The contact process seen from a typical infected site.

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October 31, 2007

Abstract

This paper studies contact processes on general countable groups. It is shown that any such contact process has a well-defined exponential growth rate, and this quantity is used to study the process. In particular, it is proved that on any nonamenable group, the critical contact process dies out.

MSC 2000. Primary: 60K35; Secondary: 82C22, 82B43.
Keywords. Critical contact process, exponential growth, amenability, Campbell law.
Acknowledgement. Research supported by GAČR grants 201/06/1323 and 201/07/0237, and the German Science Foundation. Part of this work was carried out when the author was employed as a postdoc at the university of Tübingen.

Contents

1 Introduction and main results

1.1 Introduction

This paper studies contact processes whose underlying lattice is a general countable group. There exists a small body of literature about contact processes on general lattices, but several basic questions have been answered only on specific lattices. In particular, a lot is known about the process on the d-dimensional integer lattice $\mathbb{Z}^d$, and on regular trees. (See [Lig99] as a general reference for contact processes on $\mathbb{Z}^d$, trees, and other lattices.)

It turns out that the contact process on regular trees behaves quite differently from the contact process on $\mathbb{Z}^d$. For the process on $\mathbb{Z}^2$, it is known that there is a critical infection rate $\lambda_{\rm c}$ such that for $\lambda \leq \lambda_{\rm c}$, the process dies out, while for $\lambda > \lambda_{\rm c}$, the process survives with positive probability, and complete convergence holds. On the other hand, on trees, there are two critical values $\lambda_{\rm c} < \lambda'_{\rm c}$ such that in the intermediate regime $\lambda_{\rm c} < \lambda \leq \lambda'_{\rm c}$, the process survives, but complete convergence does not hold. The situation is quite similar to the situation for (unoriented) percolation on general transitive lattices, where it is known that one has uniqueness of the infinite cluster whenever the lattice is amenable, while it is conjectured (and proved in several special cases) that on any nonamenable lattice there exists an intermediate parameter regime where there are infinitely many infinite clusters.

While a lot is known nowadays about percolation on general transitive graphs, the same cannot be said for the contact process. In particular, it is not known what is the essential difference between $\mathbb{Z}^2$ and trees that causes the observed difference in behavior on these lattices. A natural guess is that the essential feature is amenability ($\mathbb{Z}^d$ being amenable, while trees are not). However, the results in the present paper may cast some doubt on this.

In the present paper, we study contact processes on general countable groups by means of their exponential growth rate. A simple subadditivity argument shows that the expected number of infected sites of a contact process on a transitive lattice, started with finitely many infected sites, grows at a well-defined exponential rate. On $\mathbb{Z}^d$, it is known that this exponential growth rate is negative for $\lambda < \lambda_{\rm c}$ [Lig99, Thm 2.48], and zero for any $\lambda > \lambda_{\rm c}$. Indeed, it is easy to see (and prove) that on $\mathbb{Z}^d$ there is simply not enough space for a contact process to grow exponentially fast (with positive exponent). On the other hand, one of our main results in this paper will be that if a contact process survives on a nonamenable group, then its exponential growth rate must be strictly positive. Intuition says that a contact process that survives with a positive exponential growth rate behaves very much like a perturbed branching process. On the other hand, contact processes that survive but have a zero exponential growth rate are different.

We do not know if (non)amenability is the essential feature here. It is known that there exist exponentially growing groups that are amenable. (A well-known example is the lamplighter group). Although we do not prove it here, it seems plausible that a contact process on such a group may have a positive exponential growth rate. Thus, contact processes on such amenable groups might in some respects show behavior that is more similar to processes on trees than on $\mathbb{Z}^d$.

Part of the present work appeared before as Chapter 4 of the author's habilitation thesis [Swa07]. In particular, Proposition 4.3 below is Theorem 4.3 (a) in [Swa07].

1.2 Set-up

We will study contact processes whose underlying lattice is a general countable groups. From the point of view of studying general transitive lattices, this is not quite as general as one might wish; in particular, such lattices are always unimodular. Assuming that the lattice is a group will simplify our proofs, however, so as a first step it seems reasonable.

Our set-up is as follows. We let Λ be a finite or countably infinite group, which we refer to as the *lattice*, with group action $(i,j) \mapsto ij$ and unit element 0, also referred to as the origin. Each site $i \in \Lambda$ can be in one of two states: healthy or infected. Infected sites become healthy with *recovery rate* $\delta \geq 0$. An infected site i infects another site j with *infection rate* $a(i,j) \geq 0$. We assume that the infection rates are invariant with respect to the left action of the group and summable:

$$\begin{aligned} &\text{(i)} \quad a(i,j) = a(ki,kj) \qquad\qquad (i,j,k \in \Lambda), \\ &\text{(ii)} \quad |a| := \sum_i a(0,i) < \infty, \end{aligned} \tag{1.1}$$

Here we adopt the convention that sums over i,j,k always run over Λ, unless stated otherwise. Note that we do *not* assume that $a(i,j) = a(j,i)$, i.e., our contact processes are in general asymmetric.

Let η_t be the set of all infected sites at time $t \geq 0$. Then $\eta = (\eta_t)_{t\geq 0}$ is a Markov process in the space $\mathcal{P}(\Lambda) := \{A : A \subset \Lambda\}$ of all subsets of Λ, called the contact process on Λ with infection rates $a = (a(i,j))_{i,j\in\Lambda}$ and recovery rate δ, or shortly the (Λ, a, δ)*-contact process.* We equip $\mathcal{P}(\Lambda) \cong \{0,1\}^\Lambda$ with the product topology and the associated Borel-σ-field $\mathcal{B}(\mathcal{P}(\Lambda))$, and let $\mathcal{P}_{\rm fin}(\Lambda) := \{A \subset \Lambda : |A| < \infty\}$ denote the subspace of finite subsets of Λ. Under the assumptions (1.1), η is a well-defined Feller process with cadlag sample paths in the compact state space $\mathcal{P}(\Lambda)$, and $\eta_0 \in \mathcal{P}_{\rm fin}(\Lambda)$ implies $\eta_t \in \mathcal{P}_{\rm fin}(\Lambda)$ for all $t \geq 0$ a.s.

