Exam Quantum Probability

June 20, 2013

Hints: You may use claims you are supposed to prove in one excercise to solve other excercises (even if you did not prove such claims).
It may be helpful to recall the following fact (Excercise 1.2.16). Let $\mathcal{H}$ be an inner product space and let $A \in \mathcal{L}(\mathcal{H})$. Then $A \geq 0$ if one (and hence all) of the following equivalent properties hold:

(i) $\langle\psi|A|\psi\rangle \geq 0$ for all $\psi \in \mathcal{H}$.

(ii) $A = B^*B$ for some $B \in \mathcal{L}(\mathcal{H})$.

(iii) A is hermitian and all its eigenvalues are nonnegative.

By definition, we write $A \leq B$ if $B - A \geq 0$.

Exercise 1 (Ordering of projections) Let $\mathcal{H}$ be an inner product space, let $\mathcal{F}$ be a subspace of $\mathcal{H}$, and let P be the orthogonal projection on $\mathcal{F}$. Recall that $P = P^2 = P^*$ and $\|P\psi\| \leq \|\psi\|$ with equality if and only if $\psi \in \mathcal{F}$. Similarly, let Q be the orthogonal projection on some different subspace $\mathcal{G}$. Prove the equivalence of the following statements.

(i) $P \leq Q$

(ii) $\mathcal{F} \subset \mathcal{G}$

(iii) $PQP = P$.

For the next excercise, it may be helpful to recall (Proposition 4.1.2) that if P is a minimal projection in some Q-algebra $\mathcal{A}$, then there exists a pure state ρ_P on $\mathcal{A}$ such that $PAP = \rho_P(A)P$ $(A \in \mathcal{A})$ and every pure state on $\mathcal{A}$ is of this form.

Exercise 2 (States concentrated on observations) Let $\mathcal{A}$ be a Q-algebra, let ρ be a state on $\mathcal{A}$, and let $P \in \mathcal{A}$ be any projection. Assume that $\rho(P) = 1$. Prove that $\rho(A) = \rho(PAP)$ $(A \in \mathcal{A})$. Hint: prove the statement for pure states first.

Exercise 3 (States with pure marginals) Let $\mathcal{A}_1, \mathcal{A}_2$ be Q-algebras, let ρ be a state on $\mathcal{A}_1 \otimes \mathcal{A}_2$, and let

$$\rho_1(A_1) := \rho(A_1 \otimes 1) \quad \text{and} \quad \rho_2(A_2) := \rho(1 \otimes A_2)$$

denote the first and second marginal of ρ, respectively. Assume that ρ_1 is a pure state. Prove that $\rho = \rho_1 \otimes \rho_2$.

Exercise 4 (Entangled marginal) Let $\mathcal{H}$ be a two-dimensional inner product space with orthonormal basis $\{e(1), e(2)\}$. We represent the product algebra $\mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H})$ in the canonical way on the product space $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. For any state ρ on $\mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H})$, let

$$\rho_{12}(A_1 \otimes A_2) := \rho(A_1 \otimes A_2 \otimes 1) \quad \text{and} \quad \rho_{23}(A_2 \otimes A_3) := \rho(1 \otimes A_2 \otimes A_3)$$

denote the marginals corresponding to the combination of the first and second, resp. second and third subsystem. For $i = 1, 2$, let $P_i := |e(i)\rangle\langle e(i)|$ denote the orthogonal projection on $e(i)$. Set

$$\rho^{(i)}(B) := \frac{\rho\big((P_i \otimes 1 \otimes 1)B(P_i \otimes 1 \otimes 1)\big)}{\rho(P_i \otimes 1 \otimes 1)} \qquad \big(B \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H})\big),$$

and

$$\rho' := p_1\rho^{(1)} + p_2\rho^{(2)} \qquad \text{with} \quad p_i := \rho(P_i \otimes 1 \otimes 1) \quad (i = 1, 2).$$

Prove that ρ' and ρ have the same marginal corresponding to the second and third subsystem, i.e.,

$$\rho'_{23} = \rho_{23}.$$

Now assume that ρ_{12} is the fully entangled state corresponding to the state vector

$$\psi = \frac{1}{\sqrt{2}}\big(e(1) \otimes e(1) + e(2) \otimes e(2)\big).$$

Show that ρ_{23} is not entangled.

