# Exam Quantum Probability

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**Exercise 1 (A normal operator)** Let $A$ be a normal operator defined on some finite dimensional complex inner product space $\mathcal{H}$. Assume that $A^2 - A = 2I$, where $I$ denotes the identity operator. Show that there exists a projection operator $P$ such that $A = 3P - I$.

**Exercise 2 (Representation of factor algebras)** Let $\mathcal{A}$ be a Q-algebra that is moreover a factor algebra. Let $\mathcal{H}$ be a representation of $\mathcal{A}$. Show that there exists an irreducible representation $\mathcal{H}_1$ of $\mathcal{A}$ and a finite dimensional complex inner product space $\mathcal{H}_2$ such that $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2$ and

$$A(\phi_1 \otimes \phi_2) = (A\phi_1) \otimes \phi_2 \qquad (\phi_1 \in \mathcal{H}_1,\ \phi_2 \in \mathcal{H}_2).$$

**Exercise 3 (Measurement destroys entanglement)** Let $\mathcal{A}_1 = \mathcal{L}(\mathcal{H}_1)$ and $\mathcal{A}_2 = \mathcal{L}(\mathcal{H}_2)$ be Q-algebras that are moreover factor algebras. Let $\{e(1), \ldots, e(n)\}$ be an orthonormal basis for $\mathcal{H}_1$ and set $P_k := |e(k)\rangle\langle e(k)|$ $(k = 1, \ldots, n)$. Let $\rho$ be any state on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Let $\rho'$ be the state of our joint system $\mathcal{A}_1 \otimes \mathcal{A}_2$ after we perform the ideal measurement $\{P_1, \ldots, P_n\}$ on the system $\mathcal{A}_1$. Show that $\rho'$ is not entangled.

# Solutions

**Excercise 1**
Since $A$ is normal, we can find an orthonormal basis $\{e(1),\ldots,e(n)\}$ of $\mathcal{H}$ such that

$$A=\sum_{k=1}^n \lambda_k |e(k)\rangle\langle e(k)|.$$

Now $A^2-A=2I$ is equivalent to

$$\sum_{k=1}^n (\lambda_k^2-\lambda_k)|e(k)\rangle\langle e(k)| = 2\sum_{k=1}^n |e(k)\rangle\langle e(k)|,$$

which means that the eigenvalues $\lambda_k$ must satisfy $\lambda_k^2-\lambda_k-2=0$ for each $k$, which is equivalent to $(\lambda_k-2)(\lambda_k+1)=0$ for each $k$, hence $\lambda_k\in\{-1,2\}$ for each $k$. Set

$$P:=\sum_{k\colon \lambda_k=2} |e(k)\rangle\langle e(k)|.$$

Then $P$ is a projection operator and $A=3P-I$.

**Excercise 2**
By Lemma 4.6.1 in the Lecture Notes, all irreducible representations of $\mathcal{A}$ are equivalent, hence by Theorem 4.6.2 in the Lecture Notes, every representation $\mathcal{H}$ of $\mathcal{A}$ is of the form

$$(*)\quad \mathcal{H}\cong \underbrace{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_1}_{m\text{ times}},$$

where $\mathcal{H}_1$ is the up to equivalence unique irreducible representation of $\mathcal{A}$.
We need to show that $\mathcal{H}\cong\mathcal{H}_1\otimes\mathcal{H}_2$ for some complex inner product space $\mathcal{H}_2$. We note that $\dim(\mathcal{H})=nm$ so we choose $\dim(\mathcal{H}_2)=m$. We make $\mathcal{H}_1\otimes\mathcal{H}_2$ into a representation of $\mathcal{A}$ by defining

$$A(\phi_1\otimes\phi_2):=(A\phi_1)\otimes\phi_2 \qquad (\phi_1\in\mathcal{H}_1,\ \phi_2\in\mathcal{H}_2).$$

Applying Theorem 4.6.2 in the Lecture Notes again, we find that

$$\mathcal{H}_1\otimes\mathcal{H}_2\cong \underbrace{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_1}_{m'\text{ times}},$$

where we must have $m=m'$ since $\dim(\mathcal{H}_1\otimes\mathcal{H}_2)=nm$.
For those who do not like this argument, we note that if $\{f(1),\ldots,f(m)\}$ is an orthonormal basis for $\mathcal{H}_2$, then the orthonormal subspaces $\mathcal{H}^{(1)},\ldots,\mathcal{H}^{(n)}\subset\mathcal{H}_1\otimes\mathcal{H}_2$ defined by

$$\mathcal{H}^{(k)}:=\mathrm{span}\{e(i)\otimes f(k):i=1,\ldots,n\}$$

correspond to the decomposition of $\mathcal{H}$ in $(*)$.

**Excercise 3**
One has

$$(*) \quad \rho'(A) = \sum_{k=1}^{n} \rho\big((P_k \otimes 1)A(P_k \otimes 1)\big)$$
$$= \sum_{k=1}^{n} \rho(P_k \otimes 1)\rho\big(A \,\big|\, (P_k \otimes 1)\big) \qquad (A \in \mathcal{A}_1 \otimes \mathcal{A}_2),$$

where we write $\rho(A|P) := \rho(PAP)/\rho(P)$ to denote a state $\rho$ conditioned on an observation $P$. Since $(*)$ expresses $\rho'$ as a convex combination of the conditioned states $\rho\big(\,\cdot\,\big|\,(P_k \otimes 1)\big)$ (with $\rho(P_k \otimes 1) \neq 0$), it suffices to show that the latter are not entangled. We observe that

$$\rho\big(A_1 \otimes A_2 \,\big|\, (P_k \otimes 1)\big) = \rho\big((P_k \otimes 1)(A_1 \otimes A_2)(P_k \otimes 1)\big)/\rho(P_k \otimes 1)$$
$$= \rho\big(P_k A_1 P_k \otimes A_2\big)/\rho(P_k \otimes 1) \qquad (A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2).$$

Since $P_k$ is a minimal projection, by Lemma 5.1.5 in the Lecture Notes, this expression is equal to

$$\rho\big(\rho_{P_k}(A_1)P_k \otimes A_2\big)/\rho(P_k \otimes 1) = \rho_{P_k}(A_1)\frac{\rho\big(P_k \otimes A_2\big)}{\rho(P_k \otimes 1)},$$

where $\rho_{P_k}$ denotes the pure state on $\mathcal{A}_1$ associated with $P_k$. Defining a state $\rho^{(k)}$ on $\mathcal{A}_2$ by

$$\rho^{(k)}(A_2) := \frac{\rho\big(P_k \otimes A_2\big)}{\rho(P_k \otimes 1)} \qquad (A_2 \in \mathcal{A}_2),$$

we see that

$$\rho\big(A_1 \otimes A_2 \,\big|\, (P_k \otimes 1)\big) = \rho_{P_k}(A_1) \cdot \rho^{(k)}(A_2) \qquad (A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2).$$

This shows that the conditioned state $\rho\big(\,\cdot\,|(P_k \otimes 1)\big)$ is a product state, hence the state $\rho'$ is not entangled.