Note that we have not assumed any additional structure on Λ, except for the group structure. In particular, we have not assumed any sort of 'nearest neighbor' structure. This may be obtained in the following special case. Assume that Λ is finitely generated and that Δ is a finite, symmetric (with respect to taking inverses), generating set for Λ. Then the (left) Cayley graph $\mathcal{G} = \mathcal{G}(\Lambda, \Delta)$ associated with Λ and Δ is the graph with vertex set $\mathcal{V}(\mathcal{G}) := \Lambda$ and edges $\mathcal{E}(\mathcal{G}) := \{\{i,j\} : i^{-1}j \in \Delta\}$. Examples of Cayley graphs are $\mathbb{Z}^d$ and regular trees. (In the case of trees, there are several possible choices for the group structure.) Setting $a(i,j) := \lambda 1_{\{i^{-1}j\in\Delta\}}$, with $\lambda > 0$, and choosing $\delta \geq 0$, then defines a nearest-neighbor contact process on the Cayley graph $\mathcal{G}(\Lambda, \Delta)$. In this case, λ is simply referred to as 'the' infection rate. If $\delta > 0$, then by rescaling time we may set $\delta = 1$, so it is customary to assume that $\delta = 1$. If $\delta = 0$, then η is a special case of first-passage percolation (see [Kes86]).

Returning to our more general set-up, we make the following observation, which is the basis of our analysis. Below, we use the notation η_t^A to denote the (Λ, a, δ)-contact process started at time zero in $\eta_0^A = A$, evaluated at time $t \geq 0$.

Lemma 1.1 (Exponential growth rate) *Let η be a (Λ, a, δ)-contact process. Then there exists a constant $r = r(\Lambda, a, \delta) \in [-\delta, |a| - \delta]$ such that*

$$\lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}\big[|\eta_t^A|\big] = r \qquad (\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)). \tag{1.2}$$

We call $r = r(\Lambda, a, \delta)$ the *exponential growth rate* of the (Λ, a, δ)-contact process. We note that r has been defined before in the specific context of nearest-neighbor processes on regular trees. Indeed, $r = \log \phi(1)$, where $\phi(\rho)$ is the function defined in [Lig99, formula (I.4.23)].

1.3 The exponential growth rate

In this section, we investigate the exponential growth rate $r(\Lambda, a, \delta)$ of a contact process defined in Lemma 1.1.

We start by recalling a few basic facts and definitions concering groups. As before, let Λ be a finite or countably infinite group. For $i \in \Lambda$ and $A, B \subset \Lambda$ we put $AB := \{ij : i \in A,\ j \in B\}$, $iA := \{i\}A$, $Ai := A\{i\}$, $A^{-1} := \{i^{-1} : i \in A\}$, $A^0 := \{0\}$, $A^n := AA^{n-1}$ $(n \geq 1)$, and $A^{-n} := (A^{-1})^n = (A^n)^{-1}$. We write $A \triangle B := (A\backslash B) \cup (B\backslash A)$ for the symmetric difference of A and B and let $|A|$ denote the cardinality of A.

By definition, we say that Λ is *amenable* if

$$\begin{array}{l}\text{For every finite nonempty } \Delta \subset \Lambda \text{ and } \varepsilon > 0, \text{ there exists a finite} \\ \text{nonempty } A \subset \Lambda \text{ such that } |(A\Delta) \triangle A| \leq \varepsilon |A|.\end{array} \tag{1.3}$$

If Λ is finitely generated, then it suffices to check (1.3) for one finite symmetric generating set Δ. In this case, $(A\Delta) \triangle A$ is the set of all $i \notin A$ for which there exists a $j \in A$ such that i and j are connected by an edge in the Cayley graph $\mathcal{G}(\Lambda, \Delta)$. Thus, we may describe (1.3) by saying that it is possible to find nonempty sets A whose surface is small compared to their volume. For example, $\mathbb{Z}^d$ is amenable, but regular trees are not.

If Λ is a finitely generated group and Δ is a finite symmetric generating set, then we let $|i|$ denote the usual graph distance of i to the origin in the Cayley graph $\mathcal{G}(\Lambda, \Delta)$, i.e., $|i| := \min\{n : i \in \Delta^n\}$. The norm $|\cdot|$ depends on the choice of Δ, but any two norms associated with different finite symmetric generating sets are equivalent. It follows from subadditivity that the limit $\lim_{n\to\infty} \frac{1}{n} |\{i \in \Lambda : |i| \leq n\}|$ exists; one says that the group Λ has exponential (resp. subexponential) growth if this limit is positive (resp. zero). Note that since norms associated with different finite symmetric generating sets are equivalent, having (sub)exponential growth is a property of the group Λ only and does not depend on the choice of Δ.

Subexponential growth implies amenabilty, but the converse is not true: as already mentioned, the lamplighter group is an amenable group with exponential growth. See [MW89, Section 5] for general facts about amenability and subexponential growth, and [LPP96] for a nice exposition of the lamplighter group.

We also need a few definitions concerning contact processes. If $a = (a(i,j))_{i,j\in\Lambda}$ are infection rates satisfying (1.1), then we define *reversed infection rates* $a^\dagger$ by $a^\dagger(i,j) := a(j,i)$ $(i, j \in \Lambda)$. We say that the (Λ, a, δ)-contact process *survives* if

$$\mathbb{P}\big[\eta^A_t \neq \emptyset\ \forall t \geq 0\big] > 0 \tag{1.4}$$

for some, and hence for all $\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)$. Using the standard coupling, it is easy to see that if $\delta < \delta'$ and the (Λ, a, δ')-contact process survives, then the (Λ, a, δ)-contact process survives. We let

$$\delta_{\rm c} = \delta_{\rm c}(\Lambda, a) := \sup\big\{\delta \geq 0 :\ \text{the } (\Lambda, a, \delta)\text{-contact process survives}\big\} \tag{1.5}$$

denote the *critical recovery rate*. By comparison with a critical branching process, it is not hard to see that $\delta_{\rm c} \leq |a|$. If Λ is finitely generated and the infection rates a are irreducible, then one may use comparison with a one-dimensional nearest-neighbor contact process to show that $0 < \delta_{\rm c}$ (see [Swa07, Lemma 4.18]).

