Exam Quantum Probability

October 6, 2008

Exercise 1 (A normal operator) Let A be a normal operator defined on some finite dimensional complex inner product space $\mathcal{H}$. Let $\text{abs}(z)$ be the function that assigns to each complex number z its absolute value $\text{abs}(z) := |z|$ and let $\text{abs}(A)$ be defined using the functional calculus for normal operators.

(a) Show that there exists a unitary operator U such that $\text{abs}(A) = UA$.

(b) Show that $|\langle\psi|A|\psi\rangle| \leq \langle\psi|\text{abs}(A)|\psi\rangle$ for all $\psi \in \mathcal{H}$.

(c) Is it true that $|\langle\psi|A|\psi\rangle| = \langle\psi|\text{abs}(A)|\psi\rangle$ for all $\psi \in \mathcal{H}$?

Exercise 2 (Greenberger-Horne-Zeilinger state) Let $\mathcal{H}$ be a 2-dimensional complex inner product space and let $\{e(1), e(2)\}$ be an orthonormal basis for $\mathcal{H}$. Let X and Y be the operators on $\mathcal{H}$ whose matrices with respect to the basis $\{e(1), e(2)\}$ are given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

(a) Show that X and Y are hermitian operators with spectra $\sigma(X) = \sigma(Y) = \{+1, -1\}$.

In view of (a), there exists orthonormal bases $\{f(1), f(2)\}$ and $\{g(1), g(2)\}$ for $\mathcal{H}$ such that

$$X = |f(1)\rangle\langle f(1)| - |f(2)\rangle\langle f(2)| \quad \text{and} \quad Y = |g(1)\rangle\langle g(1)| - |g(2)\rangle\langle g(2)|.$$

(b) On the product space $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, consider the operators $M_1 := X \otimes X \otimes X$, $M_2 := X \otimes Y \otimes Y$, $M_3 := Y \otimes X \otimes Y$, and $M_4 := Y \otimes Y \otimes X$. Show that for each $k = 1, 2, 3, 4$, the operator M_k is a hermitian operator with spectrum $\sigma(M_k) = \{+1, -1\}$. Let $\mathcal{F}_i$ be the eigenspace of M_i corresponding to the eigenvalue $+1$. Show that

$$\begin{aligned}
\mathcal{F}_1 = \text{span}\big\{ & f(1) \otimes f(1) \otimes f(1)\,,\ f(1) \otimes f(2) \otimes f(2), \\
& f(2) \otimes f(1) \otimes f(2)\,,\ f(2) \otimes f(2) \otimes f(1)\big\}, \\
\mathcal{F}_2 = \text{span}\big\{ & f(1) \otimes g(1) \otimes g(1)\,,\ f(1) \otimes g(2) \otimes g(2), \\
& f(2) \otimes g(1) \otimes g(2)\,,\ f(2) \otimes g(2) \otimes g(1)\big\}, \\
\mathcal{F}_3 = \text{span}\big\{ & g(1) \otimes f(1) \otimes g(1)\,,\ g(1) \otimes f(2) \otimes g(2), \\
& g(2) \otimes f(1) \otimes g(2)\,,\ g(2) \otimes f(2) \otimes g(1)\big\}, \\
\mathcal{F}_4 = \text{span}\big\{ & g(1) \otimes g(1) \otimes f(1)\,,\ g(1) \otimes g(2) \otimes f(2), \\
& g(2) \otimes g(1) \otimes f(2)\,,\ g(2) \otimes g(2) \otimes f(1)\big\}.
\end{aligned}$$

(c) Let ψ be the pure state on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ given by

$$\psi := \tfrac{1}{\sqrt{2}}\big(e(1) \otimes e(1) \otimes e(1) + e(2) \otimes e(2) \otimes e(2)\big).$$

Show that $M_1\psi = \psi$, $M_2\psi = -\psi$, $M_3\psi = -\psi$, and $M_4\psi = -\psi$.

(d) For $k = 1, 2, 3, 4$, let P_k denote the orthogonal projection on the eigenspace $\mathcal{F}_k$, and set $Q_k := 1 - P_k$. Show that under the state ψ, the observations P_1, Q_2, Q_3, and Q_4 each have probability one.

