μ_n, μ, ν_n, ν be locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\mathrm{fin},+}$. Assume that the μ_n converge locally on $\mathcal{P}_{\mathrm{fin},+}$ to μ and the ν_n converge locally on $\mathcal{P}_{\mathrm{fin},+}$ to ν. Then the intersection measures $\psi(\mu_n, \nu_n)$ converge locally on $\mathcal{P}_{\mathrm{fin},+}$ to $\psi(\mu, \nu)$.*

Proof By the definition of local convergence on $\mathcal{P}_{\mathrm{fin},+}$, for each $i \in \Lambda$, the restricted measures $\mu_n|_{\mathcal{P}_{\mathrm{fin},i}}$ converge weakly to $\mu|_{\mathcal{P}_{\mathrm{fin},i}}$ with respect to the discrete topology on $\mathcal{P}_{\mathrm{fin},i}$ and likewise with μ_n, μ replaced by ν_n, ν. We observe that $\psi(\mu,\nu)|_{\mathcal{P}_{\mathrm{fin},i}}$ is the image of the product measure $\mu|_{\mathcal{P}_{\mathrm{fin},i}} \otimes \nu|_{\mathcal{P}_{\mathrm{fin},i}}$ under the map $(A, B) \mapsto A \cap B$ (which is obviously continuous w.r.t. to the discrete topologies on $\mathcal{P}_{\mathrm{fin},i} \times \mathcal{P}_{\mathrm{fin},i}$ and $\mathcal{P}_{\mathrm{fin},i}$). Now, letting $\Rightarrow$ denote weak convergence with respect to the discrete topology on $\mathcal{P}_{\mathrm{fin},i}$, we see that $\mu_n|_{\mathcal{P}_{\mathrm{fin},i}} \Rightarrow \mu|_{\mathcal{P}_{\mathrm{fin},i}}$ and $\nu_n|_{\mathcal{P}_{\mathrm{fin},i}} \Rightarrow \nu|_{\mathcal{P}_{\mathrm{fin},i}}$ imply that $\mu_n|_{\mathcal{P}_{\mathrm{fin},i}} \otimes \nu_n|_{\mathcal{P}_{\mathrm{fin},i}} \Rightarrow \mu|_{\mathcal{P}_{\mathrm{fin},i}} \otimes \nu|_{\mathcal{P}_{\mathrm{fin},i}}$ and therefore $\psi(\mu_n,\nu_n)|_{\mathcal{P}_{\mathrm{fin},i}} \Rightarrow \psi(\mu,\nu)|_{\mathcal{P}_{\mathrm{fin},i}}$ for each $i \in \Lambda$. ■

Lemma 26 (Continuity of differential formula) *Under the assumptions of Theorem 2, the right-hand side of (1.15) is continuous in δ on $(\delta_{\mathrm{c}}, \infty)$.*

Proof We may rewrite the right-hand side of (1.15) as

$$\frac{\int \psi(\mathring{\nu}_\delta, \mathring{\nu}_\delta^\dagger)(\mathrm{d}C) 1_{\{C=\{0\}\}}}{\int \psi(\mathring{\nu}_\delta, \mathring{\nu}_\delta^\dagger)(\mathrm{d}C) |C|^{-1} 1_{\{0 \in C\}}}, \tag{3.24}$$

where $\psi(\mathring{\nu}_\delta, \mathring{\nu}_\delta^\dagger)$ denotes the intersection measure of $\mathring{\nu}_\delta$ and $\mathring{\nu}_\delta^\dagger$, defined in (2.76). By Proposition 24, the maps $\delta \mapsto \mathring{\nu}_\delta$ and $\delta \mapsto \mathring{\nu}_\delta^\dagger$ are continuous on $(\delta_{\mathrm{c}}, \infty)$ in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$, which by Lemma 25 implies that also $\delta \mapsto \psi(\mathring{\nu}_\delta, \mathring{\nu}_\delta^\dagger)$ is continuous on $(\delta_{\mathrm{c}}, \infty)$ in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$. Since $C \mapsto 1_{\{C=\{0\}\}}$ and $C \mapsto |C|^{-1}$ are bounded functions on $\mathcal{P}_{\mathrm{fin},0}$, the expression in (3.24) is continuous on $(\delta_{\mathrm{c}}, \infty)$. ■

