Subcritical contact processes seen from a typical infected site

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Abstract

What is the long-time behavior of the law of a contact process started with a single infected site, distributed according to counting measure on the lattice? This question is related to the configuration as seen from a typical infected site and gives rise to the definition of so-called eigenmeasures, which are possibly infinite measures on the set of nonempty configurations that are preserved under the dynamics up to a multiplicative constant. In this paper, we study eigenmeasures of contact processes on general countable groups in the subcritical regime. We prove that in this regime, the process has a unique spatially homogeneous eigenmeasure. As an application, we show that the exponential growth rate is continuously differentiable and strictly decreasing as a function of the recovery rate, and we give a formula for the derivative in terms of the eigenmeasures of the contact process and its dual.

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1 Introduction and main results

1.1 Introduction

It is known that contact processes on regular trees behave quite differently from contact processes on the d-dimensional integer lattice $\mathbb{Z}^d$. Indeed, if λ_c and λ'_c denote the critical infection rates associated with global and local survival, respectively, then one has $\lambda_c < \lambda'_c$ on trees while $\lambda_c = \lambda'_c$ on $\mathbb{Z}^d$. For $\lambda > \lambda'_c$, the process exhibits complete convergence and the upper invariant law is the only nontrivial invariant law, while on trees, in the intermediate regime $\lambda_c < \lambda \leq \lambda'_c$, there is a multitude of (not spatially homogeneous) invariant laws. The situation is reminiscent of what is known about unoriented percolation on transitive graphs, where one has uniqueness of the infinite cluster if the graph is amenable, while it is conjectured, and proved in some cases, that on nonamenable graphs there is an intermediate parameter regime with infinitely many infinite clusters. We refer to [Lig99] as a general reference to contact processes on $\mathbb{Z}^d$ and trees and [Hag11] for percolation beyond $\mathbb{Z}^d$.

In general, it is not hard (but also not very interesting) to determine the limit behavior of contact processes started from a spatially homogeneous (i.e., translation invariant) initial law. On the other hand, it seems much more difficult to study the process started with a finite number of infected sites. For example, it seems quite difficult to prove that $\lambda_c = \lambda'_c$ on any amenable transitive graph. As an intermediate problem, in [Swa09, Problem 1 from Section 1.5], it has been proposed to study the process started with a single infected site, chosen uniformly from the lattice. For infinite lattices, the resulting 'law' at time t will be an infinite measure. However, as shown in [Swa09, Lemma 4.2], conditioning such a measure on the origin being infected yields a probability law, which can be interpreted as the process seen from a typical infected site.

There is a close connection between the law of the process seen from a typical infected site and the exponential growth rate r of the expected number of infected sites of a contact process. This can be understood by realizing that the number of healthy sites surrounding a typical infected site determines the number of infections that can be made and hence the speed at which the infection grows. In the context of infinite laws, which cannot be normalized, it is natural to generalize the concept of an invariant measure to an 'eigenmeasure', which is a measure on the set of nonempty configurations that is preserved under time evolution up to a multiplicative constant. Alternatively, such eigenmeasures can be thought of as the equivalent of a quasi-stationary law (as introduced in [DS67]) in the setting of interacting particle systems. In particular, if the suitably rescaled 'law' at time t of the process started with a single, uniformly distributed site has a nontrivial long-time limit, then it follows from results in [Swa09] that such a limit 'law' must be an eigenmeasure whose eigenvalue is the exponential growth rate r of the process.

In the present paper, we study eigenmeasures of subcritical contact processes on general countable groups. Our set-up includes translation-invariant contact processes on $\mathbb{Z}^d$ and on regular trees, as well as long-range processes and asymmetric processes. We will show that such processes have a unique homogeneous eigenmeasure which is the vague limit of the rescaled law at time t of the process started in any homogeneous, possibly infinite, initial law. As an application of our results, we give an expression for the derivative of the exponential growth rate as a function of the recovery rate in terms of the eigenmeasures of the process and its dual, and we use this to show that this derivative is strictly negative and continuous.

1.2 Contact processes on groups

We need to define the class of contact processes that we will be interested in, fix notation, and recall some well-known facts. Let Λ be a finite or countably infinite group with group action $(i,j)\mapsto ij$, inverse operation $i\mapsto i^{-1}$, and unit element 0 (also refered to as the origin). Let $a:\Lambda\times\Lambda\to[0,\infty)$ be a function such that $a(i,i)=0$ $(i\in\Lambda)$ and

$$\begin{array}{lll}\text{(i)} & a(i,j)=a(ki,kj) & (i,j,k\in\Lambda),\\ \text{(ii)} & |a|:=\displaystyle\sum_{i\in\Lambda}a(0,i)<\infty, & \end{array}\tag{1.1}$$

and let $\delta\geq 0$. By definition, the (Λ,a,δ)-*contact process* is the Markov process $\eta=(\eta_t)_{t\geq 0}$, taking values in the space $\mathcal{P}=\mathcal{P}(\Lambda):=\{A:A\subset\Lambda\}$ consisting of all subsets of Λ, with the formal generator

$$\begin{aligned}Gf(A):=&\sum_{i,j\in\Lambda}a(i,j)1_{\{i\in A\}}1_{\{j\notin A\}}\{f(A\cup\{j\})-f(A)\}\\ &+\delta\sum_{i\in\Lambda}1_{\{i\in A\}}\{f(A\backslash\{i\})-f(A)\}.\end{aligned}\tag{1.2}$$

If $i\in\eta_t$, then we say that the site i is infected at time t; otherwise it is healthy. Then (1.2) says that an infected site i infects another site j with *infection rate* $a(i,j)\geq 0$, and infected sites become healthy with *recovery rate* $\delta\geq 0$.

We will usually assume that the infection rates are irreducible in some sense or another. To make this precise, let us write $i\stackrel{a}{\hookrightarrow}j$ if the site j can be infected through a chain of infections starting from i. Then we say that a is *irreducible* if $i\stackrel{a}{\hookrightarrow}j$ for all $i,j\in\Lambda$. Equivalently, this says that for all $\Lambda'\subset\Lambda$ with $\Lambda'\neq\emptyset,\Lambda$, there exist $i\in\Lambda'$ and $j\in\Lambda\backslash\Lambda'$ such that $a(i,j)>0$. Similarly, we say that a is *weakly irreducible* if for all $\Lambda'\subset\Lambda$ with $\Lambda'\neq\emptyset,\Lambda$, there exist $i\in\Lambda'$ and $j\in\Lambda\backslash\Lambda'$ such that $a(i,j)\vee a(j,i)>0$. Finally, we will sometimes need the intermediate condition

$$\forall i,j\in\Lambda:\ \exists k,l\in\Lambda:\ k\stackrel{a}{\hookrightarrow}i,\ k\stackrel{a}{\hookrightarrow}j,\ i\stackrel{a}{\hookrightarrow}l,\ j\stackrel{a}{\hookrightarrow}l.\tag{1.3}$$

In words, this says that for any two sites i,j there exists a site k from which both i and j can be infected, and a site l that can be infected both from i and from j. If the rates a are symmetric, or more generally if one has $a(i,j)>0$ iff $a(j,i)>0$, then all three conditions are equivalent. In general, irreducibility implies (1.3) which implies weak irreducibility, but none of the converse implications hold.

It is well-known that contact processes can be constructed by a graphical representation. Let $\omega=(\omega^{\rm r},\omega^{\rm i})$ be a pair of independent, locally finite random subsets of $\Lambda\times\mathbb{R}$ and $\Lambda\times\Lambda\times\mathbb{R}$, respectively, produced by Poisson point processes with intensity δ and $a(i,j)$, respectively. This is usually visualized by plotting Λ horizontally and $\mathbb{R}$ vertically, marking points $(i,s)\in\omega^{\rm r}$ with a recovery symbol (e.g., $*$), and drawing an infection arrow from (i,t) to (j,t) for each $(i,j,t)\in\omega^{\rm i}$. For any $(i,s),(j,u)\in\Lambda\times\mathbb{R}$ with $s\leq u$, by definition, an *open path* from (i,s) to (j,u) is a cadlag function $\pi:[s,u]\to\Lambda$ such that $\{(\pi(t),t):t\in[s,u]\}\cap\omega^{\rm r}=\emptyset$ and $(\pi(t-),\pi(t),t)\in\omega^{\rm i}$ whenever $\pi(t-)\neq\pi(t)$. Thus, open paths must avoid recovery symbols and may follow infection arrows. We write $(i,s)\rightsquigarrow(j,u)$ to indicate the presence of an open path from (i,s) to (j,u). Then, for any $s\in\mathbb{R}$, we can construct a (Λ,a,δ)-contact process started in an initial state $A\in\mathcal{P}$ by setting

$$\eta^{A,s}_t:=\{j\in\Lambda:(i,s)\rightsquigarrow(j,s+t)\text{ for some }i\in A\}\qquad(A\in\mathcal{P},\ s\in\mathbb{R},\ t\geq 0).\tag{1.4}$$

In particular, we set $\eta^A_t := \eta^{A,0}_t$. Note that this construction defines contact processes with different initial states on the same probability space, i.e., the graphical representation provides a natural coupling between such processes. Moreover, the graphical representation shows that the contact process is essentially a sort of oriented percolation model (in continuous time but discrete space).

Since the graphical representation is also defined for negative times we can, in analogy to (1.4), define 'backward' or 'dual' processes by

$$\eta^{\dagger\,A,s}_t := \{j \in \Lambda : (j, s-t) \rightsquigarrow (i,s) \text{ for some } i \in A\} \qquad (A \in \mathcal{P},\ s \in \mathbb{R},\ t \geq 0). \tag{1.5}$$

In particular, we set $\eta^{\dagger\,A}_t := \eta^{\dagger\,A,0}_t$. It is not hard to see that $(\eta^{\dagger\,A,s}_t)_{t\geq 0}$ is a $(\Lambda, a^\dagger, \delta)$-contact process, where we define *reversed infection rates* as $a^\dagger(i,j) := a(j,i)$. Since

$$\big\{\eta^A_t \cap B \neq \emptyset\big\} = \big\{(i,0) \rightsquigarrow (j,t) \text{ for some } i \in A,\ j \in B\big\} = \big\{\eta^A_0 \cap \eta^{\dagger\,B,t}_t \neq \emptyset\big\} \qquad (0 \leq s \leq t) \tag{1.6}$$

and the process $\eta^{\dagger\,B,t}$ is equal in law with $\eta^{\dagger\,B}$, we see that the (Λ, a, δ)-contact process and $(\Lambda, a^\dagger, \delta)$-contact process are dual in the sense that

$$\mathbb{P}[\eta^A_t \cap B \neq \emptyset] = \mathbb{P}[A \cap \eta^{\dagger\,B}_t \neq \emptyset] \qquad (A, B \in \mathcal{P},\ t \geq 0). \tag{1.7}$$

We note that unless $a = a^\dagger$ or the group Λ is abelian, the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact processes have in general different dynamics and need to be distinguished. (If Λ is abelian, then the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact processes can be mapped into each other by the transformation $i \mapsto i^{-1}$.)

We say that the (Λ, a, δ)-contact process *survives* if $\mathbb{P}[\eta^A_t \neq \emptyset\ \forall t \geq 0] > 0$ for some, and hence for all nonempty A of finite cardinality $|A|$. We call

$$\delta_{\rm c} = \delta_{\rm c}(\Lambda, a) := \sup\big\{\delta \geq 0 :\ \text{the } (\Lambda, a, \delta)\text{-contact process survives}\big\} \tag{1.8}$$

the *critical recovery rate*. It is known that $\delta_{\rm c} < \infty$. If Λ is finitely generated, then moreover $\delta_{\rm c} > 0$ provided a is weakly irreducible [Swa07, Lemma 4.18], but for non-finitely generated groups irreducibility is in general not enough to guarantee $\delta_{\rm c} > 0$ [AS10]. It is well-known that

$$\mathbb{P}\big[\eta^\Lambda_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}, \tag{1.9}$$

where $\overline{\nu}$ is an invariant law of the (Λ, a, δ)-contact process, known as the *upper invariant law.* Using duality, it is not hard to prove that $\overline{\nu} = \delta_\emptyset$ if the $(\Lambda, a^\dagger, \delta)$-contact process dies out, while $\overline{\nu}$ is concentrated on the nonempty subsets of Λ if the process survives.

It follows from subadditivity (see [Swa09, Lemma 1.1]) that any (Λ, a, δ)-contact process has a well-defined exponential growth rate, i.e., there exists a constant $r = r(\Lambda, a, \delta)$ with $-\delta \leq r \leq |a| - \delta$ such that

$$r = \lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}\big[|\eta^A_t|\big] \qquad (0 < |A| < \infty). \tag{1.10}$$

In this article, we are concerned with subcritical contact processes for which $r < 0$. The following theorem lists some properties of the function $r(\Lambda, a, \delta)$.

Theorem 0 (Properties of the exponential growth rate)
For any (Λ, a, δ)-contact process:

(a) $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$.

(b) *The function* $\delta \to r(\Lambda, a, \delta)$ *is nonincreasing and Lipschitz continuous on* $[0,\infty)$, *with Lipschitz constant 1.*

(c) *If* $r(\Lambda, a, \delta) > 0$, *then the* (Λ, a, δ)*-contact process survives.*

(d) $\{\delta \geq 0 : r(\Lambda, a, \delta) < 0\} = (\delta_c, \infty)$.

The (easy) proofs of parts (a)–(c) can be found in [Swa09, Theorem 1.2]. The analogue of part (d) for unoriented percolation on $\mathbb{Z}^d$ was first proved by Menshikov [Men86] and Aizenman and Barsky [AB87]. Using the approach of the latter paper, Bezuidenhout and Grimmett [BG91, formula (1.13)] proved the statement in part (d) for contact processes on $\mathbb{Z}^d$. This has been generalized to processes on general transitive graphs in [AJ07]. As we point out in Appendix A, their arguments are not restricted to graphs but apply in the generality we need here. We note that it follows from parts (a) and (d) that $\delta_c(\Lambda, a) = \delta_c(\Lambda, a^\dagger)$. In general, it is not known if survival of a (Λ, a, δ)-contact process implies survival of the dual $(\Lambda, a^\dagger, \delta)$-contact process but any counterexample would have to be at $\delta = \delta_c$, while by [Swa09, Corollary 1.3], Λ would have to be amenable. If Λ is a finitely generated group of subexponential growth and the infection rates satisfy an exponential moment condition (for example, if $\Lambda = \mathbb{Z}^d$ and a is nearest-neighbor), then $r \leq 0$ [Swa09, Thm 1.2 (e)], but in general (e.g. on trees), it is possible that $r > 0$. Indeed, one of the main results of [Swa09] says that if Λ is nonamenable, the (Λ, a, δ)-contact process survives, and the infection rates satisfy the irreducibility condition (1.3), then $r > 0$ [Swa09, Thm. 1.2 (f)].

1.3 Locally finite starting measures

We will be interested in the contact process started in initial 'laws' that are infinite measures. To do this properly, we need a bit of theory. Recall that $\mathcal{P} = \mathcal{P}(\Lambda)$ denotes the space of all subsets of Λ. We let $\mathcal{P}_+ := \{A : |A| > 0\}$ and $\mathcal{P}_{\rm fin} := \{A : |A| < \infty\}$ denote the subspaces consisting of all nonempty, respectively finite subsets of Λ, and write $\mathcal{P}_{\rm fin,+} := \mathcal{P}_{\rm fin} \cap \mathcal{P}_+$. We observe that $\mathcal{P} \cong \{0,1\}^\Lambda$ and equip it with the product topology and Borel-σ-field. Note that since $\mathcal{P}$ is compact, $\mathcal{P}_+ = \mathcal{P} \backslash \{\emptyset\}$ is a locally compact space. Recall that a measure on a locally compact space is *locally finite* if it gives finite mass to compact sets, and that a sequence of locally finite measures converges vaguely if the integrals of all compactly supported, continuous functions converge. We cite the following simple facts from [Swa09, Lemmas 3.1 and 3.2].

Lemma 1.1 (Locally finite measures) *Let* μ *be a measure on* $\mathcal{P}_+$. *Then the following statements are equivalent:*

(i) μ *is locally finite.*

(ii) $\int \mu(\mathrm{d}A) 1_{\{i \in A\}} < \infty$ *for all* $i \in \Lambda$.

(iii) $\int \mu(\mathrm{d}A) 1_{\{A \cap B \neq \emptyset\}} < \infty$ *for all* $B \in \mathcal{P}_{\rm fin,+}$.

Moreover, if μ_n, μ *are locally finite measures on* $\mathcal{P}_+$, *then the* μ_n *converge vaguely to* μ *if and only if*

$$\int \mu_n(\mathrm{d}A) 1_{\{A \cap B \neq \emptyset\}} \underset{n\to\infty}{\longrightarrow} \int \mu(\mathrm{d}A) 1_{\{A \cap B \neq \emptyset\}} \qquad (B \in \mathcal{P}_{\rm fin,+}). \tag{1.11}$$

We will sometimes deal with locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\rm fin}$. We will refer to such measures as 'locally finite measures on $\mathcal{P}_{{\rm fin},+}$' (even though 'locally finite' refers to the topology on $\mathcal{P}_+$). For such measures, we will sometimes need another, stronger form of convergence than vague convergence. For each $i\in\Lambda$, we define

$$\mathcal{P}_i:=\{A\in\mathcal{P}:i\in A\}\quad\text{and}\quad\mathcal{P}_{{\rm fin},\,i}:=\mathcal{P}_{\rm fin}\cap\mathcal{P}_i. \tag{1.12}$$

Note that $\mathcal{P}_{{\rm fin},\,i}$ is a countable set. We let $\mu|_{\mathcal{P}_{{\rm fin},\,i}}$ denote the restriction of a measure μ to $\mathcal{P}_{{\rm fin},\,i}$. If μ_n,μ are locally finite measures on $\mathcal{P}_{{\rm fin},+}$, then we say that the μ_n converge to μ *locally on* $\mathcal{P}_{{\rm fin},+}$, if for each $i\in\Lambda$, the $\mu_n|_{\mathcal{P}_{{\rm fin},\,i}}$ converge weakly to $\mu|_{\mathcal{P}_{{\rm fin},\,i}}$ with respect to the discrete topology on $\mathcal{P}_{{\rm fin},\,i}$. It can be shown that local convergence on $\mathcal{P}_{{\rm fin},+}$ implies vague convergence (see Proposition 2.1 below), but the converse is not true. For example, if $\Lambda=\mathbb{Z}$, then using Lemma 1.1 it is not hard to see that we have the vague convergence

$$\sum_{i\in\mathbb{Z}}\delta_{\{i,i+n\}}=:\mu_n\underset{n\to\infty}{\Longrightarrow}\mu:=2\sum_{i\in\mathbb{Z}}\delta_{\{i\}}, \tag{1.13}$$

(where δ_A denotes the delta-measure at a point $A\in\mathcal{P}_+$) but the μ_n do not converge locally on $\mathcal{P}_{{\rm fin},+}$.

We now turn our attention to contact processes started in infinite initial 'laws'. For a given (Λ,a,δ)-contact process, we define subprobability kernels P_t $(t\geq 0)$ on $\mathcal{P}_+$ by

$$P_t(A,\,\cdot\,):=\mathbb{P}\big[\eta^A_t\in\cdot\,\big]\big|_{\mathcal{P}_+}\qquad(t\geq 0), \tag{1.14}$$

where $|_{\mathcal{P}_+}$ denotes restriction to $\mathcal{P}_+$, and we define $P^\dagger_t$ similarly for the dual $(\Lambda,a^\dagger,\delta)$-contact process. For any measure μ on $\mathcal{P}_+$, we write

$$\mu P_t:=\int\mu(\mathrm{d}A)P_t(A,\,\cdot\,)\qquad(t\geq 0), \tag{1.15}$$

which is the restriction to $\mathcal{P}_+$ of the 'law' at time t of the (Λ,a,δ)-contact process started in the initial (possibly infinite) 'law' μ.

For $A\subset\Lambda$ and $i\in\Lambda$, we write $iA:=\{ij:j\in A\}$, and for any $\mathcal{A}\subset\mathcal{P}$ we write $i\mathcal{A}:=\{iA:A\in\mathcal{A}\}$. We say that a measure μ on $\mathcal{P}$ is (spatially) *homogeneous* if it is invariant under the left action of the group, i.e., if $\mu(\mathcal{A})=\mu(i\mathcal{A})$ for each $i\in\Lambda$ and measurable $\mathcal{A}\subset\mathcal{P}$. If μ is a homogeneous, locally finite measure on $\mathcal{P}_+$, then μP_t is a homogeneous, locally finite measure on $\mathcal{P}_+$ for each $t\geq 0$ (see [Swa09, Lemma 3.3] or Lemma 2.4 below).

For processes started in homogeneous, locally finite measures, we have a useful sort of analogue of the duality formula (1.7). To formulate this, we need two more definitions. For any measure μ on $\mathcal{P}_+$, we define

$$\langle\!\langle\mu\rangle\!\rangle:=\int\mu(\mathrm{d}A)|A|^{-1}1_{\{0\in A\}}, \tag{1.16}$$

where $|A|^{-1}:=0$ if A is infinite. Note that if each set $A\in\mathcal{P}_{{\rm fin},+}$ carries mass $\mu(\{A\})$, and this mass is distributed evenly among all points in A, then $\langle\!\langle\mu\rangle\!\rangle$ is the mass received at the origin.