Solutions

Ex 1

We observe that $P \leq Q \Leftrightarrow \langle\psi|P|\psi\rangle \leq \langle\psi|Q|\psi\rangle$ for all $\psi \in \mathcal{H} \Leftrightarrow \langle P\psi|P\psi\rangle \leq \langle Q\psi|Q\psi\rangle$ for all $\psi \in \mathcal{H} \Leftrightarrow \|P\psi\| \leq \|Q\psi\|$ for all $\psi \in \mathcal{H}$.
(ii)$\Rightarrow$(i) & (iii): If $\mathcal{F} \subset \mathcal{G}$, then $P\psi \in \mathcal{F} \subset \mathcal{G}$ for all ψ and hence $QP\psi = P\psi$ so $QP = P$. Now also $P = P^* = (QP)^* = P^*Q^* = PQ$. It follows that $\|P\psi\| = \|PQ\psi\| \leq \|Q\psi\|$ for all ψ, so $P \leq Q$. Also $PQP = PP = P$.
(i)&(iii)$\Rightarrow$(ii): If $\mathcal{F} \not\subset \mathcal{G}$, then there exists $0 \neq \psi \in \mathcal{F}$, $\psi \notin \mathcal{G}$. Now $\|P\psi\| = \|\psi\| > \|Q\psi\|$ so $P \not\leq Q$. Also $\|PQP\psi\| = \|PQ\psi\| \leq \|Q\psi\| < \|\psi\| = \|P\psi\|$ so $PQP \neq P$.

Ex 2

If ρ is a pure state, then $\rho = \rho_Q$ for some mininal projection Q. Now $\rho_Q(P) = 1$ implies $QPQ = \rho_Q(P)Q = Q$ hence $Q \leq P$ by Excercise 1. It follows that $\rho_Q(PAP)Q = QPAPQ = QAQ = \rho_Q(A)Q$ and hence $\rho_Q(PAP) = \rho_Q(A)$ for all $A \in \mathcal{A}$. A general state can be written as a convex combination of pure states, $\rho = \sum_{k=1}^n p_k \rho_{Q_n}$ with $p_k > 0$ and $\sum_{k=1}^n p_k = 1$. Now $\rho(P) = 1$ implies $\rho_{Q_k}(P) = 1$ for all k and hence by what we have just proved $\rho(PAP) = \sum_{k=1}^n p_k \rho_{Q_n}(PAP) = \sum_{k=1}^n p_k \rho_{Q_n}(A) = \rho(A)$.

Ex 3

If ρ_1 is a pure state, then $\rho_1 = \rho_P$ for some minimal projection $P \in \mathcal{A}_1$. Now $P \otimes 1$ is a projection in $\mathcal{A}_1 \otimes \mathcal{A}_2$ (though not a minimal one, except in trivial cases) with the property that $\rho(P \otimes 1) = \rho_1(P) = 1$, so Excercise 2 tells us that $\rho(A \otimes B) = \rho\big((P \otimes 1)(A \otimes B)(P \otimes 1)\big) = \rho(PAP \otimes B) = \rho\big((\rho_P(A)P) \otimes B\big) = \rho_P(A)\rho(P \otimes B) = \rho_P(A)\rho\big((P \otimes 1)(1 \otimes B)(P \otimes 1)\big) = \rho_P(A)\rho(1 \otimes B) = \rho_1(A)\rho_2(B)$.