Here, we say that infection rates a on Λ are *irreducible* if

$$\bigcup_{n\geq 0} (A \cup A^{-1})^n = \Lambda \quad \text{where } A := \{i \in \Lambda : a(0,i) > 0\}. \tag{1.6}$$

At some point, we will need an assumption that is a bit stronger than this. More precisely, we will occasionally use the following assumption:

$$\bigcup_{n\geq 0,\ m\geq 0} A^n A^{-m} = \bigcup_{n\geq 0,\ m\geq 0} A^{-n} A^m = \Lambda, \quad \text{where } A := \{i \in \Lambda : a(0,i) > 0\}. \tag{1.7}$$

Note that this says that for any two sites i, j there exists a site k from which both i and j can be infected, and a site k' that can be infected both from i and from j.

With these definitions, we are ready to formulate our main result.

Theorem 1.2 (Properties of the exponential growth rate) *Let Λ be a finite or countably infinite group, let $a = (a(i,j))_{i,j\in\Lambda}$ be infection rates satisfying (1.1), and $\delta \geq 0$. Let $r = r(\Lambda, a, \delta)$ be the exponential growth rate of the (Λ, a, δ)-contact process, defined in (1.2). Then:*

(a) $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$

(b) *The function $\delta \to r(\Lambda, a, \delta)$ is nonincreasing and Lipschitz continuous on $[0, \infty)$, with Lipschitz constant 1.*

(c) *If the (Λ, a, δ)-contact process survives, then $r \geq 0$.*

(d) *If $r > 0$, then the (Λ, a, δ)-contact process survives.*

(e) *If Λ is finitely generated and has subexponential growth, and the infection rates satisfy $\sum_i a(0,i)e^{\varepsilon|i|} < \infty$ for some $\varepsilon > 0$, then $r \leq 0$.*

(f) *If Λ is nonamenable, the (Λ, a, δ)-contact process survives, and the infection rates satisfy the irreducibility condition (1.7), then $r > 0$.*

Parts (a), (b), and (c) of this theorem are easy. Part (d) follows from a variance calculation, while (e) is proved by some simple large deviation estimates. The proof of part (f) is rather involved. Basically, the idea is as follows. If the exponential growth rate of a contact process is zero, then this means that for the process started with one infected site, a 'typical' infected site at a 'typical' late time produces no net offspring, i.e., the mean number of sites it infects per unit of time is just enough to balance the probability that the site itself recovers. We prove that this implies that the local configuration as seen from this 'typical' site is distributed as the upper invariant measure, assuming that the latter is nontrivial. If Λ is nonamenable, this leads to a contradiction, since for any finite collection of particles on a nonamenable lattice, a positive fraction of the particles must lie on the 'outer boundary' of the collection, hence must see something different from the upper invariant measure.

Parts (b), (d), and (f) of Theorem 1.2 yield the following corollary.

Corollary 1.3 (The critical contact process on a nonamenable lattice dies out) *If Λ is nonamenable and the infection rates satisfy the irreducibility condition (1.7), then $\delta_{\rm c} = \delta_{\rm c}(\Lambda, a) > 0$ and the $(\Lambda, a, \delta_{\rm c})$-contact process dies out.*

For nearest-neighbor contact processes on regular trees, this result is known, see [Lig99, Proposition I.4.39].

1.4 The process seen from a typical site

There is an intimate relation between the survival probability of a contact process and its upper invariant law. Similarly, there is a relation between the exponential growth rate and certain infinite measures on the space of nonempty subsets of Λ, which we explain now.

Let $\mathcal{P}_+(\Lambda) := \{A \subset \Lambda : A \neq \emptyset\}$ denote the set of all nonempty subsets of Λ. Note that since $\mathcal{P}_+(\Lambda)$ is a locally compact space in the induced topology from $\mathcal{P}(\Lambda)$. We say that a measure μ on $\mathcal{P}(\Lambda)$ or $\mathcal{P}_+(\Lambda)$ is (spatially) *homogeneous* if $\mu \circ T_i^{-1} = \mu$ for all $i \in \Lambda$, where $T_i : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$ is defined as $T_i(A) := iA$. We say that a measure μ on $\mathcal{P}_+(\Lambda)$ is an *eigenmeasure* of the (Λ, a, δ)-contact process if μ is nonzero and there exists a constant $\lambda \in \mathbb{R}$ such that

$$\int \mu(\mathrm{d}A)\mathbb{P}[\eta_t^A \in \cdot\,] = e^{\lambda t}\mu \qquad (t \geq 0). \tag{1.8}$$

We call λ the associated *eigenvalue*. As a motivation for this terminology, we observe that if G is the generator of the (Λ, a, δ)-contact process, then formally $G^*\mu = \lambda\mu$. Note that if $\lambda = 0$, then μ is an invariant measure (though not necessarily a probability measure) for the (Λ, a, δ)-contact process.

Proposition 1.4 (Exponential growth rate and eigenmeasures) *For each (Λ, a, δ)-contact process, the set*

$$\mathcal{E}(\Lambda, a, \delta) := \big\{\lambda \in \mathbb{R} : \textit{there exists a homogeneous, locally finite eigenmeasure} \\ \textit{of the } (\Lambda, a, \delta)\textit{-contact process with eigenvalue } \lambda\big\} \tag{1.9}$$

is a nonempty compact subset of $\mathbb{R}$, and $r(\Lambda, a, \delta) = \max \mathcal{E}(\Lambda, a, \delta)$.

In particular, Proposition 1.4 implies that each (Λ, a, δ)-contact process has a homogeneous, locally finite eigenmeasure with eigenvalue $r(\Lambda, a, \delta)$. Although I cannot prove this, I conjecture that this eigenmeasure is always unique and the long-time limit of the (suitably rescaled) law of the process started with one infected site, distributed according to counting measure on Λ. If we condition such an eigenmeasure on the origin being infected, then we can view the resulting probability measure as describing the contact process as seen from a typical infected site, at late times. (See Corollary 3.4 and Lemma 4.2 below.)

