Exam Quantum Probability

June 25, 2008

Exercise 1 (Commuting projections) Let $\mathcal{H}$ be a finite-dimensional inner product space over $\mathbb{C}$ and let $P_1, P_2 \in \mathcal{L}(\mathcal{H})$ be projection operators. Show that $P_1 P_2$ is a projection operator if and only if P_1 commutes with P_2.

Exercise 2 (Contraction of tensors) Let $\mathcal{V}$ be a finite-dimensional linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{V}'$ be its dual space, i.e., $\mathcal{V}'$ is the space of all linear forms $l : \mathcal{V} \to \mathbb{K}$. Let $\{e(1), \ldots, e(n)\}$ be a basis for $\mathcal{V}$ and let $\{f(1), \ldots, f(n)\}$ be the associated dual basis of $\mathcal{V}'$, i.e., the $f(i)$'s are the linear forms defined by $f(i)(e(j)) = \delta_{ij}$. Consider the tensor product space $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}'$. A basis for this space is formed by all vectors of the form $e(i) \otimes e(j) \otimes f(k)$, hence each vector $A \in \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}'$ can uniquely be written in terms of this basis as

$$A = \sum_{ijk} A^{ij}{}_k e(i) \otimes e(j) \otimes f(k),$$

where $A^{ij}_k \in \mathbb{K}$ are the *coordinates* of the *tensor* A. Likewise, each vector $\phi \in \mathcal{V}$ can uniquely be written as

$$\phi = \sum_i \phi_i e(i)$$

where ϕ_i are the *coordinates* of ϕ. Obviously, the coordinates of A and ϕ depend on the choice of the basis $\{e(1), \ldots, e(n)\}$ (which then uniquely determines its dual basis $\{f(1), \ldots, f(n)\}$). Show that for each $A \in \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}'$ there exists a $\phi \in \mathcal{V}$ such that one has

$$\phi_i = \sum_j A^{ij}{}_j,$$

and this formula holds *for any choice of the basis* $\{e(1), \ldots, e(n)\}$. (**Hints** Consider first the case that A has the form $A = \psi \otimes \chi \otimes l$ where $\psi, \chi \in \mathcal{V}$ and $l \in \mathcal{V}'$. In this case, can you express the coordinates of A in terms of the coordinates of ψ, χ, and l? Still in this special case, if you define $\phi_i := \sum_j A^{ij}{}_j$ and $\phi := \sum_i \phi_i e(i)$, then can you give a nice expression for ϕ? Now how do you generalize to the case when A is not of the form $A = \psi \otimes \chi \otimes l$?)

Exercise 3 (Measurement) Let $\mathcal{A}, \mathcal{B}$ be Q-algebras and assume that both $\mathcal{A}$ and $\mathcal{B}$ are factor algebras. Let $H \in \mathcal{A} \otimes \mathcal{B}$ be a hermitian operator. For each state ρ on $\mathcal{A} \otimes \mathcal{B}$ and $t \geq 0$, define

$$S_t\rho(A) := \rho(e^{-itH} A e^{itH}) \qquad (A \in \mathcal{A} \otimes \mathcal{B}).$$

(a) Show that $S_t\rho$ is a state on $\mathcal{A} \otimes \mathcal{B}$. We interpret $S_t\rho$ as the state ρ evolved during a time interval of length t.

(b) Show that $S_t S_t \rho = S_{s+t}\rho$.

(c) Let σ be a fixed state on $\mathcal{B}$, and, for each state ρ on $\mathcal{A}$ and $t \geq 0$, define

$$T_t\rho(A) := S_t(\rho \otimes \sigma)(A \otimes 1) \qquad (A \in \mathcal{A}).$$

Show that there exist $V_t(1), \ldots, V_t(n) \in \mathcal{A}$ such that $\sum_{m=1}^{n} V_t(m)V_t(m)^* = 1$ and

$$T_t\rho(A) = \sum_{m=1}^{n} \rho(V_t(m) A V_t(m)^*) \qquad (A \in \mathcal{A}).$$

We may interpret $\mathcal{A}$ as our physical system of interest, $\mathcal{B}$ as our measuring equipment, and T_t as the effect of performing a measurement on the system $\mathcal{A}$.

(d) Is it true that $T_s T_t \rho = T_{s+t}\rho$?