## Exam Markov Chains

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In a small patch of forest there grow $N$ trees. Some of them are spruce (in Czech: smrk), which we denote by 0 , while the others fir (in Czech: jedle) denoted by 1 . At the end of each year, one tree dies and is replaced at the beginning of the next year by a seedling from one of the other trees. Let $\left(I_{k}, J_{k}\right)_{k \geq 1}$ be i.i.d. random variables that are uniformly distributed on the set

$$
\{(i, j): 1 \leq i, j \leq N, i \neq j\}
$$

Let $z \in\{0,1\}^{N}$ be deterministic and let $Z=\left(Z_{k}\right)_{k \geq 0}$ be the stochastic process defined inductively by $Z_{0}=z$ and

$$
Z_{k+1}(i):= \begin{cases}Z_{k}\left(J_{k+1}\right) & \text { if } i=I_{k+1} \\ Z_{k}(i) & \text { otherwise }\end{cases}
$$

We interpret $Z_{k}(i)$ as the species of the $i$-th tree in the $k$-th year. Further, $I_{k}$ is the number of the tree that died in the previous year and that is replaced in the $k$-th year by a seedling whose parent is the $J_{k}$-th tree in the forest. We set $x:=\sum_{i=1}^{N} z(i)$ and let

$$
X_{k}:=\sum_{i=1}^{N} Z_{k}(i) \quad(k \geq 0)
$$

denote number of fir threes in year $k$.
Problem 1 Prove that $Z=\left(Z_{k}\right)_{k \geq 0}$ is a Markov chain.
Problem 2 Prove that $X=\left(X_{k}\right)_{k \geq 0}$ is a Markov chain. Is $X$ autonomous?
Problem 3 (a) Prove that there exists a random variable $X_{\infty}$ such that

$$
X_{k} \underset{k \rightarrow \infty}{\longrightarrow} X_{\infty} \quad \text { a.s. }
$$

(b) Prove that $\mathbb{P}\left[X_{\infty} \in\{0, N\}\right]=1$.
(c) Determine the function

$$
h(x):=\mathbb{P}^{x}\left[\lim _{k \rightarrow \infty} X_{k}=N\right] \quad(0 \leq x \leq N) .
$$

Problem 4 Let $P$ be the transition kernel of the Markov chain $Z$ and let $f:\{0,1\}^{N} \rightarrow$ $\mathbb{R}$ be the function

$$
f(z):=\frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1}^{N} z(i)(1-z(j)),
$$

which is the probability that of two trees chosen at random (without replacement), the first one is a fir tree and the second one a spruce. Calculate $P f$.
Problem 5 Show that the limit

$$
g(x):=\lim _{n \rightarrow \infty}(1-2 /(N(N-1)))^{-n} \mathbb{P}^{x}\left[0<X_{n}<N\right] \quad(0 \leq x \leq N)
$$

exists and calculate $g$ up to a multiplicative constant. What would need to be done to determine the constant?

## Solutions

Problem 1 Obvious, since $Z$ is defined by a random mapping representation.
Problem $2 X$ is indeed an autonomous Markov chain (as a function of $Z$ ) since

$$
\mathbb{P}\left[X_{k+1}=x \mid Z_{k}\right]= \begin{cases}\frac{X_{k}}{N} \frac{N-X_{k}}{N-1} & \text { if } x=X_{k}-1 \\ \frac{N-X_{k}}{N} \frac{X_{k}}{N-1} & \text { if } x=X_{k}+1\end{cases}
$$

and one minus these probabilities if $x=X_{k}$. In particular, these probabilities depend on $Z_{k}$ only through $X_{k}$.
Problem 3 (a) Let $\left(\mathcal{F}_{k}^{Z}\right)_{k \geq 0}$ be the filtration generated by $Z$. We observe that

$$
\mathbb{E}\left[X_{k+1} \mid \mathcal{F}_{k}^{Z}\right]=X_{k},
$$

which proves that $X$ is a martingale. Since $X$ is moreover bounded, it follows that $X_{k}$ converges a.s. as $k \rightarrow \infty$.
(b) The state space of $X$ is $\{0, \ldots, N\}$, which is finite. There are two traps, 0 and $N$, while all other states are equivalent. Since the equivalence class $\{1, \ldots, N-1\}$ is not a closed set (it is possible get out of this set), while recurrent equivalence classes are always closed, it follows that all states in this set must be transient. Since the state space is finite, this implies that the process must eventually end up in one of the traps. Alternatively, this follows from the principle "what can happen must eventually happen".
(c) Since $X$ is a bounded martingale, it is certainly uniformly integrable so its a.s. limit is also its $L_{1}$-limit. In particular,

$$
h(x)=\mathbb{P}^{x}\left[X_{\infty}=N\right]=N^{-1} \mathbb{E}^{x}\left[X_{\infty}\right]=N^{-1} x \quad(0 \leq x \leq N)
$$