Recall that the upper invariant measure $\overline{\nu}$ of a contact process is the long-time limit law of the process started with all sites infected. It follows from duality that the upper invariant measure of the (Λ, a, δ)-contact process is nontrivial (i.e., gives zero probability to the empty set) if and only if the $(\Lambda, a^\dagger, \delta)$-contact process survives. The next result is an important ingredient in the proof of Theorem 1.2 (f).

Theorem 1.5 (Eigenmeasures with eigenvalue zero) *Assume that the infection rates satisfy the irreducibility condition (1.7). If the upper invariant measure $\overline{\nu}$ of the (Λ, a, δ)-contact process is nontrivial, then any homogeneous, locally finite eigenmeasure μ with eigenvalue zero satisfies $\mu = c\overline{\nu}$ for some $c > 0$.*

We prove Theorem 1.5 by extending well-known techniques for showing that $\overline{\nu}$ is the only nontrivial homogeneous invariant probability measure of a contact process.

1.5 Discussion, open problems, and outline

The work in this paper started with the question whether it is possible to prove something like 'uniqueness of the infinite cluster' in the context of oriented percolation or the (very similar) graphical representation of the contact process. This question is still very much open. See Grimmett and Hiemer [GH02] for a weak statement that is proved only on $\mathbb{Z}^d$ and Wu and Zhang [WZ06, Thm 1.4] or [Swa07, Lemma 4.5] for a stronger statement that is proved only in the nearest-neighbor, one-dimensional case.

Whether the methods in the present paper can shed some light on this question I do not know. I have tried to prove the weak statement of Grimmett and Hiemer assuming (only) subexponential growth, but ran into the problem that I would need to replace a size-biased law by a law conditioned on survival, which I do not know how to do (see [Swa07, Prop 4.4]).

In fact, although this is not obvious from the presentation above, size-biased laws and Campbell measures, well-known objects from branching theory, are closely related to the eigenmeasures introduced above. (For this connection, see Section 4.3 below.) An interesting feature of the (potentially infinite) eigenmeasures is that they allow one to use some of the simplifications that come from spatial homogeneity while studying processes started in finite initial states.

There are lots of open problems concerning contact processes on general transitive lattices, so we mention just a few.

1. Prove that the (Λ, a, δ)-contact process has a unique homogeneous, locally finite eigenmeasure with eigenvalue $r(\Lambda, a, \delta)$, which is the long-time limit law of the process started with one infected site distributed according to counting measure on Λ.

2. Prove that $\frac{\partial}{\partial\delta} r(\Lambda, a, \delta) < 0$ on $\{\delta : r(\Lambda, a, \delta) \neq 0\}$. Prove the same statement for all δ if Λ is nonamenable. Adapt the proof of [Lig99, Thm 2.48], which says that $r(\Lambda, a, \delta) < 0$ for all $\delta > \delta_c$, to general lattices.

3. Prove that $\delta_c > 0$ for some (Λ, a, δ)-contact process on a group Λ that is not finitely generated, e.g. the hierarchical group.

4. Study contact processes on transitive lattices Λ that are not groups. In this context, if Λ is not unimodular, it is possible that a (Λ, a, δ)-contact process survives but its dual $(\Lambda, a^\dagger, \delta)$-contact process dies out. It is an open problem to prove this cannot happen in the unimodular case.

5. Prove some version of uniqueness of the infinite cluster assuming that the exponential growth rate is zero.

The outline of the rest of the paper is as follows. In Section 2, we introduce some basic tools, such as the graphical representation and a martingale problem. In Section 3, we prove Theorem 1.2 (a)–(c), Proposition 1.4, and Theorem 1.5. These results are then used in Section 4 to prove Theorem 1.2 (d)–(f) and Corollary 1.3. Appendix A contains a spatial ergodic theorem that is used in the proof of Theorem 1.2 (f).

Acknowledgements The author thanks Geoffrey Grimmett, Olle Häggström, Russel Lyons, and Roberto Schonmann for useful email conversations about the contact process, oriented percolation, and amenability.

2 Construction and basic properties

2.1 Graphical representation

We will, of course, use the graphical representation of the contact process. Let $\Lambda\times\mathbb{R} := \{(i,t) : i\in\Lambda,\ t\in\mathbb{R}\}$ and $\Lambda\times\Lambda\times\mathbb{R} := \{(i,j,t) : i,j\in\Lambda,\ t\in\mathbb{R}\}$, where t is the time coordinate. Let $\omega = (\omega^{\rm r},\omega^{\rm i})$ be a pair of independent, locally finite random subsets of $\Lambda\times\mathbb{R}$ and $\Lambda\times\Lambda\times\mathbb{R}$, respectively, produced by Poisson point processes with intensity δ and $a(i,j)$, respectively. We visualize this by plotting Λ horizontally and $\mathbb{R}$ vertically, marking points $(i,s)\in\omega^{\rm r}$ with a recovery symbol $*$, and drawing an infection arrow from (i,t) to (j,t) for each $(i,j,t)\in\omega^{\rm i}$. For $C,D\subset\Lambda\times\mathbb{R}$, say that there is a *path* from C to D, denoted by $C\rightsquigarrow D$, if there exist $n\geq 0$, $i_0,\ldots,i_n\in\Lambda$, and $t_0\leq\cdots\leq t_{n+1}$ with $(i_0,t_0)\in C$ and $(i_n,t_{n+1})\in D$, such that $(\{i_k\}\times[t_k,t_{k+1}])\cap\omega^{\rm r}=\emptyset$ for all $k=0,\ldots,n$ and $(i_{k-1},i_k,t_k)\in\omega^{\rm i}$ for all $k=1,\ldots,n$. Thus, a path must walk upwards in time, may follow arrows, and must avoid recoveries. For $C\subset\Lambda\times\mathbb{R}$, we write $C\rightsquigarrow\infty$ if there is an infinite path with times $t_k\uparrow\infty$ starting in C. We define $-\infty\rightsquigarrow C$ analogously. Instead of $\{(i,s)\}\rightsquigarrow$ and $\rightsquigarrow\{(j,t)\}$, simply write $(i,s)\rightsquigarrow$ and $\rightsquigarrow(j,t)$.