3.3 Differential formulas

We continue to write $(\eta_t^{\delta,A})_{t\geq 0}$ for the (Λ, a, δ)-contact process with initial state A. Define (as before) $\pi_t(\delta) := \mathbb{E}[|\eta_t^{\delta,\{0\}}|]$. Then $r(\delta) = \lim_{t\to\infty} \frac{1}{t} \log \pi_t(\delta)$ by (1.8). In analogy with (2.16), let us define

$$\mu_{t,\delta} := \sum_{i\in\Lambda} \mathbb{P}[\eta_t^{\delta,\{i\}} \in \cdot\,]\big|_{\mathcal{P}_+} \quad \text{and} \quad \mu_{t,\delta}^\dagger := \sum_{i\in\Lambda} \mathbb{P}[\eta_t^{\dagger\,\delta,\{i\}} \in \cdot\,]\big|_{\mathcal{P}_+} \qquad (t \geq 0). \tag{3.25}$$

We begin with the following result:

Lemma 27 (Differential formula) *The function* $[0,\infty) \ni \delta \mapsto \pi_t(\delta)$ *is continuously differentiable and*

$$\frac{1}{t}\frac{\partial}{\partial\delta}\log\pi_t(\delta) = -\frac{\dfrac{1}{t}\displaystyle\int_0^t \mathrm{d}s \int \mu_{s,\delta}(\mathrm{d}A)\int \mu^{\dagger}_{t-s,\delta}(\mathrm{d}B)\,1_{\{A\cap B=\{0\}\}}}{\displaystyle\int \mu_{t/2,\delta}(\mathrm{d}A)\int \mu^{\dagger}_{t/2,\delta}(\mathrm{d}B)|A\cap B|^{-1}1_{\{0\in A\cap B\}}}. \tag{3.26}$$

Proof We use the graphical representation and write $(0,0) \rightsquigarrow_{(j,s)} (i,t)$ to denote the event that there is an open path from $(0,0)$ to (i,t), and all such paths lead through (j,s), i.e., (j,s) is *pivotal*. Then, by [Swa09, formula (3.10)], for $0 \le s \le t$,

$$\begin{aligned}\tfrac{\partial}{\partial\delta}\pi_t(\delta) &= -\sum_{i,j}\int_0^t \mathrm{d}s\,\mathbb{P}[(0,0)\rightsquigarrow_{(j,s)}(i,t)] = -\sum_{i,j}\int_0^t \mathrm{d}s\,\mathbb{P}[(j^{-1},-s)\rightsquigarrow_{(0,0)}(j^{-1}i,t-s)]\\ &= -\sum_{i,j}\int_0^t \mathrm{d}s\,\mathbb{P}[\eta^{\{i\}}_s\cap\eta^{\dagger\,\{j\}}_{t-s}=\{0\}] = -\int_0^t \mathrm{d}s\int\mu_{s,\delta}(\mathrm{d}A)\int\mu^{\dagger}_{t-s,\delta}(\mathrm{d}B)1_{\{A\cap B=\{0\}\}},\end{aligned} \tag{3.27}$$

where we have used translation invariance and changed the summation order. Similarly,