Next, for any measures μ,ν on $\mathcal{P}_+$, we let $\mu\circledast\nu$ denote the restriction to $\mathcal{P}_+$ of the image of the product measure $\mu\otimes\nu$ under the map $(A,B)\mapsto A\cap B$. Note that

$$\int\mu\circledast\nu\,(\mathrm{d}C)f(C):=\int\mu(\mathrm{d}A)\int\nu(\mathrm{d}B)f\big(A\cap B\big) \tag{1.17}$$

for any bounded measurable $f : \mathcal{P} \to \mathbb{R}$ satisfying $f(\emptyset) = 0$. We call $\mu \boxtimes \nu$ the *intersection measure* of μ and ν. It is not hard to show (see Lemma 2.2 below) that $\mu \boxtimes \nu$ is locally finite if μ and ν are. Note that if μ and ν are probability measures, then $\mu \boxtimes \nu$ is the law of the intersection of two independent random sets with laws μ and ν, restricted to the event that this intersection is nonempty. In particular, normalizing $\mu \boxtimes \nu$ yields the conditional law given this event.

With these definitions, we have the following lemma, the proof of which can be found in Section 3.2.

Lemma 1.2 (Duality for infinite initial laws) *Let μ, ν be homogeneous, locally finite measures on $\mathcal{P}_+$. Then*

$$\langle\!\langle \mu P_t \boxtimes \nu \rangle\!\rangle = \langle\!\langle \mu \boxtimes \nu P_t^\dagger \rangle\!\rangle \qquad (t \geq 0), \tag{1.18}$$

and $\mu P_t \boxtimes \nu$ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$ if and only if $\mu \boxtimes \nu P_t^\dagger$ is.

Remark If $|\mu| := \mu(\mathcal{P}_+)$ denotes the total mass of a finite measure on $\mathcal{P}_+$, then the duality formula (1.7) is easily seen to imply that $|\mu P_t \boxtimes \nu| = |\mu \boxtimes \nu P_t^\dagger|$ for any finite measures μ, ν on $\mathcal{P}_+$. One can think of (1.18) as an analogue of this for infinite (but homogeneous) measures.

1.4 Eigenmeasures

Following [Swa09], we say that a measure μ on $\mathcal{P}_+$ is an *eigenmeasure* of the (Λ, a, δ)-contact process if μ is nonzero, locally finite, and there exists a constant $\lambda \in \mathbb{R}$ such that

$$\mu P_t = e^{\lambda t} \mu \qquad (t \geq 0). \tag{1.19}$$

We call λ the associated *eigenvalue.*

It follows from [Swa09, Prop. 1.4] that each (Λ, a, δ)-contact process has a homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue $r = r(\Lambda, a, \delta)$. In general, it is not known if $\mathring{\nu}$ is (up to a multiplicative constant) unique. Under the irreducibility condition (1.3), it has been shown in [Swa09, Thm. 1.5] that if the upper invariant measure $\overline{\nu}$ of a (Λ, a, δ)-contact process is concentrated on $\mathcal{P}_+$ and $r(\Lambda, a, \delta) = 0$, then $\mathring{\nu}$ is unique up to a multiplicative constant and in fact $\mathring{\nu} = c\,\overline{\nu}$ for some $c > 0$. The main aim of the present paper is to investigate eigenmeasures in the subcritical case $r < 0$. Here is our first main result.

Theorem 1 (Eigenmeasures in the subcritical case) *Assume that the infection rates satisfy the irreducibility condition (1.3) and that the exponential growth rate from (1.10) satisfies $r < 0$. Then there exist, up to multiplicative constants, unique homogeneous eigenmeasures $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ of the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact processes, respectively. These eigenmeasures have eigenvalue r and are concentrated on $\mathcal{P}_{\mathrm{fin}}$. If μ is any nonzero, homogeneous, locally finite measure on $\mathcal{P}_+$, then*

$$e^{-rt} \mu P_t \underset{t\to\infty}{\Longrightarrow} c\,\mathring{\nu}, \tag{1.20}$$

where $\Rightarrow$ denotes vague convergence of locally finite measures on $\mathcal{P}_+$ and $c > 0$ is a constant, given by

$$c = \frac{\langle\!\langle \mu \boxtimes \mathring{\nu}^\dagger \rangle\!\rangle}{\langle\!\langle \mathring{\nu} \boxtimes \mathring{\nu}^\dagger \rangle\!\rangle}. \tag{1.21}$$

If μ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$, then (1.20) holds in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$.

The proof of Theorem 1.20 will be completed in Section 2.5.

Remark Since $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ are infinite measures, their normalizations are somewhat arbitrary. For definiteness, we will usually adopt the convention that $\int \mathring{\nu}(\mathrm{d}A)1_{\{0\in A\}} = 1 = \int \mathring{\nu}^\dagger(\mathrm{d}A)1_{\{0\in A\}}$. Theorem 1 holds regardless of the choice of normalization.

1.5 The process seen from a typical infected site

We next set out to explain the connection of eigenmeasures and the process as seen from a typical infected site, and formulate our second main result, which gives a formula for the derivative of the exponential growth rate.

Let $(\eta^{\{0\}}_t)_{t\geq 0}$ be a (Λ, a, δ)-contact process, started with a single infected site at the origin, where $\eta^{\{0\}}_t = \eta^{\{0\}}_t(\omega)$ is defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, for each $t \geq 0$, we can define a new probability law $\hat{\mathbb{P}}_t$ on a suitably enriched probability space $\hat{\Omega}$ that also contains a Λ-valued random variable ι, by setting

$$\hat{\mathbb{P}}_t\big[\omega \in \mathcal{A},\ \iota = i\big] := \frac{\mathbb{P}[\omega \in \mathcal{A},\ i \in \eta^{\{0\}}_t(\omega)]}{\mathbb{E}[|\eta^{\{0\}}_t|]} \qquad (\mathcal{A} \in \mathcal{F},\ i \in \Lambda). \tag{1.22}$$

The law $\hat{\mathbb{P}}_t$ is a Campbell law (closely related to the more well-known Palm laws). In words, $\hat{\mathbb{P}}_t$ is obtained from the original law $\mathbb{P}$ by size-biasing on the number $|\eta^{\{0\}}_t|$ of infected sites at time t and then choosing one site ι from $\eta^{\{0\}}_t$ with equal probabilities.

Let $\mu_t := \sum_{i\in\Lambda} \mathbb{P}[\eta^{\{i\}}_t \in \cdot\,]|_{\mathcal{P}_+}$ be the infinite 'law' of the process started with a single infection at a uniformly chosen site in the lattice. Then, defining conditional probabilities for infinite measures in the natural way, it has been shown in [Swa09, Lemma 4.2] that

$$\mu_t\big(\,\cdot\,\big|\,\{A : 0 \in A\}\big) = \hat{\mathbb{P}}_t\big[\iota^{-1}\eta^{\{0\}}_t \in \cdot\,\big], \tag{1.23}$$

i.e., μ_t conditioned on the origin being infected describes the distribution of $\eta^{\{0\}}_t$ under the Campbell law $\hat{\mathbb{P}}_t$ with the 'typical infected site' ι shifted to the origin.

In view of this, Theorem 1 gives information about the long-time limit law of the process seen from a typical infected site. Indeed, it is easy to see that Theorem 1 implies the weak convergence of the probability measures in (1.23) to $\mathring{\nu}(\,\cdot\,|\,\{A : 0 \in A\})$.

To see the connection of this with the derivative of the exponential growth rate, let $\eta^{\delta,\{0\}}_t$ denote the process with a given recovery rate δ (and (Λ, a) fixed), constructed with the graphical representation. A version of Russo's formula (see [Swa09, formula (3.10)] and compare [Gri99, Thm 2.25]) tells us that

$$-\frac{\partial}{\partial\delta}\frac{1}{t}\log\mathbb{E}\big[|\eta^{\delta,\{0\}}_t|\big] = \frac{1}{t}\int_0^t \hat{\mathbb{P}}_t\big[\exists j \in \Lambda \text{ s.t. } (0,0) \rightsquigarrow_{(j,s)} (\iota, t)\big]\mathrm{d}s, \tag{1.24}$$

where $(0,0) \rightsquigarrow_{(j,s)} (\iota, t)$ denotes the event that in the graphical representation, all open paths from $(0,0)$ to (ι, t) lead through (j,s). In other words, the right-hand side of (1.24) is the fraction of time that there is a *pivotal* site on the way from $(0,0)$ to the typical site (ι, t).

By grace of Theorem 1, we are able to control the long-time limit of formula (1.24), leading to the following result, whose proof will be completed at the end of Section 2.7.

Theorem 2 (Derivative of the exponential growth rate) *Assume that the infection rates satisfy the irreducibility condition (1.3). For* $\delta \in (\delta_{\rm c}, \infty)$*, let* $\mathring{\nu}_\delta$ *and* $\mathring{\nu}^\dagger_\delta$ *denote the homogeneous eigenmeasures of the* (Λ, a, δ)*- and* $(\Lambda, a^\dagger, \delta)$*-contact processes, respectively, normalized such that* $\int \mathring{\nu}_\delta(\mathrm{d}A) 1_{\{0 \in A\}} = 1 = \int \mathring{\nu}^\dagger_\delta(\mathrm{d}A) 1_{\{0 \in A\}}$*. Then the map* $(\delta_{\rm c}, \infty) \ni \delta \mapsto \mathring{\nu}_\delta$ *is continuous with respect to local convergence on* $\mathcal{P}_{\rm fin, +}$*, and similarly for* $\mathring{\nu}^\dagger_\delta$*. Moreover, the function* $\delta \mapsto r(\Lambda, a, \delta)$ *is continuously differentiable on* $(\delta_{\rm c}, \infty)$ *and satisfies*

$$-\frac{\partial}{\partial \delta} r(\Lambda, a, \delta) = \frac{\mathring{\nu}_\delta \boxtimes \mathring{\nu}^\dagger_\delta (\{0\})}{\langle\!\langle \mathring{\nu}_\delta \boxtimes \mathring{\nu}^\dagger_\delta \rangle\!\rangle} > 0 \qquad \big(\delta \in (\delta_{\rm c}, \infty)\big). \tag{1.25}$$

Remark The continuity of $\mathring{\nu}_\delta$ and $\mathring{\nu}^\dagger_\delta$ as a function of δ in the sense of local convergence on $\mathcal{P}_{\rm fin, +}$ is easily seen to imply the continuity of the right-hand side of (1.25) in δ. On the other hand, no such conclusion could be drawn from continuity in the sense of vague convergence, since the functions $A \mapsto 1_{\{A = \{0\}\}}$ and $A \mapsto |A|^{-1} 1_{\{0 \in A\}}$ (which occur in the definition of $\langle\!\langle \cdot \rangle\!\rangle$) are not continuous with respect to the topology on $\mathcal{P}_+$.

The differentiability of the exponential growth rate in the subcritical regime is expected. Indeed, for normal (unoriented) percolation in the subcritical regime, it is even known that the number of open clusters per vertex and the mean size of the cluster at the origin depend analytically on the percolation parameter. This result is due to Kesten [Kes81]; see also [Gri99, Section 6.4]. For oriented percolation in one plus one dimension in the *supercritical* regime, Durrett [Dur84, Section 14] has shown that the percolation probability is infinitely differentiable as a function of the percolation parameter. It is not immediately clear, however, if the methods in these papers can be adapted to cover the exponential growth rate. At any rate, they would not give very explicit information about the derivative such as positivity.

In principle, if for a given lattice one can show that the right-hand side of (1.25) stays positive uniformly as $\delta \downarrow \delta_{\rm c}$, then this would imply that $r(\delta) \sim (\delta - \delta_{\rm c})^1$ as $\delta \downarrow \delta_{\rm c}$, i.e., the critical exponent associated with the function r is one. But this is probably difficult in the most interesting cases, such as $\mathbb{Z}^d$ above the critical dimension.

1.6 Discussion and outlook

This paper is part of a larger program, initiated in [Swa09], which aims to describe all homogeneous eigenmeasures of (Λ, a, δ)-contact processes and to prove convergence for suitable starting measures. There are several regimes of interest: the subcritical regime $\delta > \delta_{\rm c}$, the critical regime $\delta = \delta_{\rm c}$, and the supercritical regime $\delta < \delta_{\rm c}$. In the supercritical regime one needs to distinguish further the case $r = 0$ (as for processes on $\mathbb{Z}^d$) and the case $r > 0$ (as for processes on trees).

In [Swa09] some first, relatively weak results have been derived for processes with $r = 0$ in the supercritical regime. In particular, it was shown that for such processes, there exists a unique homogeneous eigenmeasure with eigenvalue zero [Swa09, Thm. 1.5], but it has not been proved whether there are homogeneous eigenmeasures with other eigenvalues, while convergence has only been shown for one special initial measure and Laplace-transformed times [Swa09, Corollary 3.4].

Our present paper treats the subcritical case fairly conclusively. Arguably, this should be the easiest regime. Indeed, our analysis is made easier by the fact that the homogeneous eigenmeasures are concentrated on finite sets, which allows us to use a 'compensated' h-transform to translate problems related to long-time behavior into positive recurrence of a continuous-time

Markov chain (see Lemma 2.11 below). In contrast, in the critical and supercritical regimes, we expect homogeneous eigenmeasures to be concentrated on infinite sets, hence these techniques are not available.

Nevertheless, our methods give some hints on what to do in some of the other regimes as well. Formula (1.25), which we expect to hold more generally, says, roughly speaking, that $-\frac{\partial}{\partial\delta}r(\Lambda, a, \delta)$ is the probability that two independent sets, which are distributed according to the eigenmeasures $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ of the forward and dual (backward) process, and which are conditioned on having nonempty intersection, intersect in a single point. In view of this, it is tempting to try to replace the fact that $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ are each concentrated on finite sets, which holds only in the subcritical regime, by the weaker assumption that the intersection measure $\mathring{\nu} \boxtimes \mathring{\nu}^\dagger$ is concentrated on finite sets. In particular, one wonders if this always holds in the regime $r > 0$.

A simpler problem, which we have not pursued in the present paper, is to investigate higher-order derivatives of $r(\Lambda, a, \delta)$ with respect to δ or derivatives with respect to the infection rates $a(i, j)$. It seems likely that the latter are strictly positive in the subcritical regime and given by a formula similar to (1.25). Controlling higher-order derivatives of $r(\Lambda, a, \delta)$ might be more difficult; in particular, we do not know if the function $\delta \mapsto r(\Lambda, a, \delta)$ is concave, or (which in view of (1.25) is a similar question), if the conditional laws $\mathring{\nu}_\delta(\,\cdot\,|\{A : 0 \in A\})$ are decreasing in the stochastic order, as a function of δ.

2 Main line of the proofs

In this section we give an overview of the main line of our arguments. In particular, we give the proofs of Theorems 1 and 2 in Sections 2.5 and 2.7 respectively. These proofs are based on a collection of lemmas and propositions which are stated here but whose proofs are in most cases postponed until later.

In short, the line of the arguments is as follows. We start in Section 2.1 by collecting some general facts about locally finite measures on $\mathcal{P}_+$. In particular, we discuss the relation between vague and local convergence, and we show that a homogeneous, locally finite measure on $\mathcal{P}_{\mathrm{fin},+}$ can be seen as the 'law' of a random finite set, shifted to a uniformly chosen position in the lattice.

In Section 2.2, we then prove the existence part of Theorem 1. Since existence of an eigenmeasure with eigenvalue r has already been proved in [Swa09], the main task is proving that there exists such an eigenmeasure that is moreover concentrated on $\mathcal{P}_{\mathrm{fin},+}$. This is achieved by a covariance calculation.

Once existence is proved, we fix an eigenmeasure $\mathring{\nu}$ that is concentrated on $\mathcal{P}_{\mathrm{fin},+}$, and likewise $\mathring{\nu}^\dagger$ for the dual process, and set out to prove the convergence in (1.20), which will then also settle uniqueness. Our strategy is to reduce the problem to the ergodicity of an irreducible, positively recurrent Markov chain.

To this aim, in Section 2.3, we transform contact processes started in finite initial states into processes that cannot die out by means of a Doob transform based on the h-function $h(A) = \int \mathring{\nu}^\dagger(dB)1_{\{A\cap B\neq\emptyset\}}$. In Section 2.4, we then show that the eigenmeasure $\mathring{\nu}$ corresponds to an invariant law for this Doob transformed process modulo shifts, and that the latter is an irreducible, positively recurrent Markov process with countable state space. For this argument, it is essential that $\mathring{\nu}$ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$.

In Section 2.5, we then use this to prove the convergence in (1.20), completing the proof of Theorem 1. We obtain vague convergence for general starting measures by duality, using the ergodicity of the Doob transform of the dual $(\Lambda, a^\dagger, \delta)$-contact process modulo shifts. For starting measures that are concentrated on $\mathcal{P}_{\mathrm{fin},+}$, we moreover obtain pointwise convergence by using the ergodicity of the Doob transformed (forward) (Λ, a, δ)-contact process modulo shifts, which together with vague convergence, by a general lemma from Section 2.1, implies local convergence on $\mathcal{P}_{\mathrm{fin},+}$.

In order to prove Theorem 2, in Section 2.6 we show continuity of the eigenmeasures $\mathring{\nu}$ in the recovery rate δ. Continuity in the sense of vague convergence follows easily from a compactness argument and uniqueness, but continuity in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$ requires more work. We use a generalization of the covariance calculation from Section 2.2 to obtain 'local tightness', which together with vague convergence, by a general lemma from Section 2.1, implies local convergence on $\mathcal{P}_{\mathrm{fin},+}$.

In Section 2.7, finally, we use the results proved so far to take the limit $t \to \infty$ in Russo's formula (1.24) and prove formula (1.25), thereby completing the proof of Theorem 2.

At this point, the proofs of our main results are complete, but they depend on a number of lemmas and propositions the proofs of which have for readability been postponed until later. We supply these in Section 3. The paper concludes with two appendices. In Appendix A we point out how the arguments in [AJ07] generalize to the class of contact processes considered in the present article. Appendix B contains a simple fact about continuous-time Markov chains used in the construction of the Doob transformed process.

2.1 More on locally finite measures

In this section, we elaborate on the discussion in Section 1.3 of (contact processes started in) locally finite measures on $\mathcal{P}_+$ by formulating some lemmas that will be useful in what follows.

Recall from Section 1.3 the definition of vague convergence and of local convergence on $\mathcal{P}_{\mathrm{fin},+}$, and recall that $\mathcal{P}_{\mathrm{fin},i} := \{A \in \mathcal{P}_{\mathrm{fin}} : i \in A\}$. If μ_n, μ are measures on $\mathcal{P}_{\mathrm{fin},+}$, then we say the μ_n converge to μ *pointwise on* $\mathcal{P}_{\mathrm{fin},+}$ if $\mu_n(\{A\}) \to \mu(\{A\})$ for all $A \in \mathcal{P}_{\mathrm{fin},+}$. We say that the $(\mu_n)_{n\geq 1}$ are *locally tight* if for each $i \in \Lambda$ and $\varepsilon > 0$ there exists a finite $\mathcal{D} \subset \mathcal{P}_{\mathrm{fin},i}$ such that $\sup_n \mu_n(\mathcal{P}_{\mathrm{fin},i} \backslash \mathcal{D}) \leq \varepsilon$. The next proposition, the proof of which can be found in Section 3.1, connects all these definitions.

Proposition 2.1 (Local convergence) *Let μ_n, μ be locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\mathrm{fin},+}$. Then the following statements are equivalent.*

(i) *$\mu_n \Rightarrow \mu$ locally on $\mathcal{P}_{\mathrm{fin},+}$.*

(ii) *$\mu_n \to \mu$ pointwise on $\mathcal{P}_{\mathrm{fin},+}$ and the $(\mu_n)_{n\geq 1}$ are locally tight.*

(iii) *$\mu_n \Rightarrow \mu$ vaguely on $\mathcal{P}_+$ and the $(\mu_n)_{n\geq 1}$ are locally tight.*

(iv) *$\mu_n \Rightarrow \mu$ vaguely on $\mathcal{P}_+$ and $\mu_n \to \mu$ pointwise on $\mathcal{P}_{\mathrm{fin},+}$.*

Recall the definition of the intersection measure $\mu \boxtimes \nu$ in (1.17). The next lemma, the proof of which can be found in Section 3.1, says that the operation $\boxtimes$ is continuous with respect to vague and local convergence.

Lemma 2.2 (Intersection measure) *If μ, ν are locally finite measures on $\mathcal{P}_+$, then $\mu \boxtimes \nu$ is a locally finite measure on $\mathcal{P}_+$. If μ_n, ν_n are locally finite measures on $\mathcal{P}_+$ that converge vaguely to μ, ν, respectively, then $\mu_n \boxtimes \nu_n$ converges vaguely to $\mu \boxtimes \nu$. If moreover either the μ_n or the ν_n are concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and converge locally on $\mathcal{P}_{\mathrm{fin},+}$, then the $\mu_n \boxtimes \nu_n$ are concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and converge locally on $\mathcal{P}_{\mathrm{fin},+}$.*

It is often useful to view a homogeneous, locally finite measure on $\mathcal{P}_{\rm fin,+}$ as the 'law' of a random finite subset of Λ, shifted to a uniformly chosen position in Λ. To formulate this precisely, we define an equivalence relation on $\mathcal{P}_{\rm fin}$ by

$$A \sim B \quad \text{iff} \quad A = iB \quad \text{for some } i \in \Lambda, \tag{2.1}$$

and we let $\tilde{\mathcal{P}}_{\rm fin} := \{\tilde{A} : A \in \mathcal{P}_{\rm fin}\}$ with $\tilde{A} := \{iA : i \in \Lambda\}$ denote the set of equivalence classes. We can think of $\tilde{\mathcal{P}}_{\rm fin}$ as the space of finite subsets of the lattice 'modulo shifts'. Recall the definition of $\langle\!\langle \mu \rangle\!\rangle$ from (1.16). We have the following simple lemma, which will be proved in Section 3.1.