For given $A\in\mathcal{P}(\Lambda)$ and $t_0\in\mathbb{R}$, put

$$\eta^{A\times\{t_0\}}_t := \{i\in\Lambda : A\times\{t_0\}\rightsquigarrow(i,t_0+t)\}\qquad(t\geq 0).\tag{2.1}$$

Then $\eta^{A\times\{t_0\}} = (\eta^{A\times\{t_0\}}_t)_{t\geq 0}$ is a (Λ,a,δ)-contact process started in $\eta^{A\times\{t_0\}}_0 = A$.

In anology with (2.1), put

$$\eta^{\dagger\,A\times\{t_0\}}_t := \{i\in\Lambda : (i,t_0-t)\rightsquigarrow A\times\{t_0\}\}\qquad(t\geq 0).\tag{2.2}$$

Then $\eta^{\dagger\,A\times\{t_0\}} = (\eta^{\dagger\,A\times\{t_0\}}_t)_{t\geq 0}$ is a $(\Lambda,a^\dagger,\delta)$-contact process started in $\eta^{\dagger\,A\times\{t_0\}}_0 = A$. Since for any $s\leq t$ and $A,B\in\mathcal{P}(\Lambda)$, the event

$$\big\{\eta^{A\times\{s\}}_{u-s}\cap\eta^{\dagger\,B\times\{t\}}_{t-u}=\emptyset\big\} = \big\{A\times\{s\}\not\rightsquigarrow B\times\{t\}\big\}\tag{2.3}$$

does not depend on $u\in[s,t]$, it follows that the (Λ,a,δ)-contact process and the $(\Lambda,a^\dagger,\delta)$-contact process are dual in the sense that

$$\mathbb{P}[\eta^A_t\cap B=\emptyset] = \mathbb{P}[A\cap\eta^{\dagger\,B}_t=\emptyset]\qquad(A,B\in\mathcal{P}(\Lambda),\ t\geq 0).\tag{2.4}$$

Here, for brevity, we write $\eta^A_t := \eta^{A\times\{0\}}_t$ and $\eta^{\dagger\,B}_t := \eta^{\dagger\,B\times\{0\}}_t$.

It is not hard to see that $|a| := \sum_i a(0,i) = \sum_i a(i,0)$ and

$$\mathbb{E}\big[|\eta^A_t|\big]\leq|A|e^{|a|t}\quad\text{and}\quad\mathbb{E}\big[|\eta^{\dagger\,A}_t|\big]\leq|A|e^{|a|t}\qquad(t\geq 0,\ A\subset\Lambda).\tag{2.5}$$

In particular, both the (Λ,a,δ)-contact process and the $(\Lambda,a^\dagger,\delta)$-contact process are well-defined and the processes started from a finite initial state are a.s. finite for all time.

For any $A\subset\Lambda$, we let

$$\rho(A) := \mathbb{P}[\eta^A_t\neq\emptyset\ \forall t\geq 0] = \mathbb{P}[A\times\{0\}\to\infty]\tag{2.6}$$

denote the survival probability of the (Λ,a,δ)-contact process started in A. Similarly, $\rho^\dagger$ denotes the survival probability of the $(\Lambda,a^\dagger,\delta)$-contact process. Setting

$$\overline{\eta}_t := \{i\in\Lambda : -\infty\rightsquigarrow(i,t)\}\qquad(t\in\mathbb{R}).\tag{2.7}$$

defines a stationary (Λ, a, δ)-contact process whose invariant law

$$\overline{\nu} := \mathbb{P}[\overline{\eta}_t \in \cdot\,] \qquad (t \in \mathbb{R}) \tag{2.8}$$

is uniquely characterized by

$$\mathbb{P}\big[\overline{\eta}_0 \cap A \neq \emptyset\big] = \rho^\dagger(A) \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda)). \tag{2.9}$$

(To see this, note that the linear span of the functions $B \mapsto 1_{\{A\cap B=\emptyset\}}$ with $A \in \mathcal{P}_{\rm fin}(\Lambda)$ forms an algebra that separates points, hence by the Stone-Weierstrass theorem is dense in the space of continuous functions on $\mathcal{P}(\Lambda)$.) It is easy to see that $\overline{\nu}$ is *nontrivial*, i.e., gives zero probability to the empty set, if and only if the $(\Lambda, a^\dagger, \delta)$-contact process survives.

2.2 Martingale problem

We will need the fact that (Λ, a, δ)-contact processes started in finite initial states solve a martingale problem. Let

$$\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda)) := \{f : \mathcal{P}_{\rm fin}(\Lambda) \to \mathbb{R} : |f(A)| \leq K|A|^k + M \text{ for some } K, M, k \geq 0\}. \tag{2.10}$$

denote the class of real functions on $\mathcal{P}_{\rm fin}(\Lambda)$ of polynomial growth. Given Λ, a, and δ, define a linear operator G with domain $\mathcal{D}(G) := \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ by

$$\begin{aligned} Gf(A) :=& \sum_{ij} a(i,j) 1_{\{i\in A\}} 1_{\{j\notin A\}} \{f(A\cup\{j\}) - f(A)\} \\ &+\delta \sum_i 1_{\{i\in A\}} \{f(A\backslash\{i\}) - f(A)\}. \end{aligned} \tag{2.11}$$

Proposition 2.1 (Martingale problem and moment estimate) *The operator G maps the space $\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ into itself. For each $f \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ and $A \in \mathcal{P}_{\rm fin}(\Lambda)$, the process*

$$M_t := f(\eta^A_t) - \int_0^t Gf(\eta^A_s)\mathrm{d}s \qquad (t \geq 0) \tag{2.12}$$

is a martingale with respect to the filtration generated by η^A. Moreover, setting $z^{\langle k\rangle} := \prod_{i=0}^{k-1}(z+i)$, one has

$$\mathbb{E}\big[|\eta^A_t|^{\langle k\rangle}\big] \leq |A|^{\langle k\rangle} e^{k(|a|-\delta)t} \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda),\ k \geq 1,\ t \geq 0). \tag{2.13}$$