$$\begin{aligned}\pi_t(\delta) &= \sum_i \mathbb{P}[(0,0)\rightsquigarrow(i,t)] = \sum_i \mathbb{P}[\eta^{\{0\}}_s\cap\eta^{\dagger\,\{i\}}_{t-s}\neq\emptyset] = \sum_{i,j}\mathbb{E}[|\eta^{\{0\}}_s\cap\eta^{\dagger\,\{i\}}_{t-s}|^{-1}1_{\{j\in\eta^{\{0\}}_s\cap\eta^{\dagger\,\{i\}}_{t-s}\}}]\\ &= \sum_{i,j}\mathbb{E}[|\eta^{\{j^{-1}\}}_s\cap\eta^{\dagger\,\{j^{-1}i\}}_{t-s}|^{-1}1_{\{0\in\eta^{\{j^{-1}\}}_s\cap\eta^{\dagger\,\{j^{-1}i\}}_{t-s}\}}]\\ &= \int\mu_{s,\delta}(\mathrm{d}A)\int\mu^{\dagger}_{t-s,\delta}(\mathrm{d}B)|A\cap B|^{-1}1_{\{0\in A\cap B\}} \qquad (0\le s\le t).\end{aligned} \tag{3.28}$$

Since $\frac{\partial}{\partial\delta}\log\pi_t(\delta) = (\frac{\partial}{\partial\delta}\pi_t(\delta))/\pi_t(\delta)$, formulas (3.27) and (3.28) imply (3.26). ∎

We will prove Theorem 2 by taking the limit $t\to\infty$ in (3.26). To justify the interchange of limit and differentiation, we will use the following lemma.

Lemma 28 (Interchange of limit and differentiation) *Let* $I\subset\mathbb{R}$ *be a compact interval and let* f_n, f, f' *be continuous real functions on* I*. Assume each* f_n *is continuously differentiable, that* $f_n(x)\to f(x)$ *and* $\frac{\partial}{\partial x}f_n(x)\to f'(x)$ *for each* $x\in I$*, and that*

$$\sup_{x\in I}\sup_n|\tfrac{\partial}{\partial x}f_n(x)|<\infty. \tag{3.29}$$

Then f *is continuously differentiable and* $\frac{\partial}{\partial x}f(x)=f'(x)$ $(x\in I)$*.*

Proof We write $I=[x_-,x_+]$ and observe that

$$\begin{aligned}f(x) &= \lim_{n\to\infty}f_n(x_-)+\lim_{n\to\infty}\int_{x_-}^x \tfrac{\partial}{\partial y}f_n(y)\,\mathrm{d}y\\ &= f(x_-)+\int_{x_-}^x\big(\lim_{n\to\infty}\tfrac{\partial}{\partial y}f_n(y)\big)\mathrm{d}y = f(x_-)+\int_{x_-}^x f'(y)\,\mathrm{d}y,\end{aligned} \tag{3.30}$$

where the interchange of limit and integration is justified by dominated convergence, using (3.29). Differentiation of (3.30) now yields the statement since f' is continuous. ∎

Proof of Theorem 2 By Lemma 27, $\log\pi_t(\delta)$ is continuously differentiable in δ and

$$-\frac{1}{t}\frac{\partial}{\partial\delta}\log\pi_t(\delta) = \frac{\int_0^1 F_{t,\delta}(u)\,\mathrm{d}u}{G_{t,\delta}(\frac{1}{2})}, \tag{3.31}$$

where for $0 \leq u \leq 1$, we define

$$\begin{aligned} F_{t,\delta}(u) &:= \int \psi\big(e^{-utr}\mu_{ut,\delta}, e^{-(1-u)tr}\mu^{\dagger}_{(1-u)t,\delta}\big)(\mathrm{d}C)\, 1_{\{C=\{0\}\}}, \\ G_{t,\delta}(u) &:= \int \psi\big(e^{-utr}\mu_{ut,\delta}, e^{-(1-u)tr}\mu^{\dagger}_{(1-u)t,\delta}\big)(\mathrm{d}C)\, |C|^{-1} 1_{\{0\in C\}}, \end{aligned} \tag{3.32}$$

and $\psi(\cdot\,,\,\cdot)$ denotes the intersection measure of two locally finite measures, defined in (2.76). By Proposition 23, there exist constants $c_\delta, c^\dagger_\delta > 0$ such that

$$e^{-rt}\mu_{t,\delta} \underset{t\to\infty}{\Longrightarrow} c_\delta\, \mathring{\nu}_\delta \quad \text{and} \quad e^{-rt}\mu^\dagger_{t,\delta} \underset{t\to\infty}{\Longrightarrow} c^\dagger_\delta\, \mathring{\nu}^\dagger_\delta \quad \text{locally on } \mathcal{P}_{\mathrm{fin},+}. \tag{3.33}$$