(e) Consider the observables $X_1 := X \otimes 1 \otimes 1$, $X_2 := 1 \otimes X \otimes 1$, $X_3 := 1 \otimes 1 \otimes X$, $Y_1 := Y \otimes 1 \otimes 1$, $Y_2 := 1 \otimes Y \otimes 1$, and $Y_3 := 1 \otimes 1 \otimes Y$. Imagine that we prepare our system in the pure state ψ and then measure the values x_1, x_2, x_3 of the observables X_1, X_2, X_3. Show that their product $x_1x_2x_3$ is always $+1$. Likewise, if we measure the values x_1, y_2, y_3 of the observables X_1, Y_2, Y_3, then their product $x_1y_2y_3$ is always -1; if we measure the values y_1, x_2, y_3 of the observables Y_1, X_2, Y_3, then their product $y_1x_2y_3$ is always -1; if we measure the values y_1, y_2, x_3 of the observables Y_1, Y_2, X_3, then their product $y_1y_2x_3$ is always -1.

(f) Let $x_1, x_2, x_3, y_1, y_2, y_3 \in \{+1, -1\}$ and assume that $x_1y_2y_3 = y_1x_2y_3 = y_1y_2x_3 = -1$. Show that $x_1x_2x_3 = -1$.

Solutions

Excercise 1 (a) Since A is normal, there exists an orthonormal basis $\{e(1), \ldots, e(n)\}$ of $\mathcal{H}$ such that

$$A = \sum_{i=1}^{n} \lambda_i |e(i)\rangle\langle e(i)|,$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the eigenvalues of A. By the definition of the functional calculus,

$$\mathrm{abs}(A) = \sum_{i=1}^{n} |\lambda_i| \, |e(i)\rangle\langle e(i)|.$$

Set $\gamma_i := |\lambda_i|\lambda_i^{-1}$ if $\lambda_i \neq 0$ and $\gamma_i := 1$ otherwise, and define a linear operator U by

$$U := \sum_{i=1}^{n} \gamma_i |e(i)\rangle\langle e(i)|.$$

Then U is unitary since $|\gamma_i| = 1$ for all i, and

$$\begin{aligned} UA &= \big(\sum_{i=1}^{n} \gamma_i |e(i)\rangle\langle e(i)|\big)\big(\sum_{j=1}^{n} \lambda_j |e(j)\rangle\langle e(j)|\big) \\ &= \sum_{ij} \gamma_i\lambda_j |e(i)\rangle\langle e(i)|e(j)\rangle\langle e(j)| = \sum_i |\lambda_i||e(i)\rangle\langle e(i)|, \end{aligned}$$

where we have used that $\gamma_i\lambda_i = |\lambda_i|$ and $\langle e(i)|e(j)\rangle = \delta_{ij}$.

(b) In coordinates with respect to the basis $\{e(1), \ldots, e(n)\}$, one has

$$\begin{aligned} |\langle\psi|A|\psi\rangle| &= \big|\sum_{ij} \psi_i^* A_{ij} \psi_j\big| = \big|\sum_i \lambda_i |\psi_i|^2\big| \le \sum_i |\lambda_i| \, |\psi_i|^2 \\ &= \sum_{ij} \psi_i^* \mathrm{abs}(A)_{ij} \psi_j = \langle\psi|\mathrm{abs}(A)|\psi\rangle. \end{aligned}$$

(c) No, this is not true. Take $\dim(\mathcal{H}) = 2$ and $\psi_1 = 1, \psi_2 = 1$, $\lambda_1 = 1, \lambda_2 = -1$. Then

$$\big|\sum_i \lambda_i |\psi_i|^2\big| = |1 - 1| = 0 < 2 = 1 + 1 = \sum_i |\lambda_i| \, |\psi_i|^2.$$

Excercise 2 (a) An operator A is hermitian if and only if its coordinates with respect to some (and hence every) orthonormal basis satisfy

$$(A_{ji})^* = A_{ij}.$$

In view of this, we see by inspection that the operators X and Y are hermitian. To find the eigenvalues of X, we must solve

$$0 = \det \begin{pmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{pmatrix} = \lambda^2 - 1,$$

which yields the eigenvalues $\lambda = +1, -1$. Likewise, for the operator Y, we solve

$$0 = \det \begin{pmatrix} 0-\lambda & -i \\ i & 0-\lambda \end{pmatrix} = \lambda^2 - 1,$$

which again yields the eigenvalues $\lambda = +1, -1$. Alternatively, we may observe that $X^2 = 1$, which implies that $\sigma(X) \subset \{+1, -1\}$. Since X is not a multiple of the identity operator, we must have $\sigma(X) = \{+1, -1\}$. The same argument applies to Y.