Proof Our proof follows the same lines as the proof of [AS05, Proposition 8]. It is not hard to see that G maps $\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ into itself. Set $f_k(A) := |A|^{\langle k\rangle}$. Then

$$\begin{aligned} Gf_k(A) =& \sum_{ij} a(i,j) 1_{\{i\in A\}} 1_{\{j\notin A\}} \{(|A|+1)^{\langle k\rangle} - |A|^{\langle k\rangle}\} + \delta \sum_i 1_{\{i\in A\}} \{(|A|-1)^{\langle k\rangle} - |A|^{\langle k\rangle}\}, \\ \leq& (|a|-\delta)|A|\{(|A|+1)^{\langle k\rangle} - |A|^{\langle k\rangle}\} = k(|a|-\delta)|A|^{\langle k\rangle}. \end{aligned} \tag{2.14}$$

Define stopping times $\tau_N := \inf\{t \geq 0 : |\eta^A_t| \geq N\}$. The stopped process $(\eta^A_{t\wedge\tau_N})_{t\geq 0}$ has bounded jump rates, and therefore standard theory tells us that for each $N \geq 1$ and $f \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, the process

$$M^N_t := f(\eta^A_{t\wedge\tau_N}) - \int_0^{t\wedge\tau_N} Gf(\eta^A_s)\mathrm{d}s \qquad (t \geq 0) \tag{2.15}$$

is a martingale. Moreover, it easily follows from (2.14) that

$$\mathbb{E}\big[|\eta^A_{t\wedge\tau_N}|^{\langle k\rangle}\big] \leq |A|^{\langle k\rangle} e^{k(|a|-\delta)t} \qquad (k \geq 1,\ t \geq 0). \tag{2.16}$$

Using the fact that G maps $\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ into itself and (2.16) for some sufficiently high k (depending on f), one can show that for fixed $t \geq 0$, the random variables $(M^N_t)_{N\geq 1}$ are uniformly integrable. Therefore, letting $N \to \infty$ in (2.15), one finds that the process in (2.12) is a martingale. Letting $N \to \infty$ in (2.16) yields (2.13). ∎

2.3 Covariance formula

By Proposition 2.1, setting

$$S_t f(A) := \mathbb{E}[f(\eta^A_t)] \qquad (f \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda)),\ A \in \mathcal{P}_{\rm fin}(\Lambda)) \tag{2.17}$$

defines a semigroup $(S_t)_{t\geq 0}$ of linear operators $S_t : \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda)) \to \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$. Let $\mathcal{M}$ be the class of probability measures on $\mathcal{P}_{\rm fin}(\Lambda)$ such that $\int |A|^k \mu(\mathrm{d}A) < \infty$ for all $k \geq 1$. For $\mu \in \mathcal{M}$ and $f \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, we write $\mu f := \int f(A)\mu(\mathrm{d}A)$. Note that if $(\eta_t)_{t\geq 0}$ is a (Λ, a, δ)-contact process started in an initial law $\mathbb{P}[\eta_0 \in \cdot\,] =: \mu \in \mathcal{M}$, then $\mathbb{P}[\eta_t \in \cdot\,] \in \mathcal{M}$ for all $t \geq 0$ and $\int \mathbb{P}[\eta_t \in \mathrm{d}A] f(A) = \mu S_t f$. For this reason, we use the notation $\mu S_t := \mathbb{P}[\eta_t \in \cdot\,]$ $(t \geq 0)$ denote the law of η_t. For any $\mu \in \mathcal{M}$ and $f, g \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, we let

$$\mathrm{Cov}_\mu(f, g) := \mu(fg) - (\mu f)(\mu g) \tag{2.18}$$

denote the covariance of f and g under μ, which is always finite.

Proposition 2.2 (Covariance formula) *For $f, g \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, let*

$$\Gamma(f, g) := \tfrac{1}{2}\big[G(fg) - (Gf)g - f(Gg)\big] \qquad (f, g \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))). \tag{2.19}$$

Then, for any $\mu \in \mathcal{M}$ and $f, g \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, one has

$$\mathrm{Cov}_{\mu S_t}(f, g) = \mathrm{Cov}_\mu(S_t f, S_t g) + 2\int_0^t \mu S_{t-s} \Gamma(S_s f, S_s g)\,\mathrm{d}s \qquad (t \geq 0). \tag{2.20}$$

Proof Set

$$H(s, t, u) := S_s\big((S_t f)(S_u g)\big). \tag{2.21}$$

We claim that

$$\begin{aligned} \tfrac{\partial}{\partial s} H(s, t, u) &= S_s G\big((S_t f)(S_u g)\big), \\ \tfrac{\partial}{\partial t} H(s, t, u) &= S_s\big((G S_t f)(S_u g)\big), \\ \tfrac{\partial}{\partial u} H(s, t, u) &= S_s\big((S_t f)(G S_u g)\big). \end{aligned} \tag{2.22}$$

It follows that

$$\tfrac{\partial}{\partial t} H(t, T-t, T-t) = 2 S_t \Gamma(S_{T-t} f, S_{T-t} g), \tag{2.23}$$

and therefore

$$\begin{aligned} &\mathrm{Cov}_{\mu S_T}(f, g) - \mathrm{Cov}_\mu(S_T f, S_T g) \\ &\quad = \big(\mu S_T(fg) - (\mu S_T f)(\mu S_T g)\big) - \big(\mu((S_T f)(S_T g)) - (\mu S_T f)(\mu S_T g)\big) \\ &\quad = \mu\big(S_T(fg) - (S_T f)(S_T g)\big) \\ &\quad = \mu\big(H(T, 0, 0) - H(0, T, T)\big) = 2\int_0^t \mu S_t \Gamma(S_{T-t} f, S_{T-t} g)\,\mathrm{d}t. \end{aligned} \tag{2.24}$$