Lemma 2.3 (Homogeneous measures on the finite sets) *Let Δ be a $\mathcal{P}_{\rm fin,+}$-valued random variable and let $c > 0$. Then*

$$\mu := c \sum_{i \in \Lambda} \mathbb{P}\big[i\Delta \in \cdot\,\big] \tag{2.2}$$

defines a nonzero, homogeneous measure on $\mathcal{P}_{\rm fin,+}$ such that $\langle\!\langle \mu \rangle\!\rangle = c$. The measure μ is locally finite if and only if $\mathbb{E}\big[|\Delta|\big] < \infty$. Conversely, any nonzero, homogeneous measure on $\mathcal{P}_{\rm fin,+}$ such that $\langle\!\langle \mu \rangle\!\rangle < \infty$ can be written in the form (2.2) with $c = \langle\!\langle \mu \rangle\!\rangle$ for some $\mathcal{P}_{\rm fin,+}$-valued random variable Δ, and the law of $\tilde{\Delta}$ is uniquely determined by μ.

We finally turn our attention to contact processes started in infinite initial 'laws'. Recall the definition of the subprobability kernels P_t in (1.14) and of the meaures μP_t in (1.15). We cite the following simple fact from [Swa09, Lemma 3.3].

Lemma 2.4 (Process started in infinite law) *If μ is a homogeneous, locally finite measure on $\mathcal{P}_+$, then μP_t is a homogeneous, locally finite measure on $\mathcal{P}_+$ for each $t \geq 0$. If μ_n, μ are homogeneous, locally finite measures on $\mathcal{P}_+$ such that $\mu_n \Rightarrow \mu$, then $\mu_n P_t \Rightarrow \mu P_t$ for all $t \geq 0$, where $\Rightarrow$ denotes vague convergence.*

2.2 Existence of eigenmeasures concentrated on finite sets

The first step in the proof of Theorem 1 is to show that the condition $r < 0$ implies existence of a homogeneous eigenmeasure that is concentrated on $\mathcal{P}_{\rm fin}$.

We start by recalling how homogeneous eigenmeasures with eigenvalue r are constructed in [Swa09]. For any (Λ, a, δ)-contact process, we can define homogeneous, locally finite measures μ_t on $\mathcal{P}_+$ by

$$\mu_t := \sum_{i \in \Lambda} \mathbb{P}[\eta^{\{i\}}_t \in \cdot\,]\big|_{\mathcal{P}_+} \qquad (t \geq 0). \tag{2.3}$$

We can think of μ_t as the law of a contact process started with one infected site, distributed according to the counting measure on Λ. It is not hard to show (see [Swa09, formulas (3.8) and (3.20)]) that

$$\mu_t(\{A : 0 \in A\}) = \mathbb{E}\big[|\eta^{\{0\}}_t|\big] =: \pi_t. \tag{2.4}$$

Let $\hat{\mu}_\lambda$ be the Laplace transform of $(\mu_t)_{t \geq 0}$, i.e.,

$$\hat{\mu}_\lambda := \int_0^\infty \mu_t \, e^{-\lambda t} {\rm d}t \qquad (\lambda > r). \tag{2.5}$$

Then

$$\hat{\mu}_\lambda(\{A : 0 \in A\}) = \int_0^\infty \pi_t \, e^{-\lambda t} \mathrm{d}t =: \hat{\pi}_\lambda \qquad (\lambda > r), \tag{2.6}$$

which is finite for $\lambda > r$ by the definition of the exponential growth rate (see (1.10)). We cite the following result from [Swa09, Corollary 3.4], which yields the existence of homogeneous eigenmeasures.

Proposition 2.5 (Convergence to eigenmeasure) *The measures $\frac{1}{\hat{\pi}_\lambda}\hat{\mu}_\lambda$ $(\lambda > r)$ are relatively compact in the topology of vague convergence of locally finite measures on $\mathcal{P}_+$, and each subsequential limit as $\lambda \downarrow r$ is a homogeneous eigenmeasure of the (Λ, a, δ)-contact process, with eigenvalue $r(\Lambda, a, \delta)$.*

We wish to show that for $r < 0$, the approximation procedure in Proposition 2.5 yields an eigenmeasure that is concentrated on $\mathcal{P}_{\mathrm{fin}}$. The key to this is the following lemma, which will be proved in Section 3.4 using a covariance calculation. Note that this lemma still holds for general $r \in \mathbb{R}$.

Lemma 2.6 (Uniform moment bound) *Let $\hat{\mu}_\lambda$ and $\hat{\pi}_\lambda$ be defined as in (2.5)–(2.6). Then, for any (Λ, a, δ)-contact process with exponential growth rate $r = r(\Lambda, a, \delta)$,*

$$\limsup_{\lambda \downarrow r} \frac{1}{\hat{\pi}_\lambda} \int \hat{\mu}_\lambda(\mathrm{d}A) 1_{\{0 \in A\}} |A| \leq (|a| + \delta) \int_0^\infty e^{-rt} \mathrm{d}t \, \mathbb{E}\big[|\eta_t^{\{0\}}|\big]^2. \tag{2.7}$$

As a consequence, we obtain the following result that completes the existence part of Theorem 1.

Lemma 2.7 (Existence of an eigenmeasure on finite configurations) *Assume that the exponential growth rate $r = r(\Lambda, a, \delta)$ of the (Λ, a, δ)-contact process satisfies $r < 0$. Then there exists a homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue r of the (Λ, a, δ)-contact process such that*

$$\int \mathring{\nu}(\mathrm{d}A) |A| 1_{\{0 \in A\}} < \infty. \tag{2.8}$$

Proof By Proposition 2.5, we can choose $\lambda_n \downarrow r$ such that the measures $\frac{1}{\hat{\pi}_{\lambda_n}}\hat{\mu}_{\lambda_n}$ converge vaguely to a homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue r. It follows from (1.10) that $\mathbb{E}[|\eta_t^{\{0\}}|] = e^{rt + o(t)}$ where $t \mapsto o(t)$ is a continuous function such that $o(t)/t \to 0$ as $t \to \infty$, hence, by (2.7), provided $r < 0$,

$$\int_0^\infty e^{-rt} \mathrm{d}t \, \mathbb{E}\big[|\eta_t^{\{0\}}|\big]^2 = \int_0^\infty e^{2rt - rt + o(t)} \mathrm{d}t < \infty \qquad (r < 0). \tag{2.9}$$

Let Λ_k be finite sets such that $0 \in \Lambda_k \subset \Lambda$ and $\Lambda_k \uparrow \Lambda$. It is easy to check that $A \mapsto f_k(A) := |A \cap \Lambda_k| 1_{\{0 \in A\}}$ is a continuous, compactly supported real function on $\mathcal{P}_+$. Therefore, by the vague convergence of $\frac{1}{\hat{\pi}_{\lambda_n}}\hat{\mu}_{\lambda_n}$ to $\mathring{\nu}$, and by (2.7),

$$\begin{aligned} \int \mathring{\nu}(\mathrm{d}A) f_k(A) &= \lim_{n \to \infty} \frac{1}{\hat{\pi}_{\lambda_n}} \int \hat{\mu}_{\lambda_n}(\mathrm{d}A) f_k(A) \\ &\leq \liminf_{n \to \infty} \frac{1}{\hat{\pi}_{\lambda_n}} \int \hat{\mu}_{\lambda_n}(\mathrm{d}A) |A| 1_{\{0 \in A\}} \leq (|a| + \delta) \int_0^\infty e^{-rt} \mathrm{d}t \, \mathbb{E}\big[|\eta_t^{\{0\}}|\big]^2. \end{aligned} \tag{2.10}$$

Letting $k \uparrow \infty$, using the fact that the right-hand side is finite by (2.9), we arrive at (2.8). ∎

2.3 A Doob transformed Markov process

Since existence of the eigenmeasure $\mathring{\nu}$ from Theorem 1 is settled, the next aim is to prove the convergence in (1.20), which will in particular imply uniqueness. The proof will proceed in three steps. First, we will use a variant of the well-known Doob transform (also known as h-transform) to transform our contact process into a process that never gets extinct, and we will transform our eigenmeasure concentrated on finite configurations into an invariant measure of this process. In the second step, we will use Lemma 2.3 to 'divide out' translations and show that the resulting Doob transformed process modulo shifts is irreducible and positively recurrent. In the third step, we use standard ergodic results for irreducible, positively recurrent Markov processes with countable state space, together with duality, to prove the convergence in (1.20).

We recall that the classical Doob transform is based on a positive harmonic function h. We will need a slight variation of this where h is a positive eigenfunction of the generator. (This is a special case of what is called a 'compensated h-transform' in [FS02, Lemma 3].) In general, a duality relation between two Markov processes translates invariant measures of one process into harmonic functions of the dual process. Similarly, we will see that each eigenmeasure of a (Λ, a, δ)-contact process gives rise to a positive eigenfunction of the generator of the dual $(\Lambda, a^\dagger, \delta)$-contact process, and vice versa. We will exploit this and use the eigenmeasure $\mathring{\nu}^\dagger$ of the dual process to construct a function h with which we can transform the 'forward' process.

To formulate this properly, we first need to say something about the space of functions on which the generator G from (1.2) is well-defined. Let

$$\mathcal{S}(\mathcal{P}_{\rm fin}) := \{f : \mathcal{P}_{\rm fin} \to \mathbb{R} : |f(A)| \leq K|A|^k + M \text{ for some } K, M, k \geq 0\}. \tag{2.11}$$

denote the class of real functions on $\mathcal{P}_{\rm fin}$ of polynomial growth. It has been shown in [Swa09, Prop. 2.1] that the operator G maps the space $\mathcal{S}(\mathcal{P}_{\rm fin})$ into itself and for each $f \in \mathcal{S}(\mathcal{P}_{\rm fin})$ and $A \in \mathcal{P}_{\rm fin}$, the process

$$M_t := f(\eta_t^A) - \int_0^t Gf(\eta_s^A)\mathrm{d}s \qquad (t \geq 0) \tag{2.12}$$

is a martingale with respect to the filtration generated by η^A.

We say that a function $f : \mathcal{P}_{\rm fin} \to \mathbb{R}$ is *shift-invariant* if $f(iA) = f(A)$ for all $i \in \Lambda$, *monotone* if $A \subset B$ implies $f(A) \leq f(B)$, and *subadditive* if $f(A \cup B) \leq f(A) + f(B)$, for all $A, B \in \mathcal{P}_{\rm fin}$. We cite the following fact from [Swa09, Lemma 3.5].

Lemma 2.8 (Eigenmeasures and harmonic functions) *If $\mu^\dagger$ is a homogeneous eigenmeasure with eigenvalue λ of the $(\Lambda, a^\dagger, \delta)$-contact process, then*

$$h(A) = h_{\mu^\dagger}(A) := \int \mu^\dagger(\mathrm{d}B)\, 1_{\{A \cap B \neq \emptyset\}} \qquad (A \in \mathcal{P}_{\rm fin}) \tag{2.13}$$

defines a shift-invariant, monotone, subadditive function such that $h(\emptyset) = 0$, $h(A) > 0$ for any $\emptyset \neq A \in \mathcal{P}_{\rm fin}$, $h \in \mathcal{S}(\mathcal{P}_{\rm fin})$, and $Gh = \lambda h$.

We are now ready to introduce the kind of Doob transformed processes that we are interested in. For each $A, B \in \mathcal{P}_{\rm fin,+}$, let $r(A, B)$ denote the rate at which the (Λ, a, δ)-contact process jumps from A to B. Let $h = h_{\mu^\dagger}$ be given by (2.13). We will be interested in the continuous-time Markov process with countable state space $\mathcal{P}_{\rm fin,+}$ and jump rates given by

$$r^h(A, B) = \frac{h(B)}{h(A)} r(A, B) \qquad \big(A, B \in \mathcal{P}_{\rm fin,+}\big). \tag{2.14}$$

Let $\xi^A = (\xi^A_t)_{t\geq 0}$ denote this process, which a priori may be defined only up to some explosion time τ (we will see shortly that $\tau = \infty$). We call ξ^A the *h-transformed* (Λ, a, δ)*-contact process,* and let

$$P^h_t(A,B) := \mathbb{P}\big[\xi^A_t = B,\ t < \tau\big] \qquad \big(t \geq 0,\ A, B \in \mathcal{P}_{\mathrm{fin},+}\big) \tag{2.15}$$

denote its transition probabilities. A priori, due to the possibility of explosion, this might be a subprobability kernel like the P_t defined in (1.14), for which we adopt the analogous notation $P_t(A,B) := P_t(A,\{B\})$. The following lemma says that this is not the case. The proof of this result can be found in Section 3.5.

Lemma 2.9 (Doob transformed process) *Let $\mu^\dagger$ be a homogeneous eigenmeasure with eigenvalue λ of the $(\Lambda, a^\dagger, \delta)$-contact process and let $h = h_{\mu^\dagger}$ be defined as in (2.13). Then the h-transformed (Λ, a, δ)-contact process does not explode and its transition kernel is given by*

$$P^h_t(A,B) = e^{-\lambda t}\,\frac{h(B)}{h(A)}\,P_t(A,B) \qquad \big(t \geq 0,\ A, B \in \mathcal{P}_{\mathrm{fin},+}\big). \tag{2.16}$$

Remark One can check that the process ξ^A solves the martingale problem for the operator given by $G^h f := G(hf)/h - \lambda f$ $(f \in \mathcal{S}(\mathcal{P}_{\mathrm{fin}}),\ f(\emptyset) = 0)$, but we will not need this.

The next lemma shows in particular that if $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ are eigenmeasures of the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact processe with properties as in Lemma 2.7 and $h = h_{\mathring{\nu}^\dagger}$, then $h\mathring{\nu}$ is an invariant measure of the h-transformed (Λ, a, δ)-contact process. The proof can be found in Section 3.5.

Lemma 2.10 (Invariant measures of the Doob transformed process) *Let $\mu^\dagger$ be a homogeneous eigenmeasure with eigenvalue λ of the $(\Lambda, a^\dagger, \delta)$-contact process and let $h = h_{\mu^\dagger}$ be defined as in (2.13). Let μ be a homogeneous, locally finite measure on $\mathcal{P}_+$ such that $\int \mu(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty$, and let $h\mu$ denote the weighted measure $h\mu(\mathrm{d}A) := h(A)\mu(\mathrm{d}A)$. Then $h\mu$ is a locally finite measure on $\mathcal{P}_+$. Moreover, μ is an eigenmeasure of the (Λ, a, δ)-contact process with eigenvalue λ if and only if $h\mu$ is an invariant measure of the h-transformed (Λ, a, δ)-contact process.*

2.4 The Doob transformed process modulo shifts

By Theorem 0 (a), the (Λ, a, δ)-contact processes and its dual $(\Lambda, a^\dagger, \delta)$-contact processes have the same exponential growth rate $r = r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$. In particular, if $r < 0$, then by Lemma 2.7, there exist homogeneous eigenmeasures $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ of the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact process, respectively, both with eigenvalue r, such that

$$\int \mathring{\nu}(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty \quad \text{and} \quad \int \mathring{\nu}^\dagger(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty. \tag{2.17}$$

We normalize $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ such that $\int \mathring{\nu}(\mathrm{d}A)1_{\{0\in A\}} = 1 = \int \mathring{\nu}^\dagger(\mathrm{d}A)1_{\{0\in A\}}$. For the moment, we do not know yet if $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ are unique. However, we simply fix any two such measures and define functions

$$h := h_{\mathring{\nu}^\dagger} \quad \text{and} \quad h^\dagger := h_{\mathring{\nu}} \tag{2.18}$$

as in (2.13), which by Lemma 2.8 satisfy $Gh = rh$ and $G^\dagger h^\dagger = rh^\dagger$, where G and $G^\dagger$ denote the generators of the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact process, respectively. Using these functions, we define an h-transformed (Λ, a, δ)-contact process $(\xi_t)_{t\geq 0}$ and $h^\dagger$-transformed $(\Lambda, a^\dagger, \delta)$-contact process $(\xi_t^\dagger)_{t\geq 0}$ with transition rates as in (2.14). By Lemma 2.10, $h\mathring{\nu}$ and $h^\dagger\mathring{\nu}^\dagger$ are locally finite invariant measures of these processes, respectively. Note that $h\mathring{\nu}$ and $h^\dagger\mathring{\nu}^\dagger$ are moreover homogeneous (by the shift-invariance of h and $h^\dagger$ and the homogeneity of $\mathring{\nu}$ and $\mathring{\nu}^d gg$) and concentrated on $\mathcal{P}_{\rm fin,+}$ (since $\mathring{\nu}$ and $\mathring{\nu}^d gg$ have this property). Therefore, by Lemma 2.3, there exist $\mathcal{P}_{\rm fin,+}$-valued random variables ξ_∞ and $\xi^\dagger_\infty$ such that

$$h\mathring{\nu} = \langle\!\langle h\mathring{\nu}\rangle\!\rangle \sum_{i\in\Lambda} \mathbb{P}\big[i\xi_\infty \in \cdot\,\big] \quad \text{and} \quad h^\dagger\mathring{\nu}^\dagger = \langle\!\langle h^\dagger\mathring{\nu}^\dagger\rangle\!\rangle \sum_{i\in\Lambda} \mathbb{P}\big[i\xi^\dagger_\infty \in \cdot\,\big]. \tag{2.19}$$

The suggestive notation that we have chosen for these $\mathcal{P}_{\rm fin,+}$-valued random variables is motivated by the fact that $h\mathring{\nu}$ and $h^\dagger\mathring{\nu}^\dagger$ are invariant measures of the processes $(\xi_t)_{t\geq 0}$ and $(\xi^\dagger_t)_{t\geq 0}$ and will be further justified by Lemma 2.11 below.

Recall from (2.1) that $\tilde{\mathcal{P}}_{\rm fin}$ denotes the space of finite subsets of Λ 'modulo shifts'. It follows from the shift-invariance of a and h that if $(\xi_t)_{t\geq 0}$ is the h-transformed (Λ, a, δ)-contact process (started in any initial law), then the $\tilde{\mathcal{P}}_{\rm fin,+}$-valued process $(\tilde{\xi}_t)_{t\geq 0}$ is also a Markov process. We call this the *h-transformed (Λ, a, δ)-contact process modulo shifts*. The $h^\dagger$-transformed $(\Lambda, a^\dagger, \delta)$-contact process modulo shifts is defined similarly. The following observation is the central ingredient for our proof of the convergence formula (1.20). Below, we use the word 'irreducible' in the sense as defined in Section 1.2, i.e., for each two states in the state space there is a positive probability of going from one to the other. For the proof we refer to Section 3.5.

Lemma 2.11 (Positive recurrence) *Assume that $r(\Lambda, a, \delta) < 0$ and let h be defined in (2.18). Assume that the infection kernel a satisfies the irreducibility condition (1.3). Then the h-transformed (Λ, a, δ)-contact process modulo shifts is a positively recurrent, irreducible Markov process with countable state space $\tilde{\mathcal{P}}_{\rm fin,+}$, and $\mathbb{P}[\tilde{\xi}_\infty \in \cdot\,]$ with ξ_∞ from (2.19) is its unique invariant law.*

2.5 Convergence to the eigenmeasure

In this section, we prove Theorem 1. We need one preparatory lemma, the proof of which can be found in Section 3.1.

Lemma 2.12 (Intersection and weighted measures) *Let μ, ν be homogeneous locally finite measures on $\mathcal{P}_+$, assume that μ is concentrated on $\mathcal{P}_{\rm fin,+}$, and let h_ν be defined as in (2.13). Then*

$$\langle\!\langle \mu \circledcirc \nu \rangle\!\rangle = \langle\!\langle h_\nu \mu \rangle\!\rangle. \tag{2.20}$$

If moreover $\int \mu(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty$, then $h_\nu\mu$ is locally finite.

Proof of Theorem 1 The existence of $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ has already been proved in Lemma 2.7, so uniqueness will follow once we prove the convergence in (1.20), with the $\mathring{\nu}$ that we fixed earlier. We need to prove two statements: vague convergence for general (nonzero, homogeneous, locally finite) initial measures μ and local convergence on $\mathcal{P}_{\rm fin,+}$ if μ is concentrated on $\mathcal{P}_{\rm fin,+}$.