(b) We have $Xf(1) = f(1)$, $Xf(2) = -f(2)$, $Yg(1) = g(1)$, and $Yg(2) = -g(2)$. Therefore,

$$\begin{aligned} M_1 f(1) \otimes f(1) \otimes f(1) &= f(1) \otimes f(1) \otimes f(1), \\ M_1 f(1) \otimes f(2) \otimes f(2) &= f(1) \otimes (-f(2)) \otimes (-f(2)) \\ &= (-1)^2 f(1) \otimes f(2) \otimes f(2) = f(1) \otimes f(2) \otimes f(2), \\ M_1 f(2) \otimes f(1) \otimes f(2) &= (-f(2)) \otimes f(1) \otimes (-f(2)) = f(2) \otimes f(1) \otimes f(2) \\ M_1 f(2) \otimes f(2) \otimes f(1) &= (-f(2)) \otimes (-f(2)) \otimes f(1) = f(2) \otimes f(2) \otimes f(1). \end{aligned}$$

On the other hand, we see that

$$\begin{aligned} M_1 f(2) \otimes f(2) \otimes f(2) &= (-f(2)) \otimes (-f(2)) \otimes (-f(2)) = -f(2) \otimes f(2) \otimes f(2), \\ M_1 f(1) \otimes f(1) \otimes f(2) &= f(1) \otimes f(1) \otimes (-f(2)) = -f(1) \otimes f(1) \otimes f(2), \\ M_1 f(1) \otimes f(2) \otimes f(1) &= f(1) \otimes (-f(2)) \otimes f(1) = -f(1) \otimes f(2) \otimes f(1), \\ M_1 f(2) \otimes f(1) \otimes f(1) &= (-f(2)) \otimes f(1) \otimes f(1) = -f(2) \otimes f(1) \otimes f(1). \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{F}_1 &:= \operatorname{span}\big\{ f(1) \otimes f(1) \otimes f(1)\,,\; f(1) \otimes f(2) \otimes f(2), \\ &\qquad\qquad f(2) \otimes f(1) \otimes f(2)\,,\; f(2) \otimes f(2) \otimes f(1) \big\} \\ \text{and} \quad \mathcal{G}_1 &:= \operatorname{span}\big\{ f(2) \otimes f(2) \otimes f(2)\,,\; f(1) \otimes f(1) \otimes f(2), \\ &\qquad\qquad f(1) \otimes f(2) \otimes f(1)\,,\; f(2) \otimes f(1) \otimes f(1) \big\} \end{aligned}$$

are eigenspaces of M_1 corresponding to the eigenvalues $+1, -1$, respectively. Since these eigenspaces are orthogomal and their span is $\mathcal{H}$, it follows that $M_1 = P_1 - Q_1$, where P_1 is the orthogonal projection on $\mathcal{F}_1$ and Q_1 is the orthogonal projection on $\mathcal{G}_1$. In particular, this shows that M_1 is hermitian with spectrum $\sigma(M_1) = \{+1, -1\}$.

The operators M_2, M_3, M_4 go in exactly the same way, where we replace $f(1), f(2)$ by $g(1), g(2)$ in the right places. Alternatively, if we only want to prove that M_k is hermitian with spectrum $\sigma(M_k) = \{+1, -1\}$, then it suffices to check that $M_k^* = M_k$ and $M_k^2 = 1$, while M_k is not a multiple of the identity.

(c) We start by noting that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which means that $Xe(1) = e(2)$ and $Xe(2) = e(1)$. Similarly

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix}$$

which says that $Ye(1) = ie(2)$ and $Ye(2) = -ie(1)$. It follows that

$$\begin{aligned} M_1\psi &= \tfrac{1}{\sqrt{2}}\big((Xe(1)) \otimes (Xe(1)) \otimes (Xe(1)) + (Xe(2)) \otimes (Xe(2)) \otimes (Xe(2))\big) \\ &= \tfrac{1}{\sqrt{2}}\big(e(2) \otimes e(2) \otimes e(2) + e(1) \otimes e(1) \otimes e(1)\big) = \psi, \end{aligned}$$

and

$$\begin{aligned} M_2\psi &= \tfrac{1}{\sqrt{2}}\big((Xe(1)) \otimes (Ye(1)) \otimes (Ye(1)) + (Xe(2)) \otimes (Ye(2)) \otimes (Ye(2))\big) \\ &= \tfrac{1}{\sqrt{2}}\big(i^2 e(2) \otimes e(2) \otimes e(2) + (-i)^2 e(1) \otimes e(1) \otimes e(1)\big) = -\psi. \end{aligned}$$

By symmetry between the three subsystems, the operators M_3 and M_4 go in exactly the same way as M_2.