These calculations are standard. However, in order to verify (2.22), we must use some special properties of our model. Let us say that a sequence of functions $f_n \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ converges 'nicely' to a limit f, if $f_n \to f$ pointwise and there exists $K, M, k \geq 0$ such that $|f_n(A)| \leq K|A|^k + M$ for all n. By (2.13) and dominated convergence, if $f_n \to f$ 'nicely', then $S_t f_n \to S_t f$ 'nicely', for each $t \geq 0$. Note also that if $f_n, f, g \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ and $f_n \to f$ 'nicely', then $f_n g \to fg$ 'nicely'. We claim that for each $f \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$,

$$\lim_{t\to 0} t^{-1}(S_t f - f) = Gf, \tag{2.25}$$

where the convergence happens 'nicely'. Indeed, by Proposition 2.1,

$$t^{-1}\big(S_t f(A) - f(A)\big) = t^{-1}\int_0^t \mathbb{E}\big[(Gf)(\eta_s^A)\big]\mathrm{d}s \underset{t\to 0}{\longrightarrow} Gf(A), \tag{2.26}$$

where the 'niceness' of the convergence follows from (2.13) and the fact that $Gf \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$. It follows that for each $f \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ and $t \geq 0$,

$$\begin{aligned} \tfrac{\partial}{\partial t} S_t f &= \lim_{\varepsilon\to 0}(S_\varepsilon - 1)S_t f = G S_t f \\ &= \lim_{\varepsilon\to 0} S_t(S_\varepsilon - 1) f = S_t G f, \end{aligned} \tag{2.27}$$

where 1 denotes the identity operator. Using (2.27) and the properties of 'nice' convergence, (2.22) follows readily. ∎

3 The exponential growth rate

3.1 Basic facts

Proof of Lemma 1.1 Let us write

$$\pi_t(A) := \mathbb{E}\big[|\eta_t^A|\big] \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda),\ t \geq 0). \tag{3.1}$$

We start by showing that

$$\pi_{s+t}(\{0\}) \leq \pi_s(\{0\})\pi_t(\{0\}) \qquad (s, t \geq 0). \tag{3.2}$$

If η^A and η^B are defined usng the same graphical representation, then $\eta_t^A \cup \eta_t^B = \eta_t^{A\cup B}$. Therefore,

$$\mathbb{E}\big[|\eta_t^A|\big] = \mathbb{E}\Big[\big|\bigcup_{i\in A}\eta_t^{\{i\}}\big|\Big] \leq \sum_{i\in A}\mathbb{E}\big[|\eta_t^{\{i\}}|\big] = |A|\mathbb{E}\big[|\eta_t^{\{0\}}|\big], \tag{3.3}$$

where in the last step we have used shift invariance. As a consequence,

$$\pi_{s+t}(\{0\}) = \int \mathbb{P}[\eta_s^{\{0\}} \in \mathrm{d}A]\mathbb{E}\big[|\eta_t^A|\big] \leq \int \mathbb{P}[\eta_s^{\{0\}} \in \mathrm{d}A]|A|\mathbb{E}\big[|\eta_t^{\{0\}}|\big] = \pi_s(\{0\})\pi_t(\{0\}). \tag{3.4}$$

This proves (3.2). It follows that $t \mapsto \log \pi_t(\{0\})$ is subadditive and therefore, by [Lig99, Theorem B.22], the limit

$$\lim_{t\to\infty} \tfrac{1}{t}\log \pi_t(\{0\}) = \inf_{t>0} \tfrac{1}{t}\log \pi_t(\{0\}) =: r \in [-\infty, \infty) \tag{3.5}$$

exists. By monotonicity and (3.3),

$$\pi_t(\{0\}) \leq \pi_t(A) \leq |A|\pi_t(\{0\}) \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda)). \tag{3.6}$$

Taking logarithms, dividing by t, and letting $t \to \infty$ we arrive at (1.2). Since η can be bounded from below by a simple death process and from above by a branching process, one has

$$e^{-\delta t} \leq \mathbb{E}\big[|\eta_t^{\{0\}}|\big] \leq e^{(|a|-\delta)t} \qquad (t \geq 0), \tag{3.7}$$

which implies that $-\delta \leq r \leq |a| - \delta$. ∎

Proof of Theorem 1.2 (a) By duality (formula (2.4)) and shift invariance,

$$\begin{aligned}\mathbb{E}\big[|\eta_t^{\{0\}}|\big] &= \sum_i \mathbb{P}\big[\eta_t^{\{0\}} \cap \{i\} \neq \emptyset\big] = \sum_i \mathbb{P}\big[\{0\} \cap \eta_t^{\dagger\,\{i\}} \neq \emptyset\big] \\ &= \sum_i \mathbb{P}\big[\{i^{-1}\} \cap \eta_t^{\dagger\,\{0\}} \neq \emptyset\big] = \mathbb{E}\big[|\eta_t^{\dagger\,\{0\}}|\big],\end{aligned} \tag{3.8}$$

which implies that $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$. ∎

Proof of Theorem 1.2 (b) Fix a countable group Λ and infection rates a satisfying (1.1), and for each $\delta \geq 0$, write $\pi(\delta, t) := \mathbb{E}\big[|\eta_t^{\{0\}}|\big]$, where $\eta_t^{\{0\}}$ is the (Λ, a, δ)-contact process. For $0 \leq \delta < \tilde\delta$, consider the graphical representations (see Section 2.1) of the (Λ, a, δ)- and $(\Lambda, a, \tilde\delta)$-contact processes, defined by Poisson processes $(\omega^{\rm r}, \omega^{\rm i})$ and $(\tilde\omega^{\rm r}, \tilde\omega^{\rm i})$, respectively. We may couple these graphical representations such that $\omega^{\rm i} = \tilde\omega^{\rm i}$ and $\omega^{\rm r} \subset \tilde\omega^{\rm r}$, where $\tilde\omega^{\rm r} \backslash \omega^{\rm r}$ is an independent Poisson point process with intensity $\tilde\delta - \delta$. Write $\rightsquigarrow$ and $\tilde\rightsquigarrow$ to indicate the existence of an open path in the graphical representations for δ and $\tilde\delta$, respectively. Then, if $\tilde\delta - \delta$ is small, then for each $t \geq 0$,

$$\begin{aligned}\pi(\tilde\delta, t) &= \sum_i \mathbb{P}[(0,0) \tilde\rightsquigarrow (i,t)] \\ &= \sum_i \mathbb{P}[(0,0) \rightsquigarrow (i,t)] \\ &\quad - (\tilde\delta - \delta) \int_0^t \sum_{ij} \mathbb{P}\big[(0,0) \rightsquigarrow (j,s) \rightsquigarrow (i,t) \text{ and there exists} \\ &\qquad\qquad \text{no } k \neq j \text{ such that } (0,0) \rightsquigarrow (k,s) \rightsquigarrow (i,t)\big] \mathrm{d}s + O((\tilde\delta - \delta)^2),\end{aligned} \tag{3.9}$$