We start with vague convergence. By Lemma 1.1, it suffices to show that

$$e^{-rt} \int \mu P_t(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} \underset{n\to\infty}{\longrightarrow} c \int \mathring{\nu}(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} \qquad (B \in \mathcal{P}_{\rm fin,+}), \tag{2.21}$$

where $c > 0$ is given in (1.21). Let h_μ be defined as in (2.13). By duality (1.7) and Lemma 2.9, we observe that for any $B \in \mathcal{P}_{\mathrm{fin},+}$,

$$\begin{aligned} & e^{-rt}\int \mu P_t(\mathrm{d}A')1_{\{A'\cap B\neq\emptyset\}} = e^{-rt}\int \mu(\mathrm{d}A)\mathbb{P}\big[\eta_t^A \cap B \neq \emptyset\big] \\ &\quad = e^{-rt}\int \mu(\mathrm{d}A)\mathbb{P}\big[A\cap \eta_t^{\dagger\,B} \neq \emptyset\big] = e^{-rt}\int \mu(\mathrm{d}A)\sum_{B'} P_t^\dagger(B,B')1_{\{A\cap B'\neq\emptyset\}} \\ &\quad = e^{-rt}\sum_{B'} P_t^\dagger(B,B')h_\mu(B') = h^\dagger(B)\sum_{B'} P_t^{\dagger\,h^\dagger}(B,B')h^\dagger(B')^{-1}h_\mu(B'), \end{aligned} \tag{2.22}$$

where $P_t^\dagger$ and $P_t^{\dagger\,h^\dagger}$ denote the transition probabilities of the $(\Lambda, a^\dagger, \delta)$-contact process and the $h^\dagger$-transformed $(\Lambda, a^\dagger, \delta)$-contact process, respectively. We observe that $h_\mu/h^\dagger$ is a shift-invariant function. Therefore, writing $(h_\mu/h^\dagger)(\tilde{B}')$ for the value of the function $h_\mu/h^\dagger$ on the equivalence class of sets $\tilde{B}'$ containing B', we can rewrite the right-hand side of (2.22) as

$$h^\dagger(B)\mathbb{E}\big[(h^\dagger/h_\mu)(\xi_t^{B\,\dagger})\big] = h^\dagger(B)\mathbb{E}\big[(h^\dagger/h_\mu)(\tilde{\xi}_t^{B\,\dagger})\big], \tag{2.23}$$

where $\tilde{\xi}^{B\,\dagger}$ denotes the $h^\dagger$-transformed $(\Lambda, a^\dagger, \delta)$-contact process modulo shifts, started in B. By Lemma 2.11, this process is irreducible and positively recurrent with unique invariant law $\mathbb{P}[\tilde{\xi}_\infty^\dagger \in \cdot\,]$, where $\xi_\infty^\dagger$ defined as in (2.19). In particular, this process is ergodic, so by (2.22) and (2.23) we may conclude that

$$e^{-rt}\int \mu P_t(\mathrm{d}A')1_{\{A'\cap B\neq\emptyset\}} = h^\dagger(B)\mathbb{E}\big[(h^\dagger/h_\mu)(\tilde{\xi}_t^{B\,\dagger})\big] \underset{t\to\infty}{\longrightarrow} h^\dagger(B)\mathbb{E}\big[(h^\dagger/h_\mu)(\tilde{\xi}_\infty^\dagger)\big], \tag{2.24}$$

provided we show that $h_\mu/h^\dagger$ is a bounded function. To see this, note that by the fact that $\mathring{\nu}$ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and Lemma 2.3, there exists a $\mathcal{P}_{\mathrm{fin},+}$-valued random variable Δ such that $\mathring{\nu}$ can be written as in (2.2). Let κ be a Λ-valued random variable such that $\kappa \in \Delta$ a.s. Then, by the definition of $h^\dagger$ in (2.18),

$$\begin{aligned} h^\dagger(A) = h_{\mathring{\nu}} &= \int \mathring{\nu}(\mathrm{d}B)1_{\{A\cap B\neq\emptyset\}} = \langle\!\langle \mathring{\nu}\rangle\!\rangle \sum_{i\in\Lambda}\mathbb{P}[A\cap i\Delta \neq \emptyset] \\ &\geq \langle\!\langle \mathring{\nu}\rangle\!\rangle \sum_{i\in\Lambda}\mathbb{P}[A\cap \{i\kappa\} \neq \emptyset] = \langle\!\langle \mathring{\nu}\rangle\!\rangle |A|. \end{aligned} \tag{2.25}$$

On the other hand, since h_μ is subadditive and shift-invariant by Lemma 2.8, we have $h_\mu(A) \leq h_\mu(\{0\})|A|$ and therefore $h_\mu(A)/h^\dagger(A) \leq h_\mu(\{0\})/\langle\!\langle \mathring{\nu}\rangle\!\rangle$.

Recalling that $h^\dagger = h_{\mathring{\nu}}$, we obtain from (2.24) that

$$e^{-rt}\int \mu P_t(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} \underset{t\to\infty}{\longrightarrow} \mathbb{E}\big[(h^\dagger/h_\mu)(\tilde{\xi}_\infty^\dagger)\big]\int \mathring{\nu}(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}. \tag{2.26}$$

Since this holds for any $B \in \mathcal{P}_{\mathrm{fin},+}$, we conclude with the help of Lemma 1.1 that

$$e^{-rt}\mu P_t \underset{t\to\infty}{\Longrightarrow} c\mathring{\nu} \quad \text{where} \quad c := \mathbb{E}\big[(h^\dagger/h_\mu)(\tilde{\xi}_\infty^\dagger)\big] > 0 \tag{2.27}$$

and $\Rightarrow$ denotes vague convergence of locally finite measures on $\mathcal{P}_+$.

It is possible to verify by direct calculation that the constant c in (2.27) equals the one in (1.21), but this is rather tedious. More easily, we may observe that by the duality relation in Lemma 1.2 and the fact that $\mathring{\nu}^\dagger$ is an eigenmeasure

$$\langle\!\langle e^{-rt}\mu P_t \boxtimes \mathring{\nu}^\dagger\rangle\!\rangle = \langle\!\langle e^{-rt}\mu \boxtimes \mathring{\nu}^\dagger P_t^\dagger\rangle\!\rangle = \langle\!\langle \mu \boxtimes \mathring{\nu}^\dagger\rangle\!\rangle \qquad (t \geq 0), \tag{2.28}$$

while by the vague convergence of $e^{-rt}\mu P_t$ to $\mathring{\nu}$, the local convergence of $\mathring{\nu}^\dagger$ to itself, and Lemma 2.2, the measures $e^{-rt}\mu P_t \bowtie \mathring{\nu}^\dagger$ converge locally on $\mathcal{P}_{\text{fin},+}$ to $c\,\mathring{\nu} \bowtie \mathring{\nu}^\dagger$. Since $\mathcal{P}_{\text{fin},0} \ni A \mapsto |A|^{-1}1_{\{0\in A\}}$ is a bounded function, the function $\mu \mapsto \langle\!\langle \mu \rangle\!\rangle$ on $\mathcal{P}_{\text{fin},+}$ is continuous with respect to local convergence on $\mathcal{P}_{\text{fin},+}$, see the definition of $\langle\!\langle \cdot \rangle\!\rangle$ in (1.16). It therefore follows that

$$\langle\!\langle \mu \bowtie \mathring{\nu}^\dagger \rangle\!\rangle = \langle\!\langle e^{-rt}\mu P_t \bowtie \mathring{\nu}^\dagger \rangle\!\rangle \underset{t\to\infty}{\longrightarrow} c\,\langle\!\langle \mathring{\nu} \bowtie \mathring{\nu}^\dagger \rangle\!\rangle, \tag{2.29}$$

proving that $c = \langle\!\langle \mu \bowtie \mathring{\nu}^\dagger \rangle\!\rangle / \langle\!\langle \mathring{\nu} \bowtie \mathring{\nu}^\dagger \rangle\!\rangle$, which is (1.21).

It remains to show that the vague convergence in (1.20) can be strengthened to local convergence on $\mathcal{P}_{\text{fin},+}$ if μ is concentrated on $\mathcal{P}_{\text{fin},+}$. By Proposition 2.1 (iv), it suffices to prove pointwise convergence. We may equivalently prove that

$$e^{-rt}h(\mu P_t) \underset{t\to\infty}{\longrightarrow} h\mathring{\nu} \tag{2.30}$$

where $h(\mu P_t)$ denotes the weighted measure $h(\mu P_t)(\{B\}) := h(B)\mu P_t(\{B\})$, and likewise $h\mathring{\nu}$ is $\mathring{\nu}$ weighted with h. Note that $h(\mu P_t)$ need not be locally finite, but by Lemmas 2.12 and 1.2 (recall (2.18)),

$$\langle\!\langle h(\mu P_t) \rangle\!\rangle = \langle\!\langle \mu P_t \bowtie \mathring{\nu}^\dagger \rangle\!\rangle = \langle\!\langle \mu \bowtie \mathring{\nu}^\dagger P_t^\dagger \rangle\!\rangle = e^{rt}\langle\!\langle \mu \bowtie \mathring{\nu}^\dagger \rangle\!\rangle < \infty \tag{2.31}$$

since $\mu \bowtie \mathring{\nu}^\dagger$ is a locally finite measure by Lemma 2.2. By Lemma 2.9,

$$e^{-rt}h(\mu P_t)(\{B\}) = e^{-rt} \sum_{A\in\mathcal{P}_{\text{fin},+}} \mu(\{A\})P_t(A,B)h(B) = \sum_{A\in\mathcal{P}_{\text{fin},+}} \mu(\{A\})h(A)P_t^h(A,B), \tag{2.32}$$

which tells us that $e^{-rt}h(\mu P_t) = (h\mu)P_t^h$. We have, due to the fact that $h(A) \le h(\{0\})|A|$ that

$$\langle\!\langle h\mu \rangle\!\rangle = \int (h\mu)(dA)|A|^{-1}1_{\{0\in A\}} \le h(\{0\}) \int \mu(dA)1_{\{0\in A\}} < \infty. \tag{2.33}$$

Thus, by Lemma 2.3, there exists a $\mathcal{P}_{\text{fin},+}$-valued random variable ξ_0 such that

$$h\mu = \langle\!\langle h\mu \rangle\!\rangle \sum_i \mathbb{P}[i\xi_0 \in \cdot\,]. \tag{2.34}$$

Now

$$e^{-rt}h(\mu P_t) = (h\mu)P_t^h = \langle\!\langle h\mu \rangle\!\rangle \sum_i \mathbb{P}[i\xi_t \in \cdot\,], \tag{2.35}$$

where $(\xi_t)_{t\ge0}$ is the h-transformed (Λ, a, δ)-contact process started in ξ_0. By Lemma 2.11, the h-transformed (Λ, a, δ)-contact process modulo shifts is ergodic with unique invariant law $\mathbb{P}[\tilde{\xi}_\infty \in \cdot\,]$, where ξ_∞ is given in (2.19). Therefore, we may conclude that

$$e^{-rt}h(\mu P_t) \underset{t\to\infty}{\longrightarrow} \frac{\langle\!\langle h\mu \rangle\!\rangle}{\langle\!\langle h\mathring{\nu} \rangle\!\rangle} h\mathring{\nu} \quad \text{pointwise on } \mathcal{P}_{\text{fin},+}, \tag{2.36}$$

where $\langle\!\langle h\mu \rangle\!\rangle / \langle\!\langle h\mathring{\nu} \rangle\!\rangle$ equals the constant $c = \langle\!\langle \mu \bowtie \mathring{\nu}^\dagger \rangle\!\rangle / \langle\!\langle \mathring{\nu} \bowtie \mathring{\nu}^\dagger \rangle\!\rangle$ in (1.21) by Lemma 2.12. ∎

2.6 Continuity in the recovery rate

The first step in proving Theorem 2 will be to show continuity of the map $(\delta_{\rm c},\infty)\ni\delta\mapsto\mathring{\nu}_\delta$. We start by proving continuity with respect to vague convergence, which is based on the following abstract result, whose proof can be found in Section 3.2.

Lemma 2.13 (Limits of eigenmeasures) *Let ν_n $(n\geq 0)$ be homogeneous eigenmeasures of (Λ,a,δ_n)-contact processes, with eigenvalues λ_n, normalized such that $\int\nu_n(\mathrm{d}A)1_{\{0\in A\}}=1$. Assume that $\lambda_n\to\lambda$ and $\delta_n\to\delta$. Then the $(\nu_n)_{n\geq 0}$ are relatively compact in the topology of vague convergence, and each vague cluster point ν is a homogeneous eigenmeasure of the (Λ,a,δ)-contact processes, with eigenvalue λ.*

Continuity of the map $(\delta_{\rm c},\infty)\ni\delta\mapsto\mathring{\nu}_\delta$ is now a simple consequence of Theorem 1 and Lemma 2.13.

Proposition 2.14 (Vague continuity of the eigenmeasure) *Assume that the infection rates satisfy the irreducibility condition (1.3). For $\delta\in(\delta_{\rm c},\infty)$, let $\mathring{\nu}_\delta$ denote the unique homogeneous eigenmeasure of the (Λ,a,δ)-contact process normalized such that $\int\mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}}=1$. Then the map $\delta\mapsto\mathring{\nu}_\delta$ is continuous on $(\delta_{\rm c},\infty)$ w.r.t. vague convergence of locally finite measures on $\mathcal{P}_+$.*

Proof Choose $\delta_n,\delta\in(\delta_{\rm c},\infty)$ such that $\delta_n\to\delta$. Since the eigenvalue $r(\Lambda,a,\delta)$ of the homogeneous eigenmeasure $\mathring{\nu}_\delta$ is continuous in δ by Theorem 0 (b), Lemma 2.13 implies that the measures $(\mathring{\nu}_{\delta_n})_{n\geq 0}$ are relatively compact in the topology of vague convergence, and each vague cluster point is a homogeneous eigenmeasure of the (Λ,a,δ)-contact processes with eigenvalue $r(\Lambda,a,\delta)$. By Theorem 1, this implies that $\mathring{\nu}_\delta$ is the only vague cluster point, hence the $\mathring{\nu}_{\delta_n}$ converge vaguely to $\mathring{\nu}_\delta$. ∎

Unfortunately, continuity with respect to vague convergence is not enough to prove continuity of the right-hand side of (1.25), and hence of the derivative $\frac{\partial}{\partial\delta}r(\Lambda,a,\delta)$. As mentioned earlier, we will remedy this by proving continuity of the map $(\delta_{\rm c},\infty)\ni\delta\mapsto\mathring{\nu}_\delta$ with respect to local convergence on $\mathcal{P}_{\rm fin,+}$. Since vague convergence is already proved, by Proposition 2.1 (iii), it suffices to prove local tightness. This is the most technical part of our proofs, since it involves estimating how 'large' the finite sets can be that $\mathring{\nu}_\delta$ is concentrated on. The first step is to introduce a suitable concept of distance. The next result will be proved in Section 3.3.

Lemma 2.15 (Slowly growing metric) *Let Λ be a countable group and let $a:\Lambda\times\Lambda\to[0,\infty)$ satisfy (1.1). Then there exists a metric d on Λ such that*

$$\begin{array}{rlr}
\text{(i)} & d(i,j)=d(ki,kj) & (i,j,k\in\Lambda),\\
\text{(ii)} & \big|\{i\in\Lambda:d(0,i)\leq M\}\big|<\infty & (0\leq M<\infty),\\
\text{(iii)} & K_\gamma(\Lambda,a):=\sum_i a(0,i)e^{\gamma d(0,i)}<\infty & (0\leq\gamma<\infty).
\end{array}\tag{2.37}$$

Next, we fix a metric d as in (2.37) and for each $0\leq\gamma<\infty$, we define a function $e_\gamma:\mathcal{P}_{\rm fin}\to[0,\infty)$ by

$$e_\gamma(A):=\sum_{i\in A}e^{\gamma d(0,i)}\qquad(\gamma\geq 0,\ A\in\mathcal{P}_{\rm fin}).\tag{2.38}$$

We note that a similar (but not entirely identical) function has proved useful in the study of contact processes on trees, see [Lig99, formula (I.4.3)]. We have in particular $e_0(A) = |A|$. The next lemma says that there is a well-defined exponential growth rate $r_\gamma(\Lambda, a, \delta)$ associated with the function e_γ, which converges to our well-known exponential growth rate $r(\Lambda, a, \delta)$ as $\gamma \downarrow 0$. The proof can be found in Section 3.3.

Lemma 2.16 (Exponential growth rates) *Let $(\eta_t^{\{0\}})_{t\geq 0}$ be the (Λ, a, δ)-contact process started in $\eta_0^{\{0\}} = \{0\}$. Let d be a metric on Λ as in Lemma 2.15, and let e_γ be the function defined in (2.38). Then, for each $0 \leq \gamma < \infty$, the limit*

$$r_\gamma = r_\gamma(\Lambda, a, \delta) := \lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}\big[e_\gamma(\eta_t^{\{0\}})\big] = \inf_{t>0} \tfrac{1}{t} \log \mathbb{E}\big[e_\gamma(\eta_t^{\{0\}})\big] \tag{2.39}$$

exists. The function $\gamma \mapsto r_\gamma$ is nondecreasing, right-continuous, and satisfies

$$-\delta \leq r_\gamma(\Lambda, a, \delta) \leq K_\gamma(\Lambda, a) \qquad (\gamma \geq 0), \tag{2.40}$$

where $K_\gamma(\Lambda, a)$ is defined in (2.37).

We can generalize the proof of Lemma 2.6 to yield a more general version of that lemma (see Lemma 3.5 below), which after taking the limit (as in (2.10)) yields the following bound on the eigenmeasures $\mathring{\nu}_\delta$. (We refer to Section 3.4 for the detailed proof.)

Lemma 2.17 (Tightness estimate) *Let $(\eta_t^{\{0\}})_{t\geq 0}$ be the (Λ, a, δ)-contact process started in $\eta_0^{\{0\}} = \{0\}$, let $r(\delta) = r(\Lambda, a, \delta)$ be its exponential growth rate, let d be a metric on Λ as in Lemma 2.15, and let e_γ be the function defined in (2.38). For $\delta \in (\delta_{\rm c}, \infty)$, let $\mathring{\nu}_\delta$ denote the unique homogeneous eigenmeasure of the (Λ, a, δ)-contact process normalized such that $\int \mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}} = 1$. Then*

$$\int \mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}} e_\gamma(A) \leq (|a| + \delta) \int_0^\infty e^{-r(\delta)t}\mathrm{d}t\, \mathbb{E}\big[e_\gamma(\eta_t^{\delta,\{0\}})\big]^2 \qquad \big(\gamma \geq 0,\ \delta \in (\delta_{\rm c}, \infty)\big). \tag{2.41}$$

With this preparation we are now ready to prove the desired local continuity.

Proposition 2.18 (Local continuity of the eigenmeasure) *Assume that the infection rates satisfy the irreducibility condition (1.3). For $\delta \in (\delta_{\rm c}, \infty)$, let $\mathring{\nu}_\delta$ denote the unique homogeneous eigenmeasure of the (Λ, a, δ)-contact process normalized such that $\int \mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}} = 1$. Then the map $\delta \mapsto \mathring{\nu}_\delta$ is continuous on $(\delta_{\rm c}, \infty)$ in the sense of local convergence on $\mathcal{P}_{\rm fin,+}$.*

Proof Vague continuity of the map $(\delta_{\rm c}, \infty) \ni \delta \mapsto \mathring{\nu}_\delta$ has been proved in Proposition 2.14, so by Proposition 2.1 (iii), it suffices to show that for any $\delta_* \in (\delta_{\rm c}, \infty)$ there exists an $\varepsilon > 0$ such that the measures $(\mathring{\nu}_\delta)_{\delta\in(\delta_*-\varepsilon,\delta_*+\varepsilon)}$ are locally tight.

By property (2.37) (ii), for each $\gamma > 0$ and $K < \infty$, the set $\{A \in \mathcal{P}_{\rm fin,0} : e_\gamma(A) \leq K\}$ is finite. Thus, by Lemma 2.17, to prove the required local tightness, it suffices to show that for each $\delta_* \in (\delta_{\rm c}, \infty)$ there exist a $\gamma > 0$ and $\varepsilon > 0$ such that

$$\sup_{\delta\in(\delta_*-\varepsilon,\delta_*+\varepsilon)} \int_0^\infty e^{-r(\delta)t}\mathrm{d}t\, \mathbb{E}\big[e_\gamma(\eta_t^{\delta,\{0\}})\big]^2 < \infty. \tag{2.42}$$

By the continuity of $\delta \mapsto r(\delta)$ (Theorem 0 (b)), we can choose $\varepsilon > 0$ such that $\delta_{\rm c} < \delta_* - \varepsilon$ and

$$r(\delta_* - \varepsilon) \leq \frac{4}{5} r(\delta_* + \varepsilon). \tag{2.43}$$

Let $r_\gamma = r_\gamma(\delta)$ be the exponential growth rate associated with the function e_γ. By Lemma 2.16, the function $\gamma \mapsto r_\gamma$ is right-continuous, so we can choose $\gamma > 0$ such that

$$r_\gamma(\delta_* - \varepsilon) \leq \frac{3}{4} r(\delta_* - \varepsilon). \tag{2.44}$$

By the fact that $r(\delta)$ is nonincreasing in δ and the law of $\eta_t^{\delta,\{0\}}$ is nonincreasing in δ with respect to the stochastic order, it follows that for all $\delta \in (\delta_* - \varepsilon, \delta_* + \varepsilon)$,

$$\begin{aligned} &\int_0^\infty e^{-r(\delta)t} \mathrm{d}t\, \mathbb{E}\big[e_\gamma(\eta_t^{\delta,\{0\}})\big]^2 \leq \int_0^\infty e^{-r(\delta_*+\varepsilon)t} \mathrm{d}t\, \mathbb{E}\big[e_\gamma(\eta_t^{\delta_*-\varepsilon,\{0\}})\big]^2 \\ &\quad = \int_0^\infty \mathrm{d}t\, e^{(2r_\gamma(\delta_*-\varepsilon) - r(\delta_*+\varepsilon))t + o(t)} \leq \int_0^\infty \mathrm{d}t\, e^{\frac{1}{5}r(\delta_*+\varepsilon)t + o(t)} < \infty, \end{aligned} \tag{2.45}$$

where $t \mapsto o(t)$ is continuous, $o(t)/t \to 0$ for $t \to \infty$ by the definition of r_γ in Lemma 2.16, and we have used that $2r_\gamma(\delta_* - \varepsilon) \leq 2 \cdot \frac{3}{4} \cdot \frac{4}{5} r(\delta_* + \varepsilon) = \frac{6}{5} r(\delta_* + \varepsilon)$. This proves (2.42) and hence the required local tightness. ∎

2.7 The derivative of the exponential growth rate

Let us define homogeneous, locally finite measures χ_A on $\mathcal{P}_{\text{fin},+}$ by

$$\chi_A := \sum_{i \in \Lambda} \delta_{iA} \qquad (A \in \mathcal{P}_{\text{fin},+}), \tag{2.46}$$

where δ_{iA} denotes the delta measure on $\mathcal{P}_{\text{fin},+}$ at the point iA. Let $(P_t^\delta)_{t\geq 0}$ and $(P_t^{\dagger\,\delta})_{t\geq 0}$ be the subprobability kernels defined in (1.14) for the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact processes, respectively, in dependence on δ. Note that $\chi_{\{0\}} P_t^\delta$ denotes the 'law' at time t of the process started with a single infected site distributed according to the counting measure on Λ. We start by rewriting Russo's formula (1.24) in terms of the objects we are working with.