By Lemma 25, this implies that for $0 < u < 1$,

$$\begin{aligned} F_{t,\delta}(u) &\underset{t\to\infty}{\longrightarrow} F_{\infty,\delta} := c_\delta c^\dagger_\delta \int \psi(\mathring{\nu}, \mathring{\nu}^\dagger)(\mathrm{d}C)\, 1_{\{C=\{0\}\}}, \\ G_{t,\delta}(u) &\underset{t\to\infty}{\longrightarrow} G_{\infty,\delta} := c_\delta c^\dagger_\delta \int \psi(\mathring{\nu}, \mathring{\nu}^\dagger)(\mathrm{d}C)\, |C|^{-1} 1_{\{0\in C\}}. \end{aligned} \tag{3.34}$$

For fixed δ, we can estimate uniformly in t and u,

$$\begin{aligned} F_{t,\delta}(u) &= \int e^{-utr}\mu_{ut,\delta}(\mathrm{d}A) \int e^{-(1-u)tr}\mu^\dagger_{(1-u)t,\delta}(\mathrm{d}B) 1_{\{A\cap B=\{0\}\}} \\ &\leq \Big(\sup_{s\geq 0} \int e^{-rs}\mu_{s,\delta}(\mathrm{d}A) 1_{\{0\in A\}}\Big)\Big(\sup_{s\geq 0} \int e^{-rs}\mu^\dagger_{s,\delta}(\mathrm{d}B) 1_{\{0\in B\}}\Big), \end{aligned} \tag{3.35}$$

which is finite since both integrals converge. It follows that $\int_0^1 F_{t,\delta}(u)\mathrm{d}u \to F_{\infty,\delta}$ as $t \to \infty$ and therefore, taking the limit in (3.31), we find that

$$-\tfrac{\partial}{\partial\delta}\tfrac{1}{t}\log \pi_t(\delta) \underset{t\to\infty}{\longrightarrow} \frac{F_{\infty,\delta}}{G_{\infty,\delta}}. \tag{3.36}$$

Since the factors $c_\delta c^\dagger_\delta$ in the nominator and denominator cancel, by (3.24), the right-hand side of this equation equals the right-hand side of (1.15), which is continuous in δ by Lemma 26. Recall that $r(\Lambda, a, \delta) = r(\delta) = \lim_{t\to\infty} \frac{1}{t}\log \pi_t(\delta)$ by (1.8). By [Swa09, formula (3.12)], $|\frac{\partial}{\partial\delta}\frac{1}{t}\log \pi_t(\delta)| \leq 1$, so applying Lemma 28, we find that $r(\delta)$ is continuously differentiable on $(\delta_{\mathrm{c}}, \infty)$ with derivative given by the right-hand side of (1.15). ∎

A Exponential decay in the subcritical regime

A.1 Statement of the result

The aim of this appendix is to show how the arguments in [AJ07], which are written down for contact processes on transitive graphs, can be extended to prove Theorem 0 (d) for the class of (Λ, a, δ)-contact processes considered in this article. To formulate this properly, only in this appendix, we will consider a class of contact processes that is more general than both the one defined in Section 1.2 and the one considered in [AJ07], and contains them both as subclasses. Indeed, only in this appendix, will we drop the assumptions that Λ has a group structure (as in the rest of this article) or that Λ has a graph structure (as in [AJ07]). The only structure on Λ that we will use is the structure given by the infection rates $(a(i,j))_{i,j\in\Lambda}$.