Problem 4 Let $\left(I^{\prime}, J^{\prime}\right)$ be uniformly distributed on the set of all $(i, j)$ with $0 \leq i, j \leq N$ and $i \neq j$, independent of the $\left(I_{k}, J_{k}\right)_{k \geq 1}$ that define $Z$. Define

$$
I^{\prime \prime}:= \begin{cases}J_{1} & \text { if } I^{\prime}=I_{1} \\ I^{\prime} & \text { otherwise }\end{cases}
$$

and likewise

$$
J^{\prime \prime}:= \begin{cases}J_{1} & \text { if } J^{\prime}=I_{1} \\ J^{\prime} & \text { otherwise }\end{cases}
$$

In other words, $\left(I^{\prime}, J^{\prime}\right)$ are trees sampled at random from the forest in year one and $I^{\prime \prime}$ (resp. $J^{\prime \prime}$ ) is either the tree $I^{\prime}$ (resp. $J^{\prime}$ ) or its parent in the year zero. Then

$$
\begin{aligned}
& P f(z)=\sum_{z^{\prime}} P\left(z, z^{\prime}\right) f\left(z^{\prime}\right)=\mathbb{E}^{z}\left[f\left(Z_{1}\right)\right]=\mathbb{P}^{z}\left[Z_{1}\left(I^{\prime}\right)=1, Z_{2}\left(J^{\prime}\right)=0\right] \\
& \quad=\mathbb{P}\left[z\left(I^{\prime \prime}\right)=1, \quad z\left(J^{\prime \prime}\right)=0\right]=\mathbb{P}\left[z\left(I^{\prime \prime}\right)=1, z\left(J^{\prime \prime}\right)=0 \mid I^{\prime \prime} \neq J^{\prime \prime}\right] \mathbb{P}\left[I^{\prime \prime} \neq J^{\prime \prime}\right] \\
& \quad=f(z)(1-2 /(N(N-1))),
\end{aligned}
$$

where we have used that $\mathbb{P}\left[I^{\prime \prime}=J^{\prime \prime}\right]=2 /(N(N-1))$ and that $\left(I^{\prime \prime}, J^{\prime \prime}\right)$, conditionally on the event $I^{\prime \prime} \neq J^{\prime \prime}$, is equally distributed with $\left(I^{\prime}, J^{\prime}\right)$.
Alternatively, we may note that if $x:=\sum_{i=1}^{N} z(i)$, then

$$
f(z)=\frac{x(N-x)}{N(N-1)}=: f^{\prime}(x)
$$

is a function of $x$ only. It follows that

$$
\begin{aligned}
\mathbb{E}^{z} & {\left[f\left(Z_{1}\right)\right]=\mathbb{E}^{x}\left[f^{\prime}\left(X_{1}\right)\right] } \\
= & f^{\prime}(x)+\frac{x(N-x)}{N(N-1)}\left(f^{\prime}(x+1)-f^{\prime}(x)\right)+\frac{x(N-x)}{N(N-1)}\left(f^{\prime}(x-1)-f^{\prime}(x)\right) \\
= & f^{\prime}(x)+\frac{x(N-x)}{N(N-1)}\left(\frac{(x+1)(N-x-1)}{N(N-1)}-\frac{x(N-x)}{N(N-1)}\right) \\
& +\frac{x(N-x)}{N(N-1)}\left(\frac{(x-1)(N-x+1)}{N(N-1)}-\frac{x(N-x)}{N(N-1)}\right) \\
= & f^{\prime}(x)+\frac{x(N-x)}{N(N-1)}\left(\frac{(N-x)-x-1}{N(N-1)}+\frac{x-(N-x)-1}{N(N-1)}\right) \\
= & f^{\prime}(x)\left(1-\frac{2}{N(N-1)}\right)=(1-2 /(N(N-1))) f(z) .
\end{aligned}
$$

Problem 5 Let $P^{\prime}$ denote the transition kernel of $X$. We conclude from the solution of the previous problem that $P^{\prime} f^{\prime}=\alpha_{N} f^{\prime}$, where $f^{\prime}$ is defined above and $\alpha_{N}:=(1-$ $2 /(N(N-1)))$. In particular, since $f^{\prime}$ is positive on the set $\{1, \ldots, N-1\}$, this is the right Perron Frobenius eigenfunction of the retriction of $P^{\prime}$ to this set. It follows from results proved in the lecture notes that the limit

$$
g(x):=\lim _{n \rightarrow \infty} \alpha_{N}^{-1} \mathbb{P}^{x}\left[0<X_{n}<N\right] \quad(0 \leq x \leq N)
$$

exists for all $0<x<N$ where $g$ is the right Perron Frobenius eigenfunction normalized such that

$$
\sum_{x} \eta(x) g(x)=1,
$$

where $\eta$ is the left eigenvector normalized such that $\sum_{x} \eta(x)=1$. Thus $g=c f^{\prime}$ for some $c>0$ where we would need to find the left Perron Frobenius eigenfunction to determine the constant $c$. (Note that $g(x)=0=c f^{\prime}(x)$ trivially for $x=0, N$.)

