Survival of contact processes on the hierarchical group

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Abstract

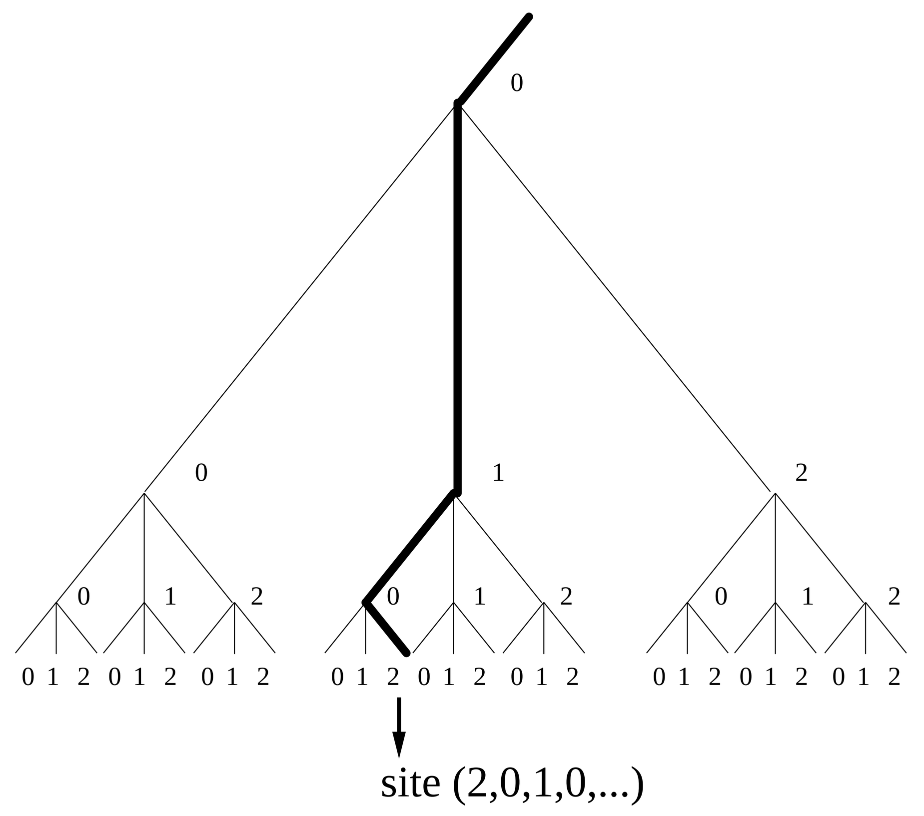
We consider contact processes on the hierarchical group, where sites infect other sites with a rate depending on their hierarchical distance, and sites become healthy with a fixed recovery rate. If the infection rates decay too fast as a function of the hierarchical distance, then we show that the critical recovery rate is zero. On the other hand, we derive sufficient conditions saying how fast the infection rates can decay while the critical recovery rate is still positive. Our proofs are based on a coupling argument that compares contact processes on the hierarchical group with contact processes on a renormalized lattice. For technical simplicity, our main argument is carried out only for the hierarhical group with freedom two.

The hierarchical group

By definition, the *hierarchical group with freedom* N is the set

$$\Omega_N := \big\{ i = (i_0, i_1, \ldots) : i_k \in \{0, \ldots, N-1\},\ i_k \neq 0 \text{ for finitely many } k \big\},$$

equipped with componentwise addition modulo N. Think of sites $i \in \Omega_N$ as the leaves of an infinite tree. Then $i_0, i_1, i_2, \ldots$ are the labels of the branches on the unique path from i to the root of the tree.