Lemma 2.19 (Differential formula) *For each $t \geq 0$, the function $[0,\infty) \ni \delta \mapsto \mathbb{E}[|\eta_t^{\delta,\{0\}}|]$ is continuously differentiable and satisfies*

$$-\frac{\partial}{\partial\delta} \frac{1}{t} \log \mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big] = \frac{1}{t} \int_0^t \mathrm{d}s\, \frac{\chi_{\{0\}} P_s^\delta \boxtimes \chi_{\{0\}} P_{t-s}^{\dagger\,\delta}(\{0\})}{\langle\!\langle \chi_{\{0\}} P_s^\delta \boxtimes \chi_{\{0\}} P_{t-s}^{\dagger\,\delta} \rangle\!\rangle}. \tag{2.47}$$

Proof By (1.24) and the definition of the Campbell law $\hat{\mathbb{P}}_t$ in (1.22)

$$-\frac{\partial}{\partial\delta} \frac{1}{t} \log \mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big] = \frac{1}{t} \int_0^t \mathrm{d}s\, \frac{1}{\mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big]} \sum_{i,j} \mathbb{P}[(0,0) \rightsquigarrow_{(j,s)} (i,t)], \tag{2.48}$$

where

$$\begin{aligned} &\sum_{i,j} \mathbb{P}[(0,0) \rightsquigarrow_{(j,s)} (i,t)] = \sum_{i,j} \mathbb{P}[(j^{-1}, -s) \rightsquigarrow_{(0,0)} (j^{-1}i, t-s)] \\ &\quad = \sum_{i,j} \mathbb{P}[\eta_s^{\delta\,\{i\}} \cap \eta_{t-s}^{\dagger\,\delta\,\{j\}} = \{0\}] = \int \chi_{\{0\}} P_s^\delta(\mathrm{d}A) \int \chi_{\{0\}} P_{t-s}^{\dagger\,\delta}(\mathrm{d}B)\, 1_{\{A \cap B = \{0\}\}}, \end{aligned} \tag{2.49}$$

and for $0 \leq s \leq t$,

$$\begin{aligned}
\mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big] &= \sum_i \mathbb{P}\big[\eta_t^{\delta\,\{0\}} \cap \{i\} \neq \emptyset\big] = \sum_i \mathbb{P}\big[\eta_t^{\delta\,\{i^{-1}\}} \cap \{0\} \neq \emptyset\big] \\
&= \sum_{i,j} \mathbb{E}\big[|\eta_t^{\delta\,\{i^{-1}\}} \cap \{j\}|^{-1} 1_{\{0 \in \eta_t^{\delta\,\{i^{-1}\}} \cap \{j\}\}}\big] \\
&= \int \chi_{\{0\}} P_t^\delta\,(\mathrm{d}A) \int \chi_{\{0\}}\,(\mathrm{d}B)\,|A \cap B|^{-1} 1_{\{0 \in A \cap B\}} \\
&= \langle\!\langle \chi_{\{0\}} P_t^\delta \boxtimes \chi_{\{0\}} \rangle\!\rangle = \langle\!\langle \chi_{\{0\}} P_s^\delta \boxtimes \chi_{\{0\}} P_{t-s}^{\dagger\,\delta} \rangle\!\rangle,
\end{aligned} \tag{2.50}$$

where we have used Lemma 1.2 in the last step. ∎

We will prove Theorem 2 by taking the limit $t \to \infty$ in (2.47). To justify the interchange of limit and differentiation, we will use the following lemma.

Lemma 2.20 (Interchange of limit and differentiation) *Let $I \subset \mathbb{R}$ be a compact interval and let f_n, f, f' be continuous real functions on I. Assume each f_n is continuously differentiable, that $f_n(x) \to f(x)$ and $\frac{\partial}{\partial x} f_n(x) \to f'(x)$ for each $x \in I$, and that*

$$\sup_{x \in I} \sup_n |\tfrac{\partial}{\partial x} f_n(x)| < \infty. \tag{2.51}$$

Then f is continuously differentiable and $\frac{\partial}{\partial x} f(x) = f'(x)$ $(x \in I)$.

Proof We write $I = [x_-, x_+]$ and observe that

$$\begin{aligned}
f(x) &= \lim_{n\to\infty} f_n(x_-) + \lim_{n\to\infty} \int_{x_-}^x \tfrac{\partial}{\partial y} f_n(y)\,\mathrm{d}y \\
&= f(x_-) + \int_{x_-}^x \big(\lim_{n\to\infty} \tfrac{\partial}{\partial y} f_n(y)\big)\mathrm{d}y = f(x_-) + \int_{x_-}^x f'(y)\,\mathrm{d}y,
\end{aligned} \tag{2.52}$$

where the interchange of limit and integration is justified by dominated convergence, using (2.51). Differentiation of (2.52) now yields the statement since f' is continuous. ∎

Proof of Theorem 2 Continuity of the map $(\delta_{\rm c}, \infty) \ni \delta \mapsto \mathring{\nu}_\delta$, and likewise for $\mathring{\nu}^\dagger_\delta$, in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$ has already been proved in Proposition 2.18. By Lemma 2.2, this implies local continuity of the map $(\delta_{\rm c}, \infty) \ni \delta \mapsto \mathring{\nu}_\delta \boxtimes \mathring{\nu}^\dagger_\delta$. Since local convergence on $\mathcal{P}_{\mathrm{fin},+}$ implies convergence of the integral of the bounded functions $A \mapsto 1_{\{A=\{0\}\}}$ and $A \mapsto |A|^{-1} 1_{\{0 \in A\}}$ (which occurs in the definition of $\langle\!\langle\,\cdot\,\rangle\!\rangle$), this implies continuity of the right-hand side of (1.25).

Note that the right-hand side of (1.24) is clearly bounded between zero and one. Therefore, since

$$\frac{1}{t} \log \mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big] \underset{t\to\infty}{\longrightarrow} r(\Lambda, a, \delta) \qquad (\delta \geq 0) \tag{2.53}$$

by the definition of the exponential growth rate in (1.10), using Lemma 2.20, we see that (1.25) follows provided we show that the right-hand side of (2.47) converges for each $\delta \in (\delta_{\rm c}, \infty)$ to the right-hand side of (1.25) as $t \to \infty$.

We rewrite the right-hand side of (2.47) as

$$\int_0^1 \mathrm{d}u\, \frac{e^{-rtu} \chi_{\{0\}} P_{tu}^\delta \boxtimes e^{-rt(1-u)} \chi_{\{0\}} P_{t(1-u)}^{\dagger\,\delta}\,(\{0\})}{\langle\!\langle e^{-rtu} \chi_{\{0\}} P_{tu}^\delta \boxtimes e^{-rt(1-u)} \chi_{\{0\}} P_{t(1-u)}^{\dagger\,\delta} \rangle\!\rangle}. \tag{2.54}$$

It is easy to see from the definition of $\langle\langle\,\cdot\,\rangle\rangle$ that the integrand is bounded between zero and one (in fact, this is the probability in (1.24)). By Theorem 1, for each $0 < u < 1$, the measures $e^{-rtu}\chi_{\{0\}}P^{\delta}_{tu}$ and $e^{-rt(1-u)}\chi_{\{0\}}P^{\dagger\,\delta}_{t(1-u)}$ converge locally on $\mathcal{P}_{\rm fin,+}$ to constant multiples of $\mathring{\nu}_\delta$ and $\mathring{\nu}^\dagger_\delta$, respectively. By Lemma 2.2 and the fact that local convergence on $\mathcal{P}_{\rm fin,+}$ implies convergence of the integral of the bounded functions $A \mapsto 1_{\{A=\{0\}\}}$ and $A \mapsto |A|^{-1}1_{\{0\in A\}}$, we see that the integrand in (2.54) converges in a bounded pointwise way with respect to u to the right-hand side of (1.25). Thus, the result follows by Lebesgue's dominated convergence theorem. ∎

3 Proof details

In this section we supply the proof of all propositions and lemmas that have not been proved yet. The organization is as follows. In Section 3.1 we prove some properties of locally finite measures and different forms of convergence, concretely Proposition 2.1 and Lemmas 2.2, 2.3 and 2.12. In Section 3.2 we consider contact processes started in infinite initial 'laws', proving Lemmas 1.2 and 2.13. In Section 3.3 we construct a metric on Λ with properties as in Lemma 2.15 and prove Lemma 2.16 on the exponential growth rate associated with the functions e_γ defined in terms of such a metric. In Section 3.4 we do a covariance calculation leading to an estimate of which Lemma 2.6 is a special case and use this to derive Lemma 2.17. In Section 3.5, finally, we prove the properties of our Doob transformed processes listed in Lemmas 2.9, 2.10 and 2.11.

3.1 Locally finite measures

In this section, we prove Proposition 2.1 as well as Lemmas 2.2, 2.3 and 2.12. Our first aim is Proposition 2.1. We start with two preparatory lemmas. Recall the definition of $\mathcal{P}_i$ from (1.12).

Lemma 3.1 (Compact classes) *If $\mathcal{C} \subset \mathcal{P}_+$ is compact, then there exists a finite $\Delta \subset \Lambda$ such that $\mathcal{C} \subset \bigcup_{i\in\Delta}\mathcal{P}_i$.*

Proof Choose $\Delta_n \uparrow \Lambda$ with Δ_n finite. If $\mathcal{C} \not\subset \bigcup_{i\in\Delta_n}\mathcal{P}_i$ for each n, then we can find $A_n \in \mathcal{C}$ such that $A_n \cap \Delta_n = \emptyset$. It follows that $A_n \to \emptyset \notin \mathcal{C}$ (in the product topology), hence $\mathcal{C}$ is not a closed subset of $\mathcal{P}$ and therefore not compact. ∎

Lemma 3.2 (Vague and weak convergence) *Let μ_n, μ be locally finite measures on $\mathcal{P}_+$. Then the μ_n converge vaguely to μ if and only if for each $i \in \Lambda$, the restricted measures $\mu_n|_{\mathcal{P}_i}$ converge weakly to $\mu|_{\mathcal{P}_i}$ with respect to the product topology.*

Proof Since $\mathcal{P}\backslash\mathcal{P}_i$ is a closed subset of $\mathcal{P}$, any continuous function $f : \mathcal{P}_i \to \mathbb{R}$ can be extended to a continuous, compactly supported function on $\mathcal{P}_+$ by putting $f(A) := 0$ for $A \in \mathcal{P}_+\backslash\mathcal{P}_i$. Therefore, if the μ_n converge vaguely to μ, it follows that the $\mu_n|_{\mathcal{P}_i}$ converge weakly to $\mu|_{\mathcal{P}_i}$. Conversely, if for each $i \in \Lambda$ the $\mu_n|_{\mathcal{P}_i}$ converge weakly to $\mu|_{\mathcal{P}_i}$, then for each $i, j \in \Lambda$ one has

$$\mu_n|_{\mathcal{P}_i\cap\mathcal{P}_j} \Rightarrow \mu|_{\mathcal{P}_i\cap\mathcal{P}_j}, \quad \mu_n|_{\mathcal{P}_i\backslash\mathcal{P}_j} \Rightarrow \mu|_{\mathcal{P}_i\backslash\mathcal{P}_j} \quad \text{and} \quad \mu_n|_{\mathcal{P}_j\backslash\mathcal{P}_i} \Rightarrow \mu|_{\mathcal{P}_j\backslash\mathcal{P}_i}, \tag{3.1}$$

where we have used that $\mathcal{P}_i\cap\mathcal{P}_j$, $\mathcal{P}_i\backslash\mathcal{P}_j$ and $\mathcal{P}_j\backslash\mathcal{P}_i$ are compact sets. Continuing this process, we see by induction that for each finite $\Delta \subset \Lambda$, the restrictions $\mu_n|_{\bigcup_{i\in\Delta}\mathcal{P}_i}$ converge weakly to

$\mu|_{\bigcup_{i\in\Delta}\mathcal{P}_i}$. By Lemma 3.1, if $f:\mathcal{P}_+\to\mathbb{R}$ is a compactly supported continuous function, then f is supported on $\bigcup_{i\in\Delta}\mathcal{P}_i$ for some finite $\Delta\subset\Lambda$. It follows that $\int\mu_n(\mathrm{d}A)f(A)\to\int\mu(\mathrm{d}A)f(A)$, proving that the μ_n converge vaguely to μ. ∎

Proof of Proposition 2.1 The equivalence of (i) and (ii) follows in a straightforward manner from Prohorov's theorem applied to the countable space $\mathcal{P}_{\mathrm{fin},\,i}$ with the discrete topology.

Since the discrete topology on $\mathcal{P}_{\mathrm{fin},\,i}$ is stronger than the product topology, weak convergence of the $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ with respect to the discrete topology implies weak convergence with respect to the product topology. By Lemma 3.2, this shows that local convergence on $\mathcal{P}_{\mathrm{fin},+}$ implies vague convergence on $\mathcal{P}_+$ and hence (i) implies also (iii).

To prove (iii)⇒(i), note that by local tightness, for each $i\in\Lambda$ the measures $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ are relatively compact in the topology of weak convergence with respect to the discrete topology. Let μ^i_* be a subsequential limit. Since weak convergence with respect to the discrete topology implies weak convergence with respect to the product topology, by Lemma 3.2, we conclude that $\mu^i_*=\mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$. Since this is true for each cluster point, we conclude that the $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ converge weakly to $\mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$ with respect to the discrete topology.

The implication (i)⇒(iv) follows from what we have already proved. To prove the reverse implication, it suffices to show local tightness. Since for each $i\in\Lambda$, the finite measures $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ converge pointwise to $\mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$, it suffices to show that their total mass satisfies

$$\limsup_{n\to\infty}\mu_n(\{A:i\in A\})\le\mu(\{A:i\in A\}).\tag{3.2}$$

By vague convergence (see Lemma 1.1), the limit superior is actually a limit and equals the right-hand side. ∎

Proof of Lemma 2.2 The local finiteness of $\mu\otimes\nu$ follows from Lemma 1.1 and the fact that

$$\begin{aligned}\int\mu\otimes\nu\,(\mathrm{d}C)1_{\{i\in C\}}&=\int\mu(\mathrm{d}A)\int\nu(\mathrm{d}B)1_{\{i\in A\cap B\}}\\&=\Big(\int\mu(\mathrm{d}A)1_{\{i\in A\}}\Big)\Big(\int\nu(\mathrm{d}B)1_{\{i\in B\}}\Big)<\infty\quad(i\in\Lambda).\end{aligned}\tag{3.3}$$

To see that $\mu_n\otimes\nu_n$ converges vaguely to $\mu\otimes\nu$ if μ_n,ν_n converge vaguely to μ,ν, respectively, by Lemma 1.1, it suffices to check that

$$\int\mu_n\otimes\nu_n(\mathrm{d}C)1_{\{C\cap D\neq\emptyset\}}\underset{n\to\infty}{\longrightarrow}\int\mu\otimes\nu(\mathrm{d}C)1_{\{C\cap D\neq\emptyset\}}\qquad(D\in\mathcal{P}_{\mathrm{fin},+}).\tag{3.4}$$

Since

$$1_{\{C\cap D\neq\emptyset\}}=1-\prod_{i\in D}1_{\{i\notin C\}}=1-\prod_{i\in D}(1-1_{\{i\in C\}})=\sum_{\substack{D'\subset D\\D'\neq\emptyset}}(-1)^{|D'|+1}\prod_{i\in D'}1_{\{i\in C\}},\tag{3.5}$$

and since $\prod_{i\in D'}1_{\{i\in C\}}=1_{\{D'\subset C\}}$ formula (3.4) is equivalent to

$$\int\mu_n\otimes\nu_n(\mathrm{d}C)1_{\{D\subset C\}}\underset{n\to\infty}{\longrightarrow}\int\mu\otimes\nu(\mathrm{d}C)1_{\{D\subset C\}}\qquad(D\in\mathcal{P}_{\mathrm{fin},+}).\tag{3.6}$$

Now

$$\begin{aligned}\int\mu_n\otimes\nu_n(\mathrm{d}C)1_{\{D\subset C\}}&=\int\mu_n(\mathrm{d}A)\int\nu_n(\mathrm{d}B)1_{\{D\subset(A\cap B)\}}\\&=\Big(\int\mu_n(\mathrm{d}A)1_{\{D\subset A\}}\Big)\Big(\int\nu_n(\mathrm{d}B)1_{\{D\subset B\}}\Big),\end{aligned}\tag{3.7}$$

which, by our assumptions that $\mu_n \Rightarrow \mu$ and $\nu_n \Rightarrow \nu$, converges to the analogue formula with μ_n, ν_n replaced by μ, ν.

To see that the vague convergence of $\mu_n \boxtimes \nu_n$ can be strengthened to local convergence on $\mathcal{P}_{\mathrm{fin},+}$ if either μ_n or ν_n converges locally on $\mathcal{P}_{\mathrm{fin},+}$, it suffices by Proposition 2.1 (iii)⇒(i) to show that the local tightness of either μ_n or ν_n implies local tightness of $\mu_n \boxtimes \nu_n$. By symmetry, it suffices to consider the case when the μ_n are locally tight. Since vague convergence of the ν_n implies convergence of $\int \nu_n(\mathrm{d}A) 1_{\{i\in A\}}$ for each $i \in \Lambda$, the statement now follows from the following lemma, that we formulate separately since it is of some interest on its own. ∎

Lemma 3.3 (Local tightness of intersection measure) *Let μ_n, ν_n $(n \geq 1)$ be locally finite measures on $\mathcal{P}_+$. Assume that the μ_n $(n \geq 1)$ are concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and that they are locally tight. Assume that the ν_n satisfy $\sup_{n\geq 1} \int \nu_n(\mathrm{d}A) 1_{\{i\in A\}} < \infty$ for all $i \in \Lambda$. Then the intersection measures $\mu_n \boxtimes \nu_n$ $(n \geq 1)$ are concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and locally tight.*

Proof Since $\mu_n \boxtimes \nu_n$ is concentrated on sets of the form $A \cap B$ with $A \in \mathcal{P}_{\mathrm{fin},+}$, it is clear that $\mu_n \boxtimes \nu_n$ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$ for each $n \geq 1$. Fix $i \in \Lambda$ and $\varepsilon > 0$, and set $K := \sup_{n\geq 1} \int \nu_n(\mathrm{d}A) 1_{\{i\in A\}}$. By the local tightness of the μ_n, there exists a finite $\mathcal{D} \subset \mathcal{P}_{\mathrm{fin},i}$ such that $\sup_n \mu_n(\mathcal{P}_{\mathrm{fin},i} \backslash \mathcal{D}) \leq \varepsilon/K$. The same obviously holds for the larger finite set $\mathcal{D}' := \mathcal{P}(D) = \{A : A \subset D\}$, where $D := \bigcup\{A : A \in \mathcal{D}\}$. Now

$$\begin{aligned}\sup_{n\geq 1} \mu_n \boxtimes \nu_n(\mathcal{P}_{\mathrm{fin},i} \backslash \mathcal{D}') &= \sup_{n\geq 1} \int \mu_n(\mathrm{d}A) \int \nu_n(\mathrm{d}B)\, 1_{\{i\in A\cap B\}} 1_{\{A\cap B \not\subset D\}} \\ &\leq \sup_{n\geq 1} \int \mu_n(\mathrm{d}A)\, 1_{\{i\in A\}} 1_{\{A\not\subset D\}} \int \nu_n(\mathrm{d}B)\, 1_{\{i\in B\}} \leq \varepsilon.\end{aligned} \tag{3.8}$$

Since $i \in \Lambda$ and $\varepsilon > 0$ are arbitrary, the claim follows. ∎

Proof of Lemma 2.3 Formula (2.2) obviously defines a nonzero, homogeneous measure on $\mathcal{P}_{\mathrm{fin},+}$. Since

$$\mu(\{A : 0 \in A\}) = c \sum_i \mathbb{P}[0 \in i\Delta] = c \sum_i \mathbb{P}[i^{-1} \in \Delta] = cE\big[|\Delta|\big], \tag{3.9}$$

it follows from Lemma 1.1 that μ is locally finite if and only if $\mathbb{E}\big[|\Delta|\big] < \infty$. If μ is given by (2.2), then

$$\langle\!\langle \mu \rangle\!\rangle = c \sum_{i\in\Lambda} \mathbb{E}\big[|i\Delta|^{-1} 1_{\{0\in i\Delta\}}\big] = c\mathbb{E}\big[|\Delta|^{-1}\big(\sum_{i\in\Lambda} 1_{\{i^{-1}\in\Delta\}}\big)\big] = c. \tag{3.10}$$

To see that every nonzero, homogeneous measure μ on $\mathcal{P}_{\mathrm{fin},+}$ with $\langle\!\langle \mu \rangle\!\rangle < \infty$ can be written in the form (2.2), define a probability law ρ on $\mathcal{P}_{\mathrm{fin},0}$ by

$$\rho(\{A\}) := \langle\!\langle \mu \rangle\!\rangle^{-1} \mu(\{A\}) |A|^{-1} 1_{\{0\in A\}}. \tag{3.11}$$

Let Δ be a random variable with law ρ. We claim that μ is given by (2.2) with $c = \langle\!\langle \mu \rangle\!\rangle$. To check this, we calculate, for $A \in \mathcal{P}_{\mathrm{fin},+}$:

$$\begin{aligned}\langle\!\langle \mu \rangle\!\rangle \sum_{i\in\Lambda} \mathbb{P}\big[i\Delta = A\big] &= \langle\!\langle \mu \rangle\!\rangle \sum_{i\in\Lambda} \mathbb{P}\big[\Delta = i^{-1}A\big] = \langle\!\langle \mu \rangle\!\rangle \sum_{i\in\Lambda} \rho(\{i^{-1}A\}) \\ &= \sum_{i\in\Lambda} \mu(\{i^{-1}A\}) |i^{-1}A|^{-1} 1_{\{0\in i^{-1}A\}} = \mu(\{A\}) |A|^{-1} \sum_{i\in\Lambda} 1_{\{i\in A\}} = \mu(\{A\}),\end{aligned} \tag{3.12}$$

where we have used the homogeneity of μ. Since

$$\begin{aligned} &\mu(\{A\}) = \langle\!\langle \mu \rangle\!\rangle \sum_{i\in\Lambda} \mathbb{P}[i\Delta = A] = \langle\!\langle \mu \rangle\!\rangle\, m(A)\mathbb{P}[\tilde\Delta = \tilde A] \\ &\quad\text{where}\quad m(A) := |\{i\in\Lambda : iA = A\}| \qquad \big(A\in\mathcal{P}_{\text{fin},+}\big), \end{aligned} \tag{3.13}$$

the law of $\tilde\Delta$ is uniquely determined by μ. ∎

Remark It is easy to see that the constant $m(A)$ defined in (3.13) satisfies $m(A) \le |A|$ and that $\{i \in \Lambda : iA = A\}$ is a finite subgroup of Λ. If every element of Λ is of infinite order (as is the case, for example, for $\Lambda = \mathbb{Z}^d$), then $m(A) = 1$ for all finite $A \subset \Lambda$.