Alternative solution (Tibor Mach): Let $A_1 \in \mathcal{A}_1$ be arbitrary and let $P_2 \in \mathcal{A}_2$ be a projection. Assume that $p := \rho(1 \otimes P_2)$ satisfies $0 < p < 1$. Then

$$\begin{aligned}\rho_1(A_1) &= \rho(A_1 \otimes 1) = \rho\big(A_1 \otimes \big(P_2 + (1 - P_2)\big)\big) = \rho(A_1 \otimes P_2) + \rho\big(A_1 \otimes (1 - P_2)\big) \\ &= p\frac{\rho(A_1 \otimes P_2)}{\rho(1 \otimes P_2)} + (1-p)\frac{\rho\big(A_1 \otimes (1 - P_2)\big)}{\rho\big(1 \otimes (1 - P_2)\big)},\end{aligned}$$

where we have used that $1 - p = \rho\big(1 \otimes (1 - P_2)\big)$. Now

$$A_1 \mapsto \frac{\rho(A_1 \otimes P_2)}{\rho(1 \otimes P_2)} \quad \text{and} \quad A_1 \mapsto \frac{\rho\big(A_1 \otimes (1 - P_2)\big)}{\rho\big(1 \otimes (1 - P_2)\big)}$$

are states on $\mathcal{A}_1$, and we have just shown that ρ_1 can be written as a nontrivial convex combination of these states. Since ρ_1 is pure, we conclude that they must both be equal to ρ_1. In particular, for any projections $P_1 \in \mathcal{A}_1$ and $P_2 \in \mathcal{A}_2$,

$$\rho_1(P_1) = \frac{\rho(P_1 \otimes P_2)}{\rho(1 \otimes P_2)},$$

which shows that

$$\rho_1(P_1)\rho_2(P_2) = \rho(P_1 \otimes P_2).$$

We claim that this formula holds even if $p = 0, 1$. Indeed, if $p := \rho(1 \otimes P_2) = 0$, then $P_1 \otimes P_2 \leq 1 \otimes P_2$ and hence $\rho(P_1 \otimes P_2) \leq \rho(1 \otimes P_2) = 0$, so

$$\rho_1(P_1)\rho_2(P_2) = \rho(P_1) \cdot 0 = 0 = \rho(P_1 \otimes P_2).$$

If $p := \rho(1 \otimes P_2) = 1$, then

$$\rho_1(P_1) = \rho(P_1 \otimes 1) = \rho(P_1 \otimes P_2) + \rho\big(P_1 \otimes (1 - P_2)\big) = \frac{\rho(P_1 \otimes P_2)}{\rho(1 \otimes P_2)} + 0.$$

Thus, we conclude that

$$\rho_1(P_1)\rho_2(P_2) = \rho(P_1 \otimes P_2)$$

for any projections $P_1 \in \mathcal{A}_1$ and $P_2 \in \mathcal{A}_2$. Since the linear span of the projections is the whole algebra, by linearity, we conclude first that $\rho_1(A_1)\rho_2(P_2) = \rho(A_1 \otimes P_2)$ for arbitrary $A_1 \in \mathcal{A}_1$ and for $P_2 \in \mathcal{A}_2$ a projection, and then by the same argument also remove the assumption that P_2 is a projection.

Ex 4

We have

$$\begin{aligned}
&\rho'(1 \otimes A_2 \otimes A_3) \\
&= \rho\big((P_1 \otimes 1 \otimes 1)(1 \otimes A_2 \otimes A_3)(P_1 \otimes 1 \otimes 1)\big)\rho\big((P_2 \otimes 1 \otimes 1)(1 \otimes A_2 \otimes A_3)(P_2 \otimes 1 \otimes 1)\big) \\
&= \rho(P_1 \otimes A_2 \otimes A_3) + \rho(P_1 \otimes A_2 \otimes A_3) = \rho(1 \otimes A_2 \otimes A_3).
\end{aligned}$$

In fact, this just says that performing a measurement on the first subsystem has no effect on the other two subsystems, which we already knew.