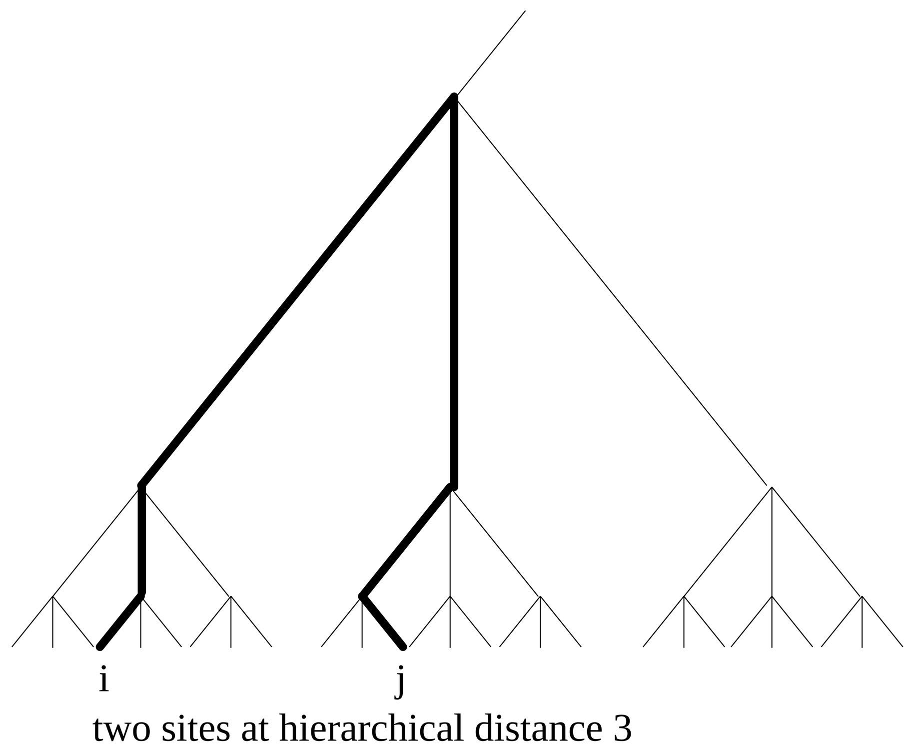


The hierarchical distance

Set

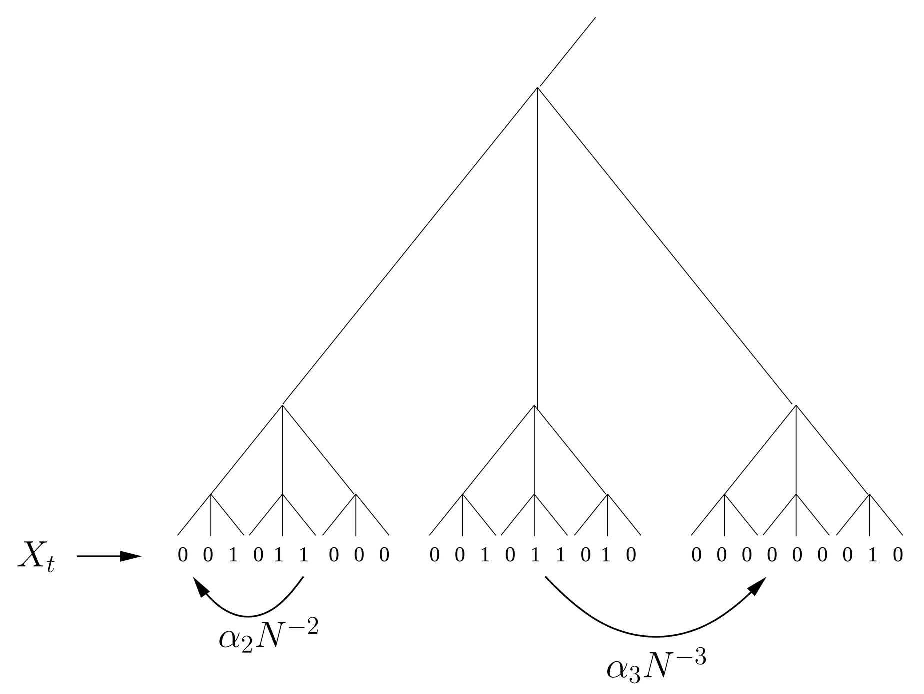
$$|i| := \inf\{k \geq 0 : i_m = 0\ \forall m \geq k\} \qquad (i \in \Omega_N).$$

Then $|i-j|$ is the *hierarchical distance* between two elements $i, j \in \Omega_N$. In the tree picture, $|i-j|$ measures how high we must go up the tree to find the last common ancestor of i and j.



Contact processes on Ω_N

Fix a *recovery rate* $\delta \geq 0$ and *infection rates* $\alpha_k \geq 0$ such that $\sum_{k=1}^\infty \alpha_k < \infty$. The *contact process* on Ω_N with these rates is the $\{0,1\}^{\Omega_N}$-valued Markov process $(X_t)_{t\geq 0}$ with the following description. If $X_t(i) = 0$ (resp. $X_t(i) = 1$), then we say that the site $i \in \Omega_N$ is *healthy* (resp. *infected*) at time $t \geq 0$. An infected site i infects a healthy site j at hierarchical distance $k := |i-j|$ with rate $\alpha_k N^{-k}$, and infected sites become healthy with rate $\delta \geq 0$.