We finish the section on locally finite measures with the still outstanding:

Proof of Lemma 2.12 We will apply the mass transport principle, compare the proof of Lemma 1.2 below. Let μ, ν be homogeneous, locally finite measures on $\mathcal{P}_+$ and assume that μ is concentrated on $\mathcal{P}_{\text{fin},+}$. For $A \in \mathcal{P}_{\text{fin}}$ and $B \in \mathcal{P}$ such that $A \cap B \neq \emptyset$, let us define a probability distribution $M_{A,B}$ on $\Lambda \times \Lambda$ by

$$M_{A,B}(i,j) := |A|^{-1}1_{\{i\in A\}}|A\cap B|^{-1}1_{\{j\in A\cap B\}}, \tag{3.14}$$

and let $f : \Lambda\times\Lambda \to [0,\infty]$ be defined by

$$f(i,j) := \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B)\, 1_{\{A\cap B\neq\emptyset\}} M_{A,B}(i,j). \tag{3.15}$$

Since μ and ν are homogeneous, we observe that $f(ki,kj) = f(i,j)$ $(i,j,k \in \Lambda)$. Moreover,

$$\sum_j f(0,j) = \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B)\, 1_{\{A\cap B\neq\emptyset\}} \frac{1}{|A|} 1_{\{0\in A\}} = \int \mu(\mathrm{d}A)\, h_\nu(A) \frac{1}{|A|} 1_{\{0\in A\}} = \langle\!\langle h_\nu \mu \rangle\!\rangle, \tag{3.16}$$

while

$$\sum_i f(i,0) = \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B)\, 1_{\{A\cap B\neq\emptyset\}} \frac{1}{|A\cap B|} 1_{\{0\in A\cap B\}} = \langle\!\langle \mu \boxtimes \nu \rangle\!\rangle. \tag{3.17}$$

Formula (2.20) now follows from the fact that $\sum_i f(i,0) = \sum_i f(0,i^{-1}) = \sum_j f(0,j)$. Note that this holds regardless of whether $h_\nu\mu$ is locally finite or not. If $\int \mu(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty$, then by the shift-invariance and subadditivity of h_ν, we see that $h_\nu(A) \le h_\nu(\{0\})|A|$ and hence $\int \mu(\mathrm{d}A) h_\nu(A) 1_{\{0\in A\}} < \infty$, proving that $h_\nu\mu$ is locally finite. ∎

3.2 Infinite starting measures

In this section we prove Lemma 1.2 on contact process duality for homogeneous, infinite starting measures. We also give the proof of Lemma 2.13, which is concerned with relative compactness and cluster points of eigenmeasures for (Λ, a, δ)-contact processes with varying δ.

Proof of Lemma 1.2 Fix $t \ge 0$ and for $A, B \in \mathcal{P}_+$, consider the events

$$\mathcal{E}_{A,B} := \{|\eta^{A,0}_t \cap B| < \infty\} \quad\text{and}\quad \mathcal{E}'_{A,B} := \{|A \cap \eta^{\dagger\, B,t}_t| < \infty\}. \tag{3.18}$$

We observe that $\mu P_t \boxtimes \nu$ (resp. $\mu \boxtimes \nu P^\dagger_t$) is concentrated on $\mathcal{P}_{\text{fin},+}$ if and only if $\mathbb{P}(\mathcal{E}_{A,B}) = 1$ (resp. $\mathbb{P}(\mathcal{E}'_{A,B}) = 1$) for a.e. A w.r.t. μ and a.e. B w.r.t. ν. Set $\Delta_0 := A \cap \eta^{\dagger\, B,t}_t$ and $\Delta_t :=$

$\eta_t^{A,0} \cap B$. Since $\eta_t^{\Delta_0,0} \supset \Delta_t$ and $\eta_t^{\dagger\,\Delta_t,t} \supset \Delta_0$, we see that the events $\mathcal{E}_{A,B}$ and $\mathcal{E}'_{A,B}$ are a.s. equal, and hence $\mu P_t \otimes \nu$ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$ if and only if $\mu \otimes \nu P_t^\dagger$ is.

We will now prove (1.18) by applying the "mass transport principle". For a given graphical representation ω and sets $A, B \in \mathcal{P}_+$ such that the events $\mathcal{E}_{A,B}$ and $\mathcal{E}'_{A,B}$ hold, we define a probability distribution $M_{A,B,\omega}$ on $\Lambda \times \Lambda$ by

$$M_{A,B,\omega}(i,j) := |\Delta_0|^{-1} 1_{\{i\in\Delta_0\}} |\Delta_t|^{-1} 1_{\{j\in\Delta_t\}}. \tag{3.19}$$

We define a function $f : \Lambda \times \Lambda \to [0,\infty]$ by

$$f(i,j) := \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B) \int \mathbb{P}(\mathrm{d}\omega)\, 1_{\mathcal{E}_{A,B}}(\omega)\, M_{A,B,\omega}(i,j). \tag{3.20}$$

Obviously, $f(ki,kj) = f(i,j)$ $(i,j,k \in \Lambda)$ due to the homogeneity of μ and ν. Moreover,

$$\begin{aligned}\sum_i f(i,0) &= \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B)\, \mathbb{E}\big[|\eta_t^{A,0} \cap B|^{-1} 1_{\{0\in\eta_t^{A,0}\cap B\}}\big] \\ &= \int \mu P_t(\mathrm{d}A') \int \nu(\mathrm{d}B) |A' \cap B|^{-1} 1_{\{0\in A'\cap B\}} = \langle\!\langle \mu P_t \otimes \nu \rangle\!\rangle.\end{aligned} \tag{3.21}$$

The same argument shows that $\sum_j f(0,j) = \langle\!\langle \mu \otimes \nu P_t^\dagger \rangle\!\rangle$ and hence

$$\langle\!\langle \mu P_t \otimes \nu \rangle\!\rangle = \sum_i f(i,0) = \sum_i f(0,i^{-1}) = \langle\!\langle \mu \otimes \nu P_t^\dagger \rangle\!\rangle, \tag{3.22}$$

where the middle step is a simple example of what is more generally known as the mass transport principle, see [Hag11]. ∎

Proof of Lemma 2.13 By the homogeneity and normalization of the ν_n, one has

$$\int \nu_n(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} \leq \sum_{i\in B} \int \nu_n(\mathrm{d}A) 1_{\{i\in A\}} = |B|. \tag{3.23}$$

Since this estimate is uniform in n, applying [Swa09, Lemma 3.2] we find that the $(\nu_n)_{n\geq 0}$ are relatively compact in the topology of vague convergence. By going to a subsequence if necessary, we may assume that the ν_n converge vaguely to a limit ν. Since the ν_n are eigenmeasures, denoting the (Λ, a, δ_n)-contact process started in A by $(\eta_t^{\delta_n,A})_{t\geq 0}$, we have

$$\int \nu_n(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta_n,A} \in \cdot\,]\big|_{\mathcal{P}_+} = e^{\lambda_n t} \nu_n \qquad (t \geq 0). \tag{3.24}$$

Since $\lambda_n \to \lambda$, the right-hand side of this equation converges vaguely to $e^{\lambda t}\nu$. To prove vague convergence of the left-hand side, by Lemma 1.1, it suffices to prove that for $B \in \mathcal{P}_{\mathrm{fin}}$,

$$\int \nu_n(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta_n,A} \cap B \neq \emptyset] \to \int \nu(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset]. \tag{3.25}$$

We estimate

$$\begin{aligned}&\Big| \int \nu_n(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta_n,A} \cap B \neq \emptyset] - \int \nu(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset] \Big| \\ &\quad\leq \int \nu_n(\mathrm{d}A) \Big| \mathbb{P}[\eta_t^{\delta_n,A} \cap B \neq \emptyset] - \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset] \Big| &(3.26)\\ &\qquad + \Big| \int \nu_n(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset] - \int \nu(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset] \Big|. &(3.27)\end{aligned}$$

The term in (3.27) tends to zero as $n \to \infty$ by Lemmas 1.1 and 2.4. By duality, we can rewrite the term in (3.26) as

$$\int \nu_n(\mathrm{d}A)\Big|\mathbb{P}[A \cap \eta_t^{\dagger\,\delta_n,B} \neq \emptyset] - \mathbb{P}[A \cap \eta_t^{\dagger\,\delta,B} \neq \emptyset]\Big|. \tag{3.28}$$

We couple the graphical representations for processes with different recovery rates in the natural way, by constructing a Poisson point process $\Omega^{\rm r}$ on $\Lambda \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity one, and letting $\omega^{\rm r}_\delta := \{(i,t) : \exists 0 \leq r \leq \delta \text{ s.t. } (i,t,r) \in \Omega^{\rm r}\}$ be the set of recovery symbols for the process with recovery rate δ. Then, letting $\eta_t^{\dagger\,0,B}$ denote the process with zero recovery rate, the quantity in (3.28) can be estimated from above by

$$\begin{aligned}
&\int \nu_n(\mathrm{d}A)\mathbb{P}\big[A \cap \eta_t^{\dagger\,0,B} \neq \emptyset,\ \eta_t^{\dagger\,\delta_n,B} \neq \eta_t^{\dagger\,\delta,B}\big] \\
&\quad = \int \mathbb{P}\big[\eta_t^{\dagger\,0,B} \in \mathrm{d}C,\ \eta_t^{\dagger\,\delta_n,B} \neq \eta_t^{\dagger\,\delta,B}\big] \int \nu_n(\mathrm{d}A) 1_{\{A\cap C\neq\emptyset\}} \\
&\quad \leq \int \mathbb{P}\big[\eta_t^{\dagger\,0,B} \in \mathrm{d}C,\ \eta_t^{\dagger\,\delta_n,B} \neq \eta_t^{\dagger\,\delta,B}\big]\,|C| = \mathbb{E}\big[|\eta_t^{\dagger\,0,B}| 1_{\{\eta_t^{\dagger\,\delta_n,B}\neq\eta_t^{\dagger\,\delta,B}\}}\big],
\end{aligned} \tag{3.29}$$

where we have used (3.23). Since the right-hand side of (3.29) tends to zero by dominated convergence, this proves the lemma. ∎

3.3 Exponential moments

Recall the function $e_\gamma(A) = \sum_{i\in A} e^{\gamma d(0,i)}$ from (2.38), which measures how 'spread out' a set $A \in \mathcal{P}_{\rm fin}$ is in terms of exponential weights and a suitably slowly growing metric d as in (2.37). In this section, we provide the proof of Lemma 2.15, showing that such a metric exists. We then give the proof of Lemma 2.16, which states that the expectation of the function e_γ of a contact process has a well defined exponential growth rate, with certain bounds.

Proof of Lemma 2.15 We can find finite $\{0\} = \Delta_1 \subset \Delta_2 \subset \cdots$ such that $\sum_{i\in\Lambda\backslash\Delta_n} a(0,i) \leq |a|e^{-(n-1)}$. Making the sets Δ_n for $n \geq 2$ larger if necessary, we can moreover choose these sets such that they are symmetric, i.e., $\{i^{-1} : i \in \Delta_n\} = \Delta_n$ and such that $\Delta_\infty := \bigcup_{n\geq1} \Delta_n$ generates Λ. (In particular, we can always choose $\Delta_\infty = \Lambda$, but for nearest-neighbor processes on graphs this leads to a somewhat unnatural metric d, which is why we only assume here that Δ_∞ generates Λ.) We set $\Delta_0 := \emptyset$ and define

$$\phi(i) := \begin{cases} n & (i \in \Delta_n\backslash\Delta_{n-1},\ n \geq 1) \\ \infty & (i \in \Lambda\backslash\Delta_\infty). \end{cases} \tag{3.30}$$

Since $a(0,i) = 0$ for $i \notin \Delta_\infty$, whe have that

$$\sum_{i\in\Lambda} a(0,i)\phi(i)^\gamma = \sum_{n\geq1} n^\gamma \sum_{i\in\Delta_n\backslash\Delta_{n-1}} a(0,i) \leq |a| \sum_{n\geq1} n^\gamma e^{-(n-2)} < \infty \tag{3.31}$$

for each $0 \leq \gamma < \infty$. Set

$$d'(i,j) = d'(0,i^{-1}j) := \log(\phi(i^{-1}j)) \qquad (i,j \in \Lambda). \tag{3.32}$$

Then d' satisfies properties (2.37) (i)–(iii), $d'(i,j) = 0$ if and only if $i = j$, and $d'(i,j) = d'(j,i)$ (by the symmetry of the sets Δ_n). Since d' need not yet be a metric, we define

$$d(i,j) := \inf\big\{\sum_{k=1}^{n} d'(i_{k-1}, i_k) : n \geq 1,\ i_0, \ldots, i_n \in \Lambda,\ i_0 = i,\ i_n = j\big\}, \tag{3.33}$$

i.e., $d(i,j)$ is a graph-style distance between i and j, defined as the shortest path from i to j where an edge from i_{k-1} to i_k has length $d'(i_{k-1}, i_k)$. Note that $d(i,j) < \infty$ for each $i,j \in \Lambda$ since Δ_∞ generates Λ and $d(i,j) > 0$ for each $i \neq j$ since $d'(i,j) \geq \log(2)$ for each $i \neq j$. It is now straightforward to check that d is a metric on Λ and that $d(i,j) = d(ki,kj)$ for all $i,j,k \in \Lambda$. Since $d(i,j) \leq d'(i,j)$, the metric d also enjoys property (2.37) (iii). Property (2.37) (ii), finally, follows from the fact that

$$\{i \in \Lambda : d(0,i) \leq M\} \subset \{j_1 \cdots j_n : 1 \leq n \leq M/\log(2),\ d'(0,j_k) \leq M\ \forall k = 1, \ldots, n\}, \tag{3.34}$$

where we use that $d'(i,j) \geq \log(2)$ for all $i \neq j$, and we observe that if $d(0,i) \leq M$ $(i \neq 0)$, then there must be some $n \geq 1$ and $0 = i_0, \ldots, i_n = i$ with $\sum_{k=1}^{n} d'(i_{k-1}, i_k) \leq M$. Setting $j_k := i_{k-1}^{-1} i_k$ we see that i must be of the form $i = j_1 \cdots j_n$ with $\sum_{k=1}^{n} d'(0, j_k) \leq M$. ∎

As a preparation for Lemma 2.16, we need one more result.

Lemma 3.4 (Existence of exponential moments) *Let $(\eta_t^A)_{t\geq 0}$ be a (Λ, a, δ)-contact process started in a finite initial state $\eta_0^A = A \in \mathcal{P}_{\rm fin}$ and let d be a metric on Λ as in Lemma 2.15. Then*

$$\mathbb{E}\big[e_\gamma(\eta_t^A)\big] \leq e^{K_\gamma t} e_\gamma(A) \quad (t \geq 0) \quad \textit{where} \quad K_\gamma := \sum_{i\in\Lambda} a(0,i) e^{\gamma d(0,i)}. \tag{3.35}$$

Proof For $\gamma = 0$ this follows from [Swa09, Prop. 2.1]. To prove the statement for $\gamma > 0$, let G be the generator of the (Λ, a, δ)-contact process as defined in (1.2). Then

$$\begin{aligned} G e_\gamma(A) &= \sum_{i\in A}\sum_{j\notin A} a(i,j) e^{\gamma d(0,j)} - \delta \sum_{i\in A} e^{-\gamma d(0,i)} \\ &\leq \sum_{i\in A}\sum_{j\in\Lambda} a(i,j) e^{\gamma(d(0,i)+d(i,j))} = K_\gamma e_\gamma(A), \end{aligned} \tag{3.36}$$

where we have used that $\sum_{j\in\Lambda} a(i,j) e^{\gamma d(i,j)} = \sum_{j\in\Lambda} a(0, i^{-1}j) e^{\gamma d(0,i^{-1}j)} = K_\gamma$ $(i \in \Lambda)$.

Set $\tau_N := \inf\{t \geq 0 : e_\gamma(\eta_t^A) \geq N\}$. Since the stopped process is a Markov process with finite state space, it follows by standard arguments from (3.36) that

$$\mathbb{E}\big[e_\gamma(\eta_{t\wedge\tau_N}^A)\big] \leq e^{K_\gamma t} e_\gamma(A) \quad (t \geq 0,\ N \geq 1), \tag{3.37}$$

which in turn implies that $\mathbb{P}[e_\gamma(\eta_{t\wedge\tau_N}^A) \geq N] \to 0$ as $N \to \infty$ and hence $\tau_N \to \infty$ a.s. Therefore, letting $N \to \infty$ in (3.37), we arrive at (3.35). ∎

Proof of Lemma 2.16 Note that $r_0(\Lambda, a, \delta) = r(\Lambda, a, \delta)$ is the exponential growth rate from (1.10). The statement for $\gamma = 0$ has been proved in [Swa09, Lemma 1.1 and formula (3.5)]. To prove the general statement, set $\pi_t^\gamma := \mathbb{E}\big[e_\gamma(\eta_t^{\{0\}})\big]$. Formula (2.39) will follow from standard facts [Lig99, Thm B.22] if we show that $t \mapsto \log \pi_t^\gamma$ is subadditive. Recalling the graphical representation of the (Λ, a, δ)-contact process, we observe that indeed

$$\begin{aligned} \pi_{s+t}^\gamma &= \sum_i \mathbb{P}[(0,0) \rightsquigarrow (i, s+t)] e^{\gamma d(0,i)} \\ &\leq \sum_{ij} \mathbb{P}[(0,0) \rightsquigarrow (j,s) \rightsquigarrow (i, s+t)] e^{\gamma(d(0,j)+d(j,i))} = \pi_s^\gamma \pi_t^\gamma, \end{aligned} \tag{3.38}$$

which implies the subadditivity of $t \mapsto \log \pi^\gamma_t$ and hence formula (2.39). Since $e_\gamma(A) \leq e_{\gamma'}(A)$ for all $\gamma \leq \gamma'$, it is clear that $\gamma \mapsto r_\gamma$ is nondecreasing. The fact that $-\delta \leq r_0$ has been proved in [Swa09, Lemma 1.1] while the estimate $r_\gamma \leq K_\gamma$ is immediate from Lemma 3.4.

To prove that the function $[0,\infty) \ni \gamma \mapsto r_\gamma$ defined in Lemma 2.16 is right-continuous, we observe that it follows from (2.39) that for any $t_n \uparrow \infty$,

$$r_\gamma = \lim_{n\to\infty} \inf_{1\leq k\leq n} \frac{1}{t_k} \log \mathbb{E}\big[e_\gamma(\eta^{\{0\}}_{t_k})\big]. \tag{3.39}$$

By dominated convergence and the finiteness of exponential moments (Lemma 3.4) we have that for each fixed $t > 0$, the function $\gamma \mapsto \frac{1}{t}\log \mathbb{E}[e_\gamma(\eta^{\{0\}}_t)]$ is continuous. Therefore, being the decreasing limit of continuous functions, $\gamma \mapsto r_\gamma$ must be upper semi-continuous. Since $\gamma \mapsto r_\gamma$ is nondecreasing, this is equivalent to continuity from the right. ∎

3.4 Covariance estimates

The next lemma gives a uniform estimate on expectations of the functions $e_\gamma(A)$ defined in (2.38) under the measures $1_{\{0\in\cdot\}}\frac{1}{\hat\pi_\lambda}\hat\mu_\lambda$. Lemma 2.6 and Lemma 2.17, which were stated and used in Sections 2.2 and 2.6 respectively, follow as corollaries to this lemma. Their proofs are given at the end of this section.

Although this is not exactly how the proof goes, the following heuristic is perhaps useful for understanding the main strategy. Since Campbell measures change second moments into first moments, what we need to control are expectations of the form $\mathbb{E}[e_\gamma(\eta^{\{0\}}_t)^2]$, which leads us to consider events of the form

$$(0,0) \rightsquigarrow (i,t) \quad \text{and} \quad (0,0) \rightsquigarrow (j,t). \tag{3.40}$$

Since in the subcritical regime, long connections are unlikely, the largest contribution to the probability of such an event comes from events of the form

$$(0,0) \rightsquigarrow (k,s) \begin{cases} \rightsquigarrow (i,t) \\ \rightsquigarrow (j,t). \end{cases} \tag{3.41}$$

where $s \in [0,t]$ is close to t and $k \in \Lambda$. Indeed, if the exponential growth rate $r = r(\Lambda, a, \delta)$ is negative, then the probability of an event of the form (3.41) is of the order $e^{rs}(e^{r(t-s)})^2$, which much smaller than the probability that $(0,0) \rightsquigarrow (i,t)$, unless $t-s$ is of order one. In view of this, if we find an infection at some late time t, then all other infected sites are likely to be close to it. Although this reasoning is only heuristic, it turns out that the covariance formula (3.45) below provides a convenient way of making such arguments precise.