Let Λ be any countable set and let $a : \Lambda \times \Lambda \to [0, \infty)$ be a function. By definition, an *automorphism* of (Λ, a) is a bijection $g : \Lambda \to \Lambda$ such that $a(gi, gj) = a(i, j)$ for each $i, j \in \Lambda$. Let $\mathrm{Aut}(\Lambda, a)$ denote the group of automorphisms of (Λ, a). We say that a subgroup $G \subset \mathrm{Aut}(\Lambda, a)$ is *(vertex) transitive* if for each $i, j \in \Lambda$ there exists a $g \in G$ such that $gi = j$. In particular, we say that (Λ, a) is transitive if $\mathrm{Aut}(\Lambda, a)$ is transitive.

Let (Λ, a) be transitive, let $a^\dagger(i, j) := a(j, i)$, and assume that

$$|a| := \sum_{j\in\Lambda} a(i, j) < \infty \quad \text{and} \quad |a^\dagger| := \sum_{j\in\Lambda} a^\dagger(i, j) < \infty, \tag{A.1}$$

where by the transitivity of (Λ, a), these definitions do not depend on the choice of $i \in \Lambda$. Then, for each $\delta \geq 0$, there exists a well-defined contact process on Λ with generator as in (1.2) and also the dual contact process with a replaced by $a^\dagger$ is well-defined. *Only in this appendix*, we will use the term (Λ, a, δ)-contact process (resp. $(\Lambda, a^\dagger, \delta)$-contact process) in this more general sense.

For any (Λ, a, δ)-contact process, as defined in this appendix, we define the critical recovery rate $\delta_c = \delta_c(\Lambda, a)$ as in (1.6), which satisfies $\delta_c < \infty$ but may be zero in the generality considered here. A straightforward extension of [Swa09, Lemma 1.1] shows that the exponential growth rate $r = r(\Lambda, a, \delta)$ in (1.8) is well-defined for the class of (Λ, a, δ)-contact processes considered here.

We will show that the arguments in [AJ07] imply the following result.

Theorem 29 (Exponential decay in the subcritical regime) *Let (Λ, a) be transitive and let a satisfy (A.1). Then $\{\delta \geq 0 : r(\Lambda, a, \delta) < 0\} = (\delta_c, \infty)$.*

We remark that Theorem 0 (a) does not hold in general for the class of (Λ, a, δ)-contact processes considered in this appendix. This is related to unimodularity. A transitive subgroup $G \subset \mathrm{Aut}(\Lambda, a)$ is *unimodular* if [BLPS99, formula (3.3)]

$$|\{gi : g \in G,\ gj = j\}| = |\{gj : g \in G,\ gi = i\}| \qquad (i, j \in \Lambda). \tag{A.2}$$

Note that this is trivially satisfied if Λ is a group and $G = \Lambda$ acts on itself by left multiplication, in which case the sets on both sides of the equation consist of a single element. Unimodularity gives rise to the *mass transport principle* which says that for any function $f : \Lambda \times \Lambda \to [0, \infty)$ such that $f(gi, gj) = f(i, j)$ $(g \in G,\ i, j \in \Lambda)$, one has $\sum_j f(i, j) = \sum_j f(j, i)$. In particular, this implies that the constants $|a|$ and $|a^\dagger|$ from (A.1) are equal and that $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$. In the nonunimodular case, this is in general no longer true and in fact it is not hard to construct examples where the critical recovery rates $\delta_c(\Lambda, a)$ and $\delta_c(\Lambda, a^\dagger)$ of a contact process and its dual are different. We remark that although in [AJ07], the authors do not always clearly distinguish between a contact process and its dual (e.g., in their formulas (1.3), (1.9) and Lemma 1.4), they do not assume that $a = a^\dagger$ and their results are valid also in the asymmetric case $a \neq a^\dagger$.