Infection rates on the hierarchical group.

The critical recovery rate

We say that a contact process $(X_t)_{t\geq 0}$ on Ω_N with given recovery and infection rates *survives* if there is a positive probability that the process started with only one infected site never recovers completely, i.e., there are infected sites at any $t \geq 0$. For given infection rates, we let

$$\delta_c := \sup\big\{\delta \geq 0 : \text{the contact process with infection rates } (\alpha_k)_{k\geq 1} \text{ and recovery rate } \delta \text{ survives}\big\}$$

denote the *critical recovery rate*. A simple monotone coupling argument shows that X survives for $\delta < \delta_c$ and dies out for $\delta > \delta_c$. It is not hard to show that $\delta_c < \infty$. The question whether $\delta_c > 0$ is more subtle.

Theorem [AS08] Assume that $\alpha_k = e^{-\theta^k}$ $(k \geq 1)$. Then:
(a) If $\theta > N$, then $\delta_c = 0$.
(b) If N is a power of two and $1 < \theta < N$, then $\delta_c > 0$.

More generally, we show that $\delta_c = 0$ if

$$\liminf_{k\to\infty} N^{-k} \log(\beta_k) = -\infty, \quad \text{where} \quad \beta_k := \sum_{n=k}^\infty \alpha_n \quad (k \geq 1), \tag{1}$$

while $\delta_c > 0$ if N is a power of two and

$$\sum_{k=m}^\infty N^{-k} \log(\alpha_k) > -\infty, \tag{2}$$

for some $m \geq 1$.

Extinction

The proof of part (a) of the theorem is rather simple. Let

$$\Omega_N^n := \big\{ i = (i_0, \ldots, i_{n-1}) : i_k \in \{0, \ldots, N-1\} \big\} \tag{3}$$

be a finite part of Ω_N, called ***n-block***, corresponding to the leaves of a finite tree of depth n. Let $(X_t^{(n)})_{t\geq 0}$ be a finite contact process on Ω_N^n with recovery rate $\delta \geq 0$ and infection rates $\alpha_1, \ldots, \alpha_n$. Rescaling time if necessary, we may assume that $\sum_k \alpha_k \leq 1$. Then $X^{(n)}$ may be stochastically bounded from above by a process $\tilde X^{(n)}$ where sites jump independently of each other from 0 to 1 with rate 1 and from 1 to 0 with rate δ. Obviously, the process $\tilde X^{(n)}$ has a unique equilibrium law, which is of product form, and if $\tilde X_\infty^{(n)}$ denotes a random variable distributed according to this law, then

$$\mathbb{P}[\tilde X_\infty^{(n)} = \underline{0}] = \Big(\frac{\delta}{1+\delta}\Big)^{N^n},$$

where $\underline{0}$ denotes the all healthy state. On the other hand, since the Markov process $\tilde X^{(n)}$ stays on average a time $(N^n)^{-1}$ in the state $\underline{0}$ every time it gets there, one has

$$\mathbb{P}[\tilde X_\infty^{(n)} = \underline{0}] = \frac{N^{-n}}{\tilde l(n) + N^{-n}} = \frac{1}{1 + N^n \tilde l(n)},$$

where

$$\tilde l(n) := \mathbb{E}^{\delta_0}\big[\inf\{t \geq 0 : \tilde X_t^{(n)} = \underline{0}\}\big]$$

denotes the expected time till extinction for the process $\tilde X^{(n)}$, started with one infected site. Denoting the analogue quantity for $X^{(n)}$ by $l(n)$, we obtain the estimate

$$l(n) \leq \tilde l(n) = N^{-n}\big((1+\delta^{-1})^{N^n} - 1\big) \leq N^{-n}(1+\delta^{-1})^{N^n}.$$

Using this, one can show that if (1) holds, then for any $\delta > 0$ there exists an $n \geq 1$ such that n-blocks get extinct faster than they get infected, hence the process dies out.

Coupling and renormalization

The proof of part (b) of the theorem is more complicated. We derive explicit upper bounds on the probability that finite systems get extinct before a fixed time t. These bounds are derived inductively, by comparing large systems with smaller systems, via a renormalization-type argument. For simplicity, the calculations are carried out only for $N = 2$.