Lemma 3.5 (Uniform exponential moment bound) *Let $\hat\mu_\lambda$ and $\hat\pi_\lambda$ be defined as in (2.5)–(2.6) and for $\gamma \geq 0$, let e_γ be the function defined in (2.38) in terms of a metric d satisfying (2.37). Then, for any (Λ, a, δ)-contact process with exponential growth rate $r = r(\Lambda, a, \delta)$,*

$$\limsup_{\lambda\downarrow r} \frac{1}{\hat\pi_\lambda} \int \hat\mu_\lambda(\mathrm{d}A) 1_{\{0\in A\}} e_\gamma(A) \leq (|a| + \delta) \int_0^\infty e^{-rt}\mathrm{d}t\, \mathbb{E}\big[e_\gamma(\eta^{\{0\}}_t)\big]^2. \tag{3.42}$$

We note that although the bound in (3.42) holds regardless of the values of γ and $r = r(\Lambda, a, \delta)$, the right-hand side will usually be infinite, unless $r < 0$ and γ is small enough (see the proofs of Lemma 2.7 and Proposition 2.18).

Proof Fix $\gamma \geq 0$ and, to ease notation, set $\psi_\gamma(i,j) := e^{\gamma d(i,j)}$ $(i,j,k \in \Lambda)$. We observe that

$$\begin{aligned}\int \hat{\mu}_\lambda(\mathrm{d}A) 1_{\{0\in A\}} e_\gamma(A) &= \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \mathbb{E}\big[1_{\{0\in\eta_t^{\{i\}}\}} 1_{\{j\in\eta_t^{\{i\}}\}} \psi_\gamma(0,j)\big] \\ &= \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \mathbb{E}\big[1_{\{i^{-1}\in\eta_t^{\{0\}}\}} 1_{\{i^{-1}j\in\eta_t^{\{0\}}\}} \psi_\gamma(i^{-1}, i^{-1}j)\big] \\ &= \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \psi_\gamma(i,j) \mathbb{P}\big[i\in\eta_t^{\{0\}},\ j\in\eta_t^{\{0\}}\big].\end{aligned} \tag{3.43}$$

Set $f_i(A) := 1_{\{i\in A\}}$. Then

$$\mathbb{P}\big[i\in\eta_t^{\{0\}},\ j\in\eta_t^{\{0\}}\big] = \mathbb{E}\big[f_i(\eta_t^{\{0\}})\big]\,\mathbb{E}\big[f_j(\eta_t^{\{0\}})\big] + \mathrm{Cov}\big(f_i(\eta_t^{\{0\}}), f_j(\eta_t^{\{0\}})\big). \tag{3.44}$$

By a standard covariance formula (see [Swa09, Prop. 2.2]), for any functions f, g of polynomial growth (as in (2.11) below), one has

$$\mathrm{Cov}\big(f(\eta_t^{\{0\}}), g(\eta_t^{\{0\}})\big) = 2\int_0^t \mathbb{E}\big[\Gamma(P_s f, P_s g)(\eta_{t-s}^{\{0\}})\big]\mathrm{d}s \qquad (t \geq 0), \tag{3.45}$$

where $(P_t)_{t\geq 0}$ denotes the semigroup of the (Λ, a, δ)-contact process and $\Gamma(f,g) = \frac{1}{2}(G(fg) - fGg - gGf)$, with G as in (1.2). A little calculation (see [Swa09, formula (4.6)]) shows that

$$\begin{aligned}2\Gamma(P_s f, P_s g)(A) = &\sum_{k\in A}\sum_{l\notin A} a(k,l)\big(P_s f(A\cup\{l\}) - P_s f(A)\big)\big(P_s g(A\cup\{l\}) - P_s g(A)\big) \\ &+\delta \sum_{k\in A}\big(P_s f(A\backslash\{k\}) - P_s f(A)\big)\big(P_s g(A\backslash\{k\}) - P_s g(A)\big).\end{aligned} \tag{3.46}$$

Applying (3.46) to the functions $f = f_i$, $g = f_j$, using the fact that, by the graphical representation,

$$\big|P_s f_i(A\cup\{l\}) - P_s f_i(A)\big| = \big|\mathbb{P}\big[i\in\eta_s^{A\cup\{l\}}\big] - \mathbb{P}\big[i\in\eta_s^{A}\big]\big| \leq \mathbb{P}\big[i\in\eta_s^{\{l\}}\big], \tag{3.47}$$

we find that

$$2\big|\Gamma(P_s f_i, P_s f_j)(A)\big| \leq \sum_{k\in A}\sum_{l\notin A} a(k,l)\mathbb{P}\big[i\in\eta_s^{\{l\}}\big]\mathbb{P}\big[j\in\eta_s^{\{l\}}\big] + \delta\sum_{k\in A}\mathbb{P}\big[i\in\eta_s^{\{k\}}\big]\mathbb{P}\big[j\in\eta_s^{\{k\}}\big], \tag{3.48}$$

which by (3.45) implies that

$$\begin{aligned}&\big|\mathrm{Cov}\big(f_i(\eta_t^{\{0\}}), f_j(\eta_t^{\{0\}})\big)\big| \\ &\quad\leq \int_0^t \sum_{k,l} a(k,l)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}},\ l\notin\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{l\}}\big]\mathbb{P}\big[j\in\eta_s^{\{l\}}\big]\mathrm{d}s \\ &\qquad+\delta\int_0^t \sum_k \mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{k\}}\big]\mathbb{P}\big[j\in\eta_s^{\{k\}}\big]\mathrm{d}s.\end{aligned} \tag{3.49}$$

Inserting this into (3.44), we obtain for the quantity in (3.43) the estimate

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_t^{\{0\}},\ j \in \eta_t^{\{0\}}\big] \\
&\quad \le \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_t^{\{0\}}\big]\mathbb{P}\big[j \in \eta_t^{\{0\}}\big] \\
&\quad\quad + \int_0^\infty e^{-\lambda t}\mathrm{d}t \int_0^t \mathrm{d}s \sum_{i,j,k,l} \psi_\gamma(i,j)a(k,l)\mathbb{P}\big[k \in \eta_{t-s}^{\{0\}},\ l \notin \eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i \in \eta_s^{\{l\}}\big]\mathbb{P}\big[j \in \eta_s^{\{l\}}\big] \\
&\quad\quad + \delta \int_0^\infty e^{-\lambda t}\mathrm{d}t \int_0^t \mathrm{d}s \sum_{i,j,k} \psi_\gamma(i,j)\mathbb{P}\big[k \in \eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i \in \eta_s^{\{k\}}\big]\mathbb{P}\big[j \in \eta_s^{\{k\}}\big].
\end{aligned}
\tag{3.50}
$$

Here

$$
\begin{aligned}
&\sum_{i,j,k} \psi_\gamma(i,j)\mathbb{P}\big[k \in \eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i \in \eta_s^{\{k\}}\big]\mathbb{P}\big[j \in \eta_s^{\{k\}}\big] \\
&\quad = \sum_{i,j,k} \psi_\gamma(k^{-1}i,k^{-1}j)\mathbb{P}\big[k \in \eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[k^{-1}i \in \eta_s^{\{0\}}\big]\mathbb{P}\big[k^{-1}j \in \eta_s^{\{0\}}\big] \\
&\quad = \Big(\sum_k \mathbb{P}\big[k \in \eta_{t-s}^{\{0\}}\big]\Big)\Big(\sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_s^{\{0\}}\big]\mathbb{P}\big[j \in \eta_s^{\{0\}}\big]\Big) \\
&\quad = \mathbb{E}\big[|\eta_{t-s}^{\{0\}}|\big] \sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_s^{\{0\}}\big]\mathbb{P}\big[j \in \eta_s^{\{0\}}\big]
\end{aligned}
\tag{3.51}
$$

and similarly

$$
\begin{aligned}
&\sum_{i,j,k,l} \psi_\gamma(i,j)a(k,l)\mathbb{P}\big[k \in \eta_{t-s}^{\{0\}},\ l \notin \eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i \in \eta_s^{\{l\}}\big]\mathbb{P}\big[j \in \eta_s^{\{l\}}\big] \\
&\quad \le \sum_{i,j,k,l} \psi_\gamma(l^{-1}i,l^{-1}j)a(k,l)\mathbb{P}\big[k \in \eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[l^{-1}i \in \eta_s^{\{0\}}\big]\mathbb{P}\big[l^{-1}j \in \eta_s^{\{0\}}\big] \\
&\quad = \Big(\sum_{k,l} a(k,l)\mathbb{P}\big[k \in \eta_{t-s}^{\{0\}}\big]\Big)\Big(\sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_s^{\{0\}}\big]\mathbb{P}\big[j \in \eta_s^{\{0\}}\big]\Big) \\
&\quad = |a|\,\mathbb{E}\big[|\eta_{t-s}^{\{0\}}|\big] \sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_s^{\{0\}}\big]\mathbb{P}\big[j \in \eta_s^{\{0\}}\big].
\end{aligned}
\tag{3.52}
$$

Inserting this into (3.50) and recalling that this is an estimate for the quantity in (3.43) yields

$$
\begin{aligned}
&\int \hat{\mu}_\lambda(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A) \\
&\quad \le \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_t^{\{0\}}\big]\mathbb{P}\big[j \in \eta_t^{\{0\}}\big] \\
&\quad\quad + (|a|+\delta)\int_0^\infty e^{-\lambda t}\mathrm{d}t \int_0^t \mathrm{d}s\,\mathbb{E}\big[|\eta_{t-s}^{\{0\}}|\big] \sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_s^{\{0\}}\big]\mathbb{P}\big[j \in \eta_s^{\{0\}}\big] \\
&\quad = \Big(1 + (|a|+\delta)\int_0^\infty e^{-\lambda t}\mathrm{d}t\,\mathbb{E}\big[|\eta_t^{\{0\}}|\big]\Big)\Big(\int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j} \psi_\gamma(i,j)\mathbb{P}\big[i \in \eta_t^{\{0\}}\big]\mathbb{P}\big[j \in \eta_t^{\{0\}}\big]\Big),
\end{aligned}
\tag{3.53}
$$

where in the last step we have changed the integration order on the set $\{(s,t) : 0 \le s \le t\}$. Using the fact that $\psi_\gamma(i,j) = e^{\gamma d(i,j)}$ where d is a metric, we may further estimate the sum in the second factor on the right-hand side of (3.53) as

$$\begin{aligned}
&\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big]=\sum_{i,j}e^{\gamma d(i,j)}\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big]\\
\leq\quad&\sum_{i,j}e^{\gamma(d(0,i)+d(0,j))}\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big]\\
=\quad&\big(\sum_i e^{\gamma d(0,i)}\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\big)^2=\mathbb{E}\big[\sum_{i\in\eta_t^{\{0\}}}e^{\gamma d(0,i)}\big]^2.
\end{aligned}\tag{3.54}$$

Inserting this into (3.53) and recalling the definition of $\hat\pi_\lambda$ in (2.6) yields

$$\int\hat\mu_\lambda(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A)\leq\Big(1+(|a|+\delta)\hat\pi_\lambda\Big)\int_0^\infty e^{-\lambda t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta_t^{\{0\}})\big]^2.\tag{3.55}$$

We note that setting $\gamma=0$ in (2.39) shows that

$$e^{rt}\leq\mathbb{E}\big[|\eta_t^{\{0\}}|\big]\qquad(t\geq0),\tag{3.56}$$

and therefore

$$\lim_{\lambda\downarrow r}\hat\pi_\lambda=\lim_{\lambda\downarrow r}\int_0^\infty e^{-\lambda t}\mathrm{d}t\,\mathbb{E}\big[|\eta_t^{\{0\}}|\big]=\infty.\tag{3.57}$$

Using this and (3.55), we arrive at (3.42). ∎

As a direct applications we obtain:

Proof of Lemma 2.6 This is special case of Lemma 3.5, where $\gamma=0$. ∎

Proof of Lemma 2.17 This is very similar to the proof of Lemma 2.7. For $\delta\in(\delta_\mathrm{c},\infty)$, let $(\eta_t^{\delta,\{0\}})_{t\geq0}$ and $\mathring{\nu}_\delta$ be as in Lemma 2.17. Let Λ_k be finite sets such that $0\in\Lambda_k\subset\Lambda$ and $\Lambda_k\uparrow\Lambda$. It is again easy to check that $A\mapsto f_k^\gamma(A):=e_\gamma(A\cap\Lambda_k)1_{\{0\in A\}}$ is a continuous, compactly supported real function on $\mathcal{P}_+$. Therefore, since (by Proposition 2.5) the $\frac{1}{\hat\pi_{\lambda_n}}\hat\mu_{\lambda_n}$ converge vaguely to $\mathring{\nu}^\delta$,

$$\begin{aligned}
\int\mathring{\nu}^\delta(\mathrm{d}A)f_k^\gamma(A)=\lim_{n\to\infty}\frac{1}{\hat\pi_{\lambda_n}}\int\hat\mu_{\lambda_n}(\mathrm{d}A)f_k^\gamma(A)\quad&\leq\quad\liminf_{n\to\infty}\frac{1}{\hat\pi_{\lambda_n}}\int\hat\mu_{\lambda_n}(\mathrm{d}A)e_\gamma(A)1_{\{0\in A\}}\\
&\leq\quad(|a|+\delta)\int_0^\infty e^{-rt}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta_t^{\delta,\{0\}})\big]^2.
\end{aligned}$$

Letting $k\uparrow\infty$ such that $f_k^\gamma\uparrow e_\gamma(A)1_{\{0\in A\}}$ we arrive at (2.41) by the monotone convergence theorem. ∎

3.5 The Doob transformed process

In this section we provide the proofs of Lemmas 2.9, 2.10 and 2.11. We start with Lemma 2.9. We need to check that the Doob transformed contact process introduced in (2.14) and (2.15) is a well defined non-explosive process. We will moreover show that the laws of the (Λ,a,δ)-contact process η^A and the Doob transformed (Λ,a,δ)-contact process ξ^A, started in the same initial state $A\in\mathcal{P}_{\mathrm{fin},+}$, are related by

$$\mathbb{P}\big[(\xi_s^A)_{0\leq s\leq t}\in\mathrm{d}w\big]=e^{-\lambda t}\frac{h(w_t)}{h(A)}\,\mathbb{P}\big[(\eta_s^A)_{0\leq s\leq t}\in\mathrm{d}w\big]\qquad(t\geq0).\tag{3.58}$$

Proof of Lemma 2.9 Instead of showing that the rates in (2.14) define a non-explosive Markov process, we will argue in the opposite direction. We will first construct a non-explosive Markov process with semigroup as in (2.16) and then apply general theory to conclude that this is the same as the one that we would have obtained by starting off with the rates in (2.14).

Thus, for the moment, we define $P^h_t(A,B)$ by (2.16) (instead of (2.15)), and start by observing that this is well-defined since $h(A)>0$ for all $A\in\mathcal{P}_{\text{fin},+}$. By duality (1.7) and the fact that $\mu^\dagger$ is an eigenmeasure, we see that

$$\begin{aligned}P_th(A)&=\sum_{B\in\mathcal{P}_{\text{fin},+}}P_t(A,B)\int\mu^\dagger(\mathrm{d}C)1_{\{B\cap C\neq\emptyset\}}=\int\mu^\dagger(\mathrm{d}C)\mathbb{P}\big[\eta^A_t\cap C\neq\emptyset\big]\\&=\int\mu^\dagger(\mathrm{d}C)\mathbb{P}\big[A\cap\eta^{\dagger\,C}_t\neq\emptyset\big]=e^{\lambda t}\int\mu^\dagger(\mathrm{d}B)1_{\{A\cap B\neq\emptyset\}}=e^{\lambda t}h(A)\qquad(A\in\mathcal{P}_{\text{fin},+}).\end{aligned}\tag{3.59}$$

It follows that

$$\sum_{B\in\mathcal{P}_{\text{fin},+}}P^h_t(A,B)=\sum_{B\in\mathcal{P}_{\text{fin},+}}e^{-\lambda t}\,\frac{h(B)}{h(A)}\,P_t(A,B)=\frac{1}{h(A)}e^{-\lambda t}\,P_th(A)=1\tag{3.60}$$

for all $A\in\mathcal{P}_{\text{fin},+}$, i.e., P^h is a probability kernel. It is now straightforward to check that for each $A\in\mathcal{P}_{\text{fin},+}$, the right-hand side of (3.58) consistently defines a probability law on the space of cadlag paths $w:[0,\infty)\to\mathcal{P}_{\text{fin},+}$ and that this is the law of a Markov process with semigroup $(P^h_t)_{t\geq0}$ and initial state A.

General theory (see Theorem B.1 in Appendix B) now tells us that both $(P_t)_{t\geq0}$ and $(P^h_t)_{t\geq0}$ are uniquely defined in terms of jump rates $r(A,B)$ and $r^h(A,B)$, which satisfy

$$r(A,B)=\tfrac{\partial}{\partial t}P_t(A,B)\big|_{t=0}\quad\text{and}\quad r^h(A,B)=\tfrac{\partial}{\partial t}P^h_t(A,B)\big|_{t=0}\qquad(A\neq B).\tag{3.61}$$

In particular, for each $A,B\in\mathcal{P}_{\text{fin},+}$ with $A\neq B$, by (2.16),

$$\begin{aligned}r^h(A,B)&=\tfrac{\partial}{\partial t}P^h_t(A,B)\big|_{t=0}\\&=\tfrac{\partial}{\partial t}\big(e^{-\lambda t}h(A)^{-1}P_t(A,B)h(B)\big)\big|_{t=0}=h(A)^{-1}r(A,B)h(B),\end{aligned}\tag{3.62}$$

which shows that the jump rates of the process with semigroup $(P^h_t)_{t\geq0}$ are given by (2.14).∎

In the next proof we relate eigenmeasures of the contact process to invariant measures of the Doob transformed contact process.

Proof of Lemma 2.10 Recall that $h=h_{\mu^\dagger}$ as in (2.13). Since $\mu^\dagger$ is homogeneous and locally finite and μ satisfies $\int\mu(\mathrm{d}A)|A|1_{\{0\in A\}}<\infty$ by assumption, it follows by Lemma 2.12 that $h\mu$ is a locally finite measure on $\mathcal{P}_+$.

The measure μ is an eigenmeasure of the (Λ,a,δ)-contact process with eigenvalue r if and only if

$$\sum_{A\in\mathcal{P}_{\text{fin},+}}\mu(\{A\})P_t(A,B)=e^{rt}\mu(\{B\})\qquad(t\geq0,\ B\in\mathcal{P}_{\text{fin},+}),\tag{3.63}$$

which by (2.16) and the fact that $h(A)>0$ for $A\in\mathcal{P}_{\text{fin},+}$ is equivalent to

$$\sum_{A\in\mathcal{P}_{\text{fin},+}}\mu(\{A\})h(A)P^h_t(A,B)=h(B)\mu(\{B\})\qquad(t\geq0,\ B\in\mathcal{P}_{\text{fin},+}),\tag{3.64}$$

i.e., $h\mu$ is an invariant law of the h-transformed (Λ, a, δ)-contact process. ∎

Finally, we can use this to show that the transformed process modulo shifts is positively recurrent with a unique invariant law that is related to the previously considered eigenmeasures.

Proof of Lemma 2.11 By Lemma 2.10, for any $c > 0$, the measure

$$\mu := c \sum_{i \in \Lambda} \mathbb{P}\big[i\xi_\infty \in \cdot\,\big] \tag{3.65}$$

is an invariant law for the h-transformed (Λ, a, δ)-contact process. It is intuitively clear that this implies that $\mathbb{P}[\tilde{\xi}_\infty \in \cdot\,]$ is an invariant law for the h-transformed (Λ, a, δ)-contact process modulo shifts, but for completeness, we prove this formally.

We can without loss of generality assume that $c = 1$. The transition probabilities of the h-transformed (Λ, a, δ)-contact process modulo shifts are given by

$$\tilde{P}^h_t(\tilde{A}, \tilde{B}) = m(B)^{-1} \sum_{i \in \Lambda} P^h_t(A, iB) \qquad \big(t \geq 0,\ A, B \in \mathcal{P}_{\text{fin},+}\big), \tag{3.66}$$

where $m(B)$ is defined as in (3.13). Let $\mathcal{R}$ be a subset of $\mathcal{P}_{\text{fin},+}$ that contains exactly one representative of each equivalence class $\tilde{A} \in \tilde{\mathcal{P}}_{\text{fin},+}$. Then we have by (3.13) that

$$\begin{aligned}
&\sum_{\tilde{A} \in \tilde{\mathcal{P}}_{\text{fin},+}} \mathbb{P}[\tilde{\xi}_\infty = \tilde{A}]\tilde{P}^h_t(\tilde{A}, \tilde{B}) = \sum_{A \in \mathcal{R}} \mathbb{P}[\tilde{\xi}_\infty = \tilde{A}]\tilde{P}^h_t(\tilde{A}, \tilde{B}) \\
&\quad = \sum_{A \in \mathcal{R}} m(A)^{-1}\mu(\{A\}) m(B)^{-1} \sum_{i \in \Lambda} P^h_t(A, iB) \\
&\quad = m(B)^{-1} \sum_{A \in \mathcal{R}} m(A)^{-1} \sum_{i \in \Lambda} \mu(\{i^{-1}A\}) P^h_t(i^{-1}A, B) \\
&\quad = m(B)^{-1} \sum_{A \in \mathcal{P}_{\text{fin},+}} \mu(\{A\}) P^h_t(A, B) = m(B)^{-1}\mu(\{B\}) = \mathbb{P}[\tilde{\xi}_\infty = \tilde{B}],
\end{aligned} \tag{3.67}$$

which shows that $\mathbb{P}[\tilde{\xi}_\infty \in \cdot\,]$ is an invariant law for the h-transformed (Λ, a, δ)-contact process modulo shifts.