(d) In (c) we have shown that $\psi \in \mathcal{F}_1$, $\psi \in \mathcal{F}_2^\perp$, $\psi \in \mathcal{F}_3^\perp$, and $\psi \in \mathcal{F}_4^\perp$. It follows that $\langle\psi|P_1|\psi\rangle = 1$, $\langle\psi|Q_2|\psi\rangle = 1$, $\langle\psi|Q_3|\psi\rangle = 1$, and $\langle\psi|Q_4|\psi\rangle = 1$.

(e) Set $P := |f(1)\rangle\langle f(1)|$, $Q := |f(2)\rangle\langle f(2)|$, $P' := |g(1)\rangle\langle g(1)|$, and $Q' := |g(2)\rangle\langle g(2)|$. Then, for example, $P \otimes 1 \otimes 1$ corresponds to the observation that the observable X_1 takes on the value $+1$, and $P' \otimes Q' \otimes Q$ is the joint observation that Y_1 takes on the value $+1$, Y_2 takes on the value -1, and X_3 takes on the value -1, to give another example. A joint ideal measurement of the observables X_1, X_2, X_3 corresponds to the partition of the identity

$$\begin{aligned} \big\{&P \otimes P \otimes P\,,\ P \otimes Q \otimes Q\,,\ Q \otimes P \otimes Q\,,\ Q \otimes Q \otimes P \\ &Q \otimes Q \otimes Q\,,\ Q \otimes P \otimes P\,,\ P \otimes Q \otimes P\,,\ P \otimes P \otimes Q\big\}. \end{aligned}$$

Of these eight possible observations, the first four yield the product of values $x_1x_2x_3 = +1$. We observe that

$$P \otimes P \otimes P + P \otimes Q \otimes Q + Q \otimes P \otimes Q + Q \otimes Q \otimes P = P_1,$$

where P_1 is the orthogonal projection on the space $\mathcal{F}_1$ from part (b). We have shown in part (d) that P_1 has probability one, hence the probabilities of $P \otimes P \otimes P$, $P \otimes Q \otimes Q$, $Q \otimes P \otimes Q$, and $Q \otimes Q \otimes P$ sum up to one.

Similarly, a joint ideal measurement of the observables X_1, Y_2, Y_3 corresponds to the partition of the identity

$$\begin{aligned}\{&P \otimes P' \otimes P' \,,\, P \otimes Q' \otimes Q' \,,\, Q \otimes P' \otimes Q' \,,\, Q \otimes Q' \otimes P' \\ &Q \otimes Q' \otimes Q' \,,\, Q \otimes P' \otimes P' \,,\, P \otimes Q' \otimes P' \,,\, P \otimes P' \otimes Q'\}.\end{aligned}$$

Of these eight possible observations, the last four yield the product of values $x_1 y_2 y_3 = -1$. We observe that

$$Q \otimes Q' \otimes Q' + Q \otimes P' \otimes P' + P \otimes Q' \otimes P' + P \otimes P' \otimes Q' = Q_2$$

where Q_2 is the orthogonal projection on the space $\mathcal{F}_2^\perp$. We have shown in part (d) that Q_2 has probability one, hence the probabilities of $Q \otimes Q' \otimes Q'$, $Q \otimes P' \otimes P'$, $P \otimes Q' \otimes P'$, and $P \otimes P' \otimes Q'$ sum up to one. The other two cases, which correspond to Q_3 and Q_4, go in the same way.

(f) This follows from the observation that

$$-1 = (-1)^3 = x_1 y_2 y_3 \cdot y_1 x_2 y_3 \cdot y_1 y_2 x_3 = (y_1)^2 (y_2)^2 (y_3)^2 x_1 x_2 x_3 = x_1 x_2 x_3.$$

Some extra calculations The eigenvectors of X are found by solving

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$$

and

$$-\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

which yields the normalized eigenvectors

$$f(1) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad f(2) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvectors of Y are found by solving

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -i\psi_2 \\ i\psi_1 \end{pmatrix}$$

and

$$-\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -i\psi_2 \\ i\psi_1 \end{pmatrix},$$

which yields the normalized eigenvectors

$$g(1) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad g(2) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$