A.2 The key differential inequalities and their consequences

The main method used in [AJ07], that in its essence goes back to [AB87] and that yields Theorem 29 and a number of related results, is the derivation of differential inequalities for certain quantities related to the process. Using the graphical representation (described in

Section 3.1) to construct a (Λ,a,δ)-contact process and its dual, we define the *susceptibility* as

$$\chi=\chi(\Lambda,a,\delta)=\mathbb{E}\Big[\int_0^\infty|\eta^{\{0\}}_t|\,\mathrm{d}t\Big],\tag{A.3}$$

which may be $+\infty$. Moreover, letting $\omega^{\rm c}$ be a Poisson point process on $\Lambda\times\mathbb{R}$ with intensity $h\geq 0$, independent of the Poisson point processes $\omega^{\rm i}$ and $\omega^{\rm r}$ corresponding to infection arrows and recovery symbols, we define

$$\theta=\theta(\Lambda,a,\delta,h):=\mathbb{P}\big[C_{(0,0)}\cap\omega^{\rm c}\neq\emptyset\big]\quad\text{where}\quad C_{(i,s)}:=\big\{(j,t):t\geq s,\ (i,s)\rightsquigarrow(j,t)\big\}.\tag{A.4}$$

Then θ can be interpreted as the density of infected sites in the upper invariant law of a (dual) "$(\Lambda,a^\dagger,\delta,h)$-contact process", which in addition to the dynamics in (1.2) exhibits spontaneous infection of healthy sites with rate h, corresponding to a term in the generator of the form $h\sum_i\{f(A\cup\{i\})-f(A)\}$.

Let Λ,a,δ be fixed and for $\lambda,h\geq 0$ let $\theta=\theta(\lambda,h):=\theta(\Lambda,\lambda a,\delta,h)$ and $\chi=\chi(\lambda):=\chi(\Lambda,\lambda a,\delta)$ be the quantities defined above. The analysis in [AJ07] centers on the deriviation of the following three differential inequalities (see [AJ07, formulas (1.17), (1.19) and (1.20)])

$$\begin{array}{rl}\text{(i)} & \frac{\partial}{\partial\lambda}\chi\leq|a|\chi^2,\\ \text{(ii)} & \frac{\partial}{\partial\lambda}\theta\leq|a|\theta\frac{\partial}{\partial h}\theta,\\ \text{(iii)} & \theta\leq h\frac{\partial}{\partial h}\theta+\big(2\lambda^2|a|\theta+h\lambda\big)\frac{\partial}{\partial\lambda}\theta+\theta^2.\end{array}\tag{A.5}$$

These differential inequalities, and their proofs, generalize without a change to the more general class of (Λ,a,δ)-contact processes discussed in this appendix.

Since $\theta\geq h(1+h)$, which follows by estimating the $(\Lambda,\lambda a^\dagger,\delta,h)$-contact process from below by a process with no infections, one has $h\leq\theta(1-\theta)$. Inserting this into (A.5) (iii) yields

$$\theta\leq h\frac{\partial}{\partial h}\theta+\Big(2\lambda^2|a|+\frac{\lambda}{1-\theta}\Big)\theta\frac{\partial}{\partial\lambda}\theta+\theta^2.\tag{A.6}$$

Abstract results of Aizenman and Barsky [AB87, Lemmas 4.1 and 5.1] allow one to draw the following conclusions from (A.5) (ii) and (A.6).

Lemma 30 (Estimates on critical exponents) *Assume that there exists some $\lambda'>0$ such that $\theta(\lambda',0)=0$ and $\lim_{h\to0}h^{-1}\theta(\lambda',h)=\infty$. Then there exist $c_1,c_2>0$ such that*

$$\begin{array}{rlr}\text{(i)} & \theta(\lambda',h)\geq c_1h^{1/2} & (h\geq 0),\\ \text{(ii)} & \theta(\lambda,0)\geq c_2(\lambda-\lambda') & (\lambda\geq\lambda').\end{array}\tag{A.7}$$

Note that this lemma (in particular, formula (A.7) (i), which depends on the assumption that $\lim_{h\to0}h^{-1}\theta(\lambda',h)=\infty$) implies in particular that if for some fixed $\lambda'>0$, one has $\theta(\lambda',h)\sim h^\alpha$ as $h\to0$, then either $\alpha\leq\frac{1}{2}$ or $\alpha\geq 1$.