Since the h-transformed (Λ, a, δ)-contact process modulo shifts has an invariant law, positive recurrence and the other statements of the proposition will follow once we prove irreducibility. It follows from (2.16) and the fact that $h(A) > 0$ for all $A \neq \emptyset$ that $P^h_t(A, B) > 0$ if and only if $P_t(A, B) > 0$ $(A, B \in \mathcal{P}_{\text{fin},+})$. Our assumption that $r < 0$ entails that $\delta > 0$. Therefore, since it may happen that all sites except one recover, for each finite set A and $i \in A$ we have $P^h_t(A, \{i\}) > 0$. On the other hand, by (1.3), for each finite set A there exists an $i \in \Lambda$ such that all sites in A can be infected from i, hence $P^h_t(\{i\}, A) > 0$. This proves the irreducibility of the h-transformed (Λ, a, δ)-contact process modulo shifts. ∎

A Exponential decay in the subcritical regime

A.1 Statement of the result

The aim of this appendix is to show how the arguments in [AJ07], which are written down for contact processes on transitive graphs, can be extended to prove Theorem 0 (d) for the class of (Λ, a, δ)-contact processes considered in this article. To formulate this properly, only in this appendix, we will consider a class of contact processes that is more general than both the one defined in Section 1.2 and the one considered in [AJ07], and contains them both as subclasses. Indeed, only in this appendix, will we drop the assumptions that Λ has a group structure (as in the rest of this article) or that Λ has a graph structure (as in [AJ07]). The only structure on Λ that we will use is the structure given by the infection rates $(a(i,j))_{i,j\in\Lambda}$.

Let Λ be any countable set and let $a : \Lambda \times \Lambda \to [0,\infty)$ be a function. By definition, an *automorphism* of (Λ, a) is a bijection $g : \Lambda \to \Lambda$ such that $a(gi, gj) = a(i,j)$ for each $i,j \in \Lambda$. Let $\mathrm{Aut}(\Lambda, a)$ denote the group of automorphisms of (Λ, a). We say that a subgroup $G \subset \mathrm{Aut}(\Lambda, a)$ is *(vertex) transitive* if for each $i,j \in \Lambda$ there exists a $g \in G$ such that $gi = j$. In particular, we say that (Λ, a) is transitive if $\mathrm{Aut}(\Lambda, a)$ is transitive.

Let (Λ, a) be transitive, let $a^\dagger(i,j) := a(j,i)$, and assume that

$$|a| := \sum_{j\in\Lambda} a(i,j) < \infty \quad \text{and} \quad |a^\dagger| := \sum_{j\in\Lambda} a^\dagger(i,j) < \infty, \tag{A.1}$$

where by the transitivity of (Λ, a), these definitions do not depend on the choice of $i \in \Lambda$. Then, for each $\delta \geq 0$, there exists a well-defined contact process on Λ with generator as in (1.2) and also the dual contact process with a replaced by $a^\dagger$ is well-defined. *Only in this appendix*, we will use the term (Λ, a, δ)-contact process (resp. $(\Lambda, a^\dagger, \delta)$-contact process) in this more general sense.

For any (Λ, a, δ)-contact process, as defined in this appendix, we define the critical recovery rate $\delta_{\mathrm{c}} = \delta_{\mathrm{c}}(\Lambda, a)$ as in (1.8), which satisfies $\delta_{\mathrm{c}} < \infty$ but may be zero in the generality considered here. A straightforward extension of [Swa09, Lemma 1.1] shows that the exponential growth rate $r = r(\Lambda, a, \delta)$ in (1.10) is well-defined for the class of (Λ, a, δ)-contact processes considered here.

We will show that the arguments in [AJ07] imply the following result.

Theorem A.1 (Exponential decay in the subcritical regime) *Let (Λ, a) be transitive and let a satisfy (A.1). Then $\{\delta \geq 0 : r(\Lambda, a, \delta) < 0\} = (\delta_{\mathrm{c}}, \infty)$.*

We remark that Theorem 0 (a) does not hold in general for the class of (Λ, a, δ)-contact processes considered in this appendix. This is related to unimodularity. A transitive subgroup $G \subset \mathrm{Aut}(\Lambda, a)$ is *unimodular* if [BLPS99, formula (3.3)]

$$|\{gi : g \in G,\ gj = j\}| = |\{gj : g \in G,\ gi = i\}| \qquad (i,j \in \Lambda). \tag{A.2}$$

Note that this is trivially satisfied if Λ is a group and $G = \Lambda$ acts on itself by left multiplication, in which case the sets on both sides of the equation consist of a single element. Unimodularity gives rise to the *mass transport principle* which says that for any function $f : \Lambda \times \Lambda \to [0,\infty)$ such that $f(gi, gj) = f(i,j)$ $(g \in G,\ i,j \in \Lambda)$, one has $\sum_j f(i,j) = \sum_j f(j,i)$. In particular, this implies that the constants $|a|$ and $|a^\dagger|$ from (A.1) are equal and that $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$. In the nonunimodular case, this is in general no longer true and in fact it is not hard to construct examples where the critical recovery rates $\delta_{\mathrm{c}}(\Lambda, a)$ and $\delta_{\mathrm{c}}(\Lambda, a^\dagger)$ of a contact process and its dual are different. We remark that although in [AJ07], the authors do not always clearly distinguish between a contact process and its dual (e.g., in their formulas (1.3), (1.9) and Lemma 1.4), they do not assume that $a = a^\dagger$ and their results are valid also in the asymmetric case $a \neq a^\dagger$.

A.2 The key differential inequalities and their consequences

The main method used in [AJ07], that in its essence goes back to [AB87] and that yields Theorem A.1 and a number of related results, is the derivation of differential inequalities for certain quantities related to the process. Using the graphical representation to construct a (Λ, a, δ)-contact process and its dual, we define the *susceptibility* as

$$\chi = \chi(\Lambda, a, \delta) = \mathbb{E}\big[\int_0^\infty |\eta_t^{\{0\}}| \, \mathrm{d}t\big], \tag{A.3}$$

which may be $+\infty$. Moreover, letting ω^{c} be a Poisson point process on $\Lambda \times \mathbb{R}$ with intensity $h \geq 0$, independent of the Poisson point processes ω^{i} and ω^{r} corresponding to infection arrows and recovery symbols, we define

$$\theta = \theta(\Lambda, a, \delta, h) := \mathbb{P}\big[C_{(0,0)} \cap \omega^{\mathrm{c}} \neq \emptyset\big] \quad \text{where} \quad C_{(i,s)} := \big\{(j,t) : t \geq s,\ (i,s) \rightsquigarrow (j,t)\big\}. \tag{A.4}$$

Then θ can be interpreted as the density of infected sites in the upper invariant law of a (dual) "$(\Lambda, a^\dagger, \delta, h)$-contact process", which in addition to the dynamics in (1.2) exhibits spontaneous infection of healthy sites with rate h, corresponding to a term in the generator of the form $h \sum_i \{f(A \cup \{i\}) - f(A)\}$.

Let Λ, a, δ be fixed and for $\lambda, h \geq 0$ let $\theta = \theta(\lambda, h) := \theta(\Lambda, \lambda a, \delta, h)$ and $\chi = \chi(\lambda) := \chi(\Lambda, \lambda a, \delta)$ be the quantities defined above. The analysis in [AJ07] centers on the deriviation of the following three differential inequalities (see [AJ07, formulas (1.17), (1.19) and (1.20)])

$$\begin{aligned} &\text{(i)} \quad \tfrac{\partial}{\partial \lambda}\chi \leq |a|\chi^2, \\ &\text{(ii)} \quad \tfrac{\partial}{\partial \lambda}\theta \leq |a|\theta \tfrac{\partial}{\partial h}\theta, \\ &\text{(iii)} \quad \theta \leq h\tfrac{\partial}{\partial h}\theta + \big(2\lambda^2|a|\theta + h\lambda\big)\tfrac{\partial}{\partial \lambda}\theta + \theta^2. \end{aligned} \tag{A.5}$$

These differential inequalities, and their proofs, generalize without a change to the more general class of (Λ, a, δ)-contact processes discussed in this appendix.

Since $\theta \geq h(1+h)$, which follows by estimating the $(\Lambda, \lambda a^\dagger, \delta, h)$-contact process from below by a process with no infections, one has $h \leq \theta(1-\theta)$. Inserting this into (A.5) (iii) yields

$$\theta \leq h\tfrac{\partial}{\partial h}\theta + \Big(2\lambda^2|a| + \frac{\lambda}{1-\theta}\Big)\theta\tfrac{\partial}{\partial \lambda}\theta + \theta^2. \tag{A.6}$$

Abstract results of Aizenman and Barsky [AB87, Lemmas 4.1 and 5.1] allow one to draw the following conclusions from (A.5) (ii) and (A.6).

Lemma A.2 (Estimates on critical exponents) *Assume that there exists some $\lambda' > 0$ such that $\theta(\lambda', 0) = 0$ and $\lim_{h \to 0} h^{-1}\theta(\lambda', h) = \infty$. Then there exist $c_1, c_2 > 0$ such that*

$$\begin{aligned} &\text{(i)} \quad \theta(\lambda', h) \geq c_1 h^{1/2} \qquad (h \geq 0), \\ &\text{(ii)} \quad \theta(\lambda, 0) \geq c_2(\lambda - \lambda') \qquad (\lambda \geq \lambda'). \end{aligned} \tag{A.7}$$

Note that this lemma (in particular, formula (A.7) (i), which depends on the assumption that $\lim_{h \to 0} h^{-1}\theta(\lambda', h) = \infty$) implies in particular that if for some fixed $\lambda' > 0$, one has $\theta(\lambda', h) \sim h^\alpha$ as $h \to 0$, then either $\alpha \leq \frac{1}{2}$ or $\alpha \geq 1$.

Remark Lemmas 4.1 and 5.1 of [AB87] are also cited in [AJ07, Thm. 4.1], but there the statement that $c_1, c_2 > 0$ is erroneously replaced by the (empty) statement that $c_1, c_2 < \infty$.

Proof of Theorem A.1 (sketch) Set

$$\begin{aligned}\lambda_{\rm c} &:= \inf\{\lambda \geq 0 : \theta(\lambda, 0) > 0\},\\ \lambda'_{\rm c} &:= \inf\{\lambda \geq 0 : \chi(\lambda) = \infty\}.\end{aligned} \tag{A.8}$$

Since $\chi(\lambda) < \infty$ implies $\theta(\lambda, 0) = 0$, obviously $\lambda'_{\rm c} \leq \lambda_{\rm c}$. Our first aim is to show that they are in fact equal. We note that it is always true that $\lambda'_{\rm c} > 0$. It may happen that $\lambda'_{\rm c} = \infty$ but in this case also $\lambda_{\rm c} = \infty$ so without loss of generality we may assume that $\lambda'_{\rm c} < \infty$.

It follows from (A.5) (i) and approximation of infinite systems by finite systems (compare [AN84, Lemma 3.1], which is written down for unoriented percolation and which is cited in [AJ07, formula (1.18)]) that $\lim_{\lambda \uparrow \lambda'_{\rm c}} \chi(\lambda) = \chi(\lambda'_{\rm c}) = \infty$, and in fact

$$\chi(\lambda) \geq \frac{|a|^{-1}}{\lambda'_{\rm c} - \lambda} \qquad (\lambda < \lambda'_{\rm c}). \tag{A.9}$$

Now either $\theta(\lambda'_{\rm c}, 0) > 0$, in which case we are done, or $\theta(\lambda'_{\rm c}, 0) = 0$. In the latter case, since

$$\chi(\lambda) = \lim_{h \to 0} h^{-1} \theta(\lambda, h) \qquad (\lambda < \lambda'_{\rm c}), \tag{A.10}$$

(see [AJ07, formula (1.11)]), using the monotonicity of θ in λ and h, it follows from (A.9) that

$$\lim_{h \to 0} h^{-1} \theta(\lambda'_{\rm c}, h) = \infty \tag{A.11}$$

and therefore Lemma A.2 implies that (A.7) holds at $\lambda' = \lambda'_{\rm c}$. In particular, (A.7) (ii) implies that $\theta(\lambda, 0) > 0$ for $\lambda > \lambda'_{\rm c}$, hence $\lambda_{\rm c} = \lambda'_{\rm c}$.

Since by a trivial rescaling of time, questions about critical values for λ can always be translated into questions about critical values for δ, we learn from this that for any (Λ, a, δ)-contact process, one has $\chi(\Lambda, a, \delta) < \infty$ if $\delta > \delta_{\rm c}(\Lambda, a)$, where the latter critical point is defined in (1.8). It follows from (3.56) that $\chi(\Lambda, a, \delta) = \infty$ if $r(\delta) = r(\Lambda, a, \delta) \geq 0$, hence we must have $r(\delta) < 0$ for $\delta \in (\delta_{\rm c}, \infty)$. Part (b) of Theorem 0 is easily generalized to the class of (Λ, a, δ)-contact processes considered in this appendix. Moreover, it is not hard to prove that $r < 0$ implies that the process does not survive. This shows that $r(\delta) \geq 0$ on $[0, \delta_{\rm c})$ while $\delta \mapsto r(\delta)$ is continuous, which allows us to conclude that $\{\delta \geq 0 : r(\delta) < 0\} = (\delta_{\rm c}, \infty)$ if $\delta_{\rm c} > 0$. If $\delta_{\rm c} = 0$ (which may happen for the general class of models considered here), then we may use the fact that $\theta(\Lambda, a, 0) = 1$ to conclude that $r(\Lambda, a, 0) \geq 0$, hence the conclusion of Theorem A.1 is also valid in this case. ∎

B A basic result about continuous-time Markov chains

In this appendix we prove a basic result about continuous-time Markov chains that we need in the construction of the Doob transformed process and for which we did no find an exact reference.

Let S be a countable set. By definition, a *transition kernel* on S is a collection of probability kernels $(P_t)_{t \geq 0}$ on S such that $P_s P_t = P_{s+t}$ $(s, t \geq 0)$ and $\lim_{t \downarrow 0} P_t(x, x) = P_0(x, x) = 1$ $(x \in S)$. By definition, a *Markov process with semigroup* $(P_t)_{t \geq 0}$ is an S-valued stochastic process $X = (X_t)_{t \geq 0}$ such that

$$\mathbb{P}\big[X_{t_{n+1}} = x \,\big|\, (X_{t_1}, \ldots, X_{t_n})\big] = P_{t_{n+1} - t_n}(X_{t_n}, x) \quad \text{a.s.} \quad (x \in S,\ 0 \leq t_1 \leq \cdots \leq t_{n+1}). \tag{B.1}$$

We say that X has *cadlag sample paths* if for each ω in the underlying probability space, the function $t \mapsto X_t(\omega)$ is cadlag (i.e., right-continuous with left limits).

By definition, a *Q-matrix* is a collection of real numbers $\{q(x,y) : x, y \in S\}$ such that $q(x,y) \geq 0$ for $x \neq y$ and

$$-q(x,x) = \sum_{y:\, y \neq x} q(x,y) < \infty \qquad (x \in S). \tag{B.2}$$

For any Q-matrix, it can be shown (see [Lig10, Thm 2.26] or [Nor97, Thm 2.8.3]) that the family of differential equations

$$\tfrac{\partial}{\partial t} P_t(x,y) = \sum_{x' \in S} q(x,x') P_t(x',y) \qquad (t \geq 0,\ x, y \in S) \tag{B.3}$$

with $P_0(x,y) = 1_{\{x=y\}}$ $(x, y \in S)$ has a unique minimal nonnegative solution. We say that this solution is *stochastic* if $\sum_{y \in S} P_t(x,y) = 1$ for all $x \in S$ and $t \geq 0$.

Theorem B.1 (Continuous-time Markov chains) *Let S be a countable set and let Q be a Q-matrix on S. Assume that the minimal nonnegative solution $(P_t)_{t\geq 0}$ of (B.3) is stochastic. Then $(P_t)_{t\geq 0}$ is a transition kernel on S and for each $z \in S$, there exists a unique (in distribution) Markov process $X^z = (X^z_t)_{t\geq 0}$ with initial state $X^z_0 = z$, semigroup $(P_t)_{t\geq 0}$, and cadlag sample paths. Conversely, if for a given transition kernel $(P_t)_{t\geq 0}$ on S and for each $z \in S$, there exists a Markov process $X^z = (X^z_t)_{t\geq 0}$ with initial state $X^z_0 = z$, semigroup $(P_t)_{t\geq 0}$, and cadlag sample paths, then there exists a Q-matrix on S such that $(P_t)_{t\geq 0}$ is the unique minimal nonnegative solution of (B.3).*

Proof The first part of the theorem, that says that the stochasticity of the minimal solution of (B.3) implies the existence of an associated Markov process with cadlag sample paths, can be found in, for example, [Lig10, Thm 2.37] or [Nor97, Thm 2.8.4]. It is well-known that the second, converse part of the theorem is false without the assumption of cadlag sample paths; see any book on the topic for counterexamples.

To see that the statement is true under the assumption of cadlag sample paths, fix $z \in S$, write $X = X^z$ and define inductively stopping times by $\sigma_0 = \sigma^\varepsilon_0 = 0$ and

$$\begin{aligned} \sigma_k &:= \inf\{t \geq \sigma_{k-1} : X_t \neq X_{\sigma_{k-1}}\} \\ \sigma^\varepsilon_k &:= \inf\{\varepsilon l \geq \sigma^\varepsilon_{k-1} : X_{\varepsilon l} \neq X_{\sigma^\varepsilon_{k-1}},\ l \in \mathbb{N}\} \qquad (\varepsilon > 0). \end{aligned} \tag{B.4}$$

Let $K_\varepsilon := \sup\{k \geq 0 : \sigma^\varepsilon_k < \infty\}$. Then, for each $\varepsilon > 0$, we may define a Markov chain $Y^\varepsilon = (Y^\varepsilon_k)_{k\geq 0}$ by setting $Y^\varepsilon_k := X_{\sigma^\varepsilon_k}$ for $k \leq K_\varepsilon$ and $Y^\varepsilon_k := X_{\sigma^\varepsilon_{K_\varepsilon}}$ for $k > K_\varepsilon$. Conditional on Y^ε, the times $(\sigma^\varepsilon_k - \sigma^\varepsilon_{k-1})$ with $1 \leq k \leq K_\varepsilon$ are independent and geometrically distributed. By the fact that X has cadlag sample paths, $\sigma^\varepsilon_k \to \sigma_k$ a.s. and $Y^\varepsilon_k \to Y_k$ a.s. as $\varepsilon \downarrow 0$ for each $k \geq 0$, where the process $Y = (Y_k)_{k\geq 0}$ is defined analogously to Y^ε with σ^ε_k replaced by σ_k.

By [Lig10, Thm 2.14 (a)], for each $x \in S$, the limit

$$Q(x) := \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\big(1 - P_\varepsilon(x,x)\big) \tag{B.5}$$

exists in $[0, \infty]$. In particular, since

$$\mathbb{P}^z\big[\sigma^\varepsilon_1 = \varepsilon k\big] = P_\varepsilon(z,z)^{k-1}\big(1 - P_\varepsilon(z,z)\big) \qquad (k \geq 1), \tag{B.6}$$

we see that either of the following three possibilities holds: 1. $Q(z)=\infty$ and $\sigma^\varepsilon_1\to\sigma_1=0$ as $\varepsilon\to 0$, 2. $0<Q(z)<\infty$ and σ_1 is exponentially distributed with parameter $Q(z)$, or 3. $Q(z)=0$ and $\sigma_1=\infty$. By the fact that X has cadlag sample paths, $X^z_t\to z$ a.s. as $t\downarrow 0$ which implies $\sigma_1>0$ a.s., so we can exclude the first possibility. Since z is arbitrary, we conclude that $Q(x)<\infty$ for all $x\in S$. Now by [Lig10, Thm 2.14 (b)] we have for each $x,y\in S$ with $x\neq y$ the existence of the limit

$$q(x,y):=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}P_\varepsilon(x,y)\quad\text{with}\quad\sum_{y\colon y\neq x}q(x,y)\leq Q(x)\qquad(x\in S).\tag{B.7}$$

If $Q(z)>0$, then we observe that Y^ε_1 is distributed according to the law

$$\mathbb{P}^z\big[Y^\varepsilon_1=y\big]=\big(1-P_\varepsilon(z,z)\big)^{-1}P_\varepsilon(z,y)\qquad(y\in S,\ y\neq z).\tag{B.8}$$

Since $Y^\varepsilon_1\to Y_1$ as $\varepsilon\to 0$, we conclude by (B.5) and (B.7) that Y_1 is distributed according to the probability law $Q(z)^{-1}q(z,\,\cdot\,)$. In particular, this shows that $\sum_{y\colon y\neq z}q(z,y)=Q(z)$. Since z is arbitrary, the same holds with z replaced by an arbitrary $x\in S$. It is now not hard to check that Y is a Markov chain that jumps from a state x with $Q(x)>0$ to a state y with probability $Q(x)^{-1}q(x,y)$, and that conditional on Y, the times $(\sigma_k-\sigma_{k-1})$ are independent and exponentially distributed with parameter $Q(Y_{k-1})$. By [Nor97, Thm 2.8.4], we conclude that $(P_t)_{t\geq 0}$ is the unique minimal nonnegative solution of (B.3). ∎

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