In view of this, to show ρ_{23} is not entangled, it suffices to prove that ρ'_{23} is not entangled. Here $\rho'_{23} = p_1\rho^{(1)}_{23} + p_2\rho^{(2)}_{23}$, so it suffices to show that $\rho^{(i)}_{23}$ $i = 1, 2$ are not entangled. Let

$$\rho^{(i)}_2(A_2) := \rho^{(i)}(1 \otimes A_2 \otimes 1)$$

denote the marginal of $\rho^{(i)}$ corresponding to the second subsystem only. We observe that

$$\begin{aligned}
\rho^{(i)}_2(P_i) &= \frac{\rho\big((P_i \otimes 1 \otimes 1)(1 \otimes P_i \otimes 1)(P_i \otimes 1 \otimes 1)\big)}{\rho(P_i \otimes 1 \otimes 1)} \\
&= \frac{\rho(P_i \otimes P_i \otimes 1)}{\rho(P_i \otimes 1 \otimes 1)} = \frac{\rho_{12}(P_i \otimes P_i)}{\rho_{12}(P_i \otimes 1)} = \frac{\langle\psi|P_i \otimes P_i|\psi\rangle}{\langle\psi|P_i \otimes 1|\psi\rangle}.
\end{aligned}$$

Here

$$\begin{aligned}
\langle\psi|P_1 \otimes 1|\psi\rangle &= \tfrac{1}{2}\langle e(1) \otimes e(1)|P_1 \otimes 1|e(1) \otimes e(1)\rangle \\
&\quad + \tfrac{1}{2}\langle e(1) \otimes e(1)|P_1 \otimes 1|e(2) \otimes e(2)\rangle \\
&\quad + \tfrac{1}{2}\langle e(2) \otimes e(2)|P_1 \otimes 1|e(1) \otimes e(1)\rangle \\
&\quad + \tfrac{1}{2}\langle e(2) \otimes e(2)|P_1 \otimes 1|e(2) \otimes e(2)\rangle \\
&= \tfrac{1}{2}\langle e(1) \otimes e(1)|e(1) \otimes e(1)\rangle + 0 \\
&\quad + \tfrac{1}{2}\langle e(2) \otimes e(2)|e(1) \otimes e(1)\rangle + 0 \\
&= \tfrac{1}{2}.
\end{aligned}$$

and similarly

$$\begin{aligned}\langle\psi|P_1 \otimes P_1|\psi\rangle &= \tfrac{1}{2}\langle e(1)\otimes e(1)|P_1 \otimes P_1|e(1)\otimes e(1)\rangle \\ &\quad +\tfrac{1}{2}\langle e(1)\otimes e(1)|P_1 \otimes P_1|e(2)\otimes e(2)\rangle \\ &\quad +\tfrac{1}{2}\langle e(2)\otimes e(2)|P_1 \otimes P_1|e(1)\otimes e(1)\rangle \\ &\quad +\tfrac{1}{2}\langle e(2)\otimes e(2)|P_1 \otimes P_1|e(2)\otimes e(2)\rangle \\ &= \tfrac{1}{2}\langle e(1)\otimes e(1)|e(1)\otimes e(1)\rangle + 0 + 0 + 0 = \tfrac{1}{2},\end{aligned}$$

which shows that

$$\rho^{(1)}(P_1) = \tfrac{1}{2}/\tfrac{1}{2} = 1 \quad \text{and similarly} \quad \rho^{(2)}(P_2) = 1.$$

By Excercise 2, it follows that for any $A \in \mathcal{L}(\mathcal{H})$,

$$\rho_2^{(i)}(A) = \rho_2^{(i)}(P_i A P_i) = \rho_2^{(i)}\big(\rho_{P_i}(A)P_i\big) = \rho_{P_i}(A)\rho_2^{(i)}(P_i) = \rho_{P_i}(A),$$

which shows that $\rho_2^{(i)} = \rho_{P_i}$ is a pure state. By Excercise 3, it follows that $\rho_{23}^{(i)} = \rho_2^{(i)} \otimes \rho_3^{(i)}$.