Remark Lemmas 4.1 and 5.1 of [AB87] are also cited in [AJ07, Thm. 4.1], but there the statement that $c_1,c_2>0$ is erroneously replaced by the (empty) statement that $c_1,c_2<\infty$.

Proof of Theorem 29 (sketch) Set

$$\begin{aligned}\lambda_{\rm c}&:=\inf\{\lambda\geq 0:\theta(\lambda,0)>0\},\\ \lambda'_{\rm c}&:=\inf\{\lambda\geq 0:\chi(\lambda)=\infty\}.\end{aligned}\tag{A.8}$$

Since $\chi(\lambda) < \infty$ implies $\theta(\lambda, 0) = 0$, obviously $\lambda'_c \leq \lambda_c$. Our first aim is to show that they are in fact equal. We note that it is always true that $\lambda'_c > 0$. It may happen that $\lambda'_c = \infty$ but in this case also $\lambda_c = \infty$ so without loss of generality we may assume that $\lambda'_c < \infty$.

It follows from (A.5) (i) and approximation of infinite systems by finite systems (compare [AN84, Lemma 3.1], which is written down for unoriented percolation and which is cited in [AJ07, formula (1.18)]) that $\lim_{\lambda \uparrow \lambda'_c} \chi(\lambda) = \chi(\lambda'_c) = \infty$, and in fact

$$\chi(\lambda) \geq \frac{|a|^{-1}}{\lambda'_c - \lambda} \qquad (\lambda < \lambda'_c). \tag{A.9}$$

Now either $\theta(\lambda'_c, 0) > 0$, in which case we are done, or $\theta(\lambda'_c, 0) = 0$. In the latter case, since

$$\chi(\lambda) = \lim_{h \to 0} h^{-1} \theta(\lambda, h) \qquad (\lambda < \lambda'_c), \tag{A.10}$$

(see [AJ07, formula (1.11)]), using the monotonicity of θ in λ and h, it follows from (A.9) that

$$\lim_{h \to 0} h^{-1} \theta(\lambda'_c, h) = \infty \tag{A.11}$$

and therefore Lemma 30 implies that (A.7) holds at $\lambda' = \lambda'_c$. In particular, (A.7) (ii) implies that $\theta(\lambda, 0) > 0$ for $\lambda > \lambda'_c$, hence $\lambda_c = \lambda'_c$.

Since by a trivial rescaling of time, questions about critical values for λ can always be translated into questions about critical values for δ, we learn from this that for any (Λ, a, δ)-contact process, one has $\chi(\Lambda, a, \delta) < \infty$ if $\delta > \delta_c(\Lambda, a)$, where the latter critical point is defined in (1.6). It follows from (2.34) that $\chi(\Lambda, a, \delta) = \infty$ if $r(\delta) = r(\Lambda, a, \delta) \geq 0$, hence we must have $r(\delta) < 0$ for $\delta \in (\delta_c, \infty)$. Part (b) of Theorem 0 is easily generalized to the class of (Λ, a, δ)-contact processes considered in this appendix. Moreover, it is not hard to prove that $r < 0$ implies that the process does not survive. This shows that $r(\delta) \geq 0$ on $[0, \delta_c)$ while $\delta \mapsto r(\delta)$ is continuous, which allows us to conclude that $\{\delta \geq 0 : r(\delta) < 0\} = (\delta_c, \infty)$ if $\delta_c > 0$. If $\delta_c = 0$ (which may happen for the general class of models considered here), then we may use the fact that $\theta(\Lambda, a, 0) = 1$ to conclude that $r(\Lambda, a, 0) \geq 0$, hence the conclusion of Theorem 29 is also valid in this case. ∎

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