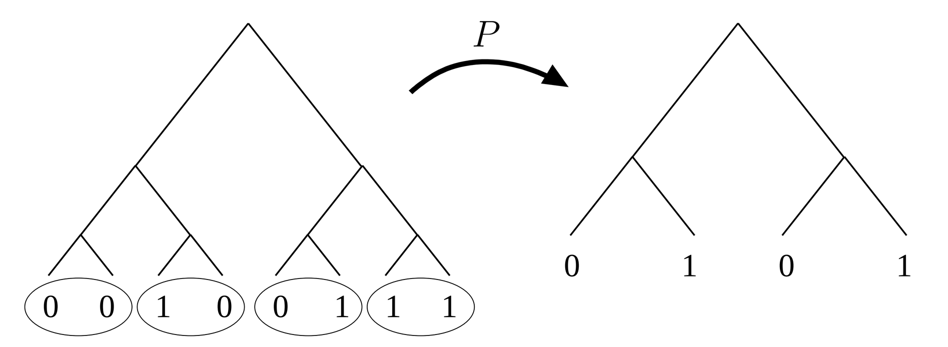
A probability kernel

Define Ω_2^n as in (3) and let $S_n := \{0,1\}^{\Omega_2^n}$ be the set of spin configurations on Ω_2^n. Let $\xi \in (0, \frac{1}{2}]$ be a constant, to be determined later. We define a probability kernel P from S_n to S_{n-1} whose aim is to 'renormalize' our system, i.e., to replace a finite contact process X_t, which is a Markov process with state space S_n, by a 'renormalized' process which takes values in the smaller space S_{n-1}. To that aim, we independently replace 1-blocks, consisting of two spins each, by a single 'renormalized' spin, according to the stochastic rules:

$$00 \longrightarrow 0, \qquad 11 \longrightarrow 1,$$

$$\text{and} \quad 01 \text{ or } 10 \longrightarrow \begin{cases} 0 & \text{with probability } \xi, \\ 1 & \text{with probability } 1-\xi, \end{cases}$$

In a picture, this looks like this:



The probability of this transition is $1 \cdot (1-\xi) \cdot \xi \cdot 1$.

A coupling

Let $(X_t)_{t\geq 0}$ be a finite contact process on Ω_2^n with recovery rate $\delta \geq 0$ and infection rates $\alpha_1, \ldots, \alpha_n$. Using a result of Rogers and Pitman [RP81] (see also [Kur98]), we can couple $(X_t)_{t\geq 0}$ to an S_{n-1}-valued process $(\tilde Y_t)_{t\geq 0}$, in such a way that

$$\mathbb{P}\big[\tilde Y_t = y \,\big|\, X_t\big] = P(X_t, y) \quad \text{a.s.} \qquad (t \geq 0,\ y \in S_{n-1}),$$

where P is the probability kernel from S_n to S_{n-1} defined above and

$$\xi := \gamma - \sqrt{\gamma^2 - \tfrac{1}{2}} \quad \text{with} \quad \gamma := \frac{1}{4}\Big(3 + \frac{\alpha_1}{2\delta}\Big).$$

Ideally, we would like $(\tilde Y_t)_{t\geq 0}$ to be a contact process itself, but we do not know how to achieve this. We have the following proposition, however, which is sufficient for our purposes:

Proposition The process $(\tilde Y_t)_{t\geq 0}$ can be coupled to a finite contact process $(Y_t)_{t\geq 0}$ on Ω_2^{n-1} with recovery rate $\delta' := 2\xi\delta$ and infection rates $\alpha'_1, \ldots, \alpha'_{n-1}$ given by $\alpha'_k := \frac{1}{2}\alpha_{k+1}$, in such a way that $\tilde Y_t \geq Y_t$ for all $t \geq 0$.

We may view the map $(\delta, \alpha_1, \ldots, \alpha_n) \mapsto (\delta', \alpha'_1, \ldots, \alpha'_{n-1})$ as a renormalization transformation. By iterating this map n times, we get a sequence of recovery rates $\delta, \delta', \delta'', \ldots$, the last of which gives a upper bound on the spectral gap of the finite contact process X on Ω_2^n. Under the condition (2), we can show that this spectral gap tends to zero as $n \to \infty$, and in fact, we can derive explicit lower bounds on the probability that finite systems survive till some fixed time t.

References

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