where the terms order $(\tilde\delta - \delta)^2$ come from events where two or more recovery symbols in $\tilde\omega^{\rm r} \backslash \omega^{\rm r}$ are needed to block all paths from $(0,0)$ to (i,t). Dividing by $\tilde\delta - \delta$ and letting $\tilde\delta \to \delta$ yields

$$\begin{aligned}&\tfrac{\partial}{\partial\delta}\pi(\tilde\delta, t) \\ &\quad = -\int_0^t \sum_i \mathbb{P}\big[\exists j \text{ s.t. } (0,0) \rightsquigarrow (j,s) \rightsquigarrow (i,t) \text{ and } \nexists k \neq j \text{ s.t. } (0,0) \rightsquigarrow (k,s) \rightsquigarrow (i,t)\big] \mathrm{d}s,\end{aligned} \tag{3.10}$$

which is an analogon of what is known as Russo's formula in percolation. Since

$$\begin{aligned}&\sum_i \mathbb{P}\big[\exists j \text{ s.t. } (0,0) \rightsquigarrow (j,s) \rightsquigarrow (i,t) \text{ and } \nexists k \neq j \text{ s.t. } (0,0) \rightsquigarrow (k,s) \rightsquigarrow (i,t)\big] \\ &\quad \leq \sum_i \mathbb{P}[(0,0) \rightsquigarrow (i,t)] = \pi(\delta, t),\end{aligned} \tag{3.11}$$

it follows that $0 \leq -\frac{\partial}{\partial\delta}\pi(\delta,t) \leq t\pi(\delta,t)$ $(t \geq 0)$, and therefore

$$0 \leq -\tfrac{\partial}{\partial\delta}\tfrac{1}{t}\log\pi(\delta,t) \leq 1. \tag{3.12}$$

Taking the limit $t \to \infty$, using (3.5), the claims follow. ∎

Proof of Theorem 1.2 (c) If the (Λ, a, δ)-contact process survives, then

$$\mathbb{E}\big[|\eta^{\{0\}}_t|\big] \geq \mathbb{P}[\eta^{\{0\}}_t \neq 0] \underset{t\to\infty}{\longrightarrow} \mathbb{P}[\eta^{\{0\}}_s \neq 0\ \forall s \geq 0] > 0, \tag{3.13}$$

which implies that $r \geq 0$. ∎

3.2 Eigenmeasures

Recall that a measure μ on a locally compact space is called locally finite if $\mu(K) < \infty$ for all compact sets K. We need a few basic facts about locally finite measures on $\mathcal{P}_+(\Lambda)$.

Lemma 3.1 (Locally finite measures) *Let μ be a measure on $\mathcal{P}_+(\Lambda)$. Then the following statements are equivalent: 1. μ is locally finite. 2. $\int\mu(\mathrm{d}A)1_{\{i\in A\}} < \infty$ for all $i \in \Lambda$. 3. $\int\mu(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} < \infty$ for all $B \in \mathcal{P}_{\rm fin}(\Lambda)$.*

Proof We will prove that 1⇒3⇒2⇒1. For each $B \in \mathcal{P}_{\rm fin}(\Lambda)$, the set $\mathcal{Q}(B) := \{A \subset \Lambda : A\cap B \neq \emptyset\}$ is a compact subset of $\mathcal{P}_+(\Lambda)$, and $A \mapsto 1_{\{A\cap B\neq\emptyset\}}$ is a continuous function with compact support $\mathcal{Q}(B)$. It follows that any locally finite measure μ satisfies $\int\mu(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} < \infty$ for each $B \in \mathcal{P}_{\rm fin}(\Lambda)$. In particular, setting $B = \{i\}$ this implies that $\int\mu(\mathrm{d}A)1_{\{i\in A\}} < \infty$ for each $i \in \Lambda$. This proves the implications 1⇒3⇒2. To see that 2 implies 1, let $\Delta_n \subset \Lambda$ be finite sets increasing to Λ. Then $\mathcal{Q}(\Delta_n)$ are compact sets increasing to $\mathcal{P}_+(\Lambda)$ and $\mu(\mathcal{Q}(\Delta_n)) = \int\mu(\mathrm{d}A)1_{\{A\cap\Delta_n\neq\emptyset\}} \leq \sum_{i\in\Delta_n}\int\mu(\mathrm{d}A)1_{\{i\in A\}} < \infty$ for each n, hence μ is locally finite. ∎

We equip the space of locally finite measures on $\mathcal{P}_+(\Lambda)$ with the vague topology, i.e., we say that a sequence of locally finite measures μ_n on $\mathcal{P}_+(\Lambda)$ converges vaguely to a limit μ, denoted as $\mu_n \Rightarrow \mu$, if $\int\mu_n(\mathrm{d}A)f(A) \to \int\mu(\mathrm{d}A)f(A)$ for each continuous compactly supported real function f on $\mathcal{P}_+(\Lambda)$.

Lemma 3.2 (Vague convergence) *Let μ_n, μ be locally finite measures on $\mathcal{P}_+(\Lambda)$. Then the μ_n converge vaguely to μ if and only if $\int\mu_n(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} \to \int\mu(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}$ for each $B \in \mathcal{P}_{\rm fin}(\Lambda)$. The sequence μ_n is relatively compact in the topology of vague convergence if and only if $\sup_n\int\mu_n(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} < \infty$ for each $B \in \mathcal{P}_{\rm fin}(\Lambda)$.*

Proof Since for each $B \in \mathcal{P}_{\rm fin}(\Lambda)$, the function $A \mapsto 1_{\{A\cap B\neq\emptyset\}}$ is continuous with compact support, the conditions for convergence and relative compactness given above are clearly necessary. To see that they are also sufficient, let $\Delta_m \subset \Lambda$ be finite sets increasing to Λ and set $f_m(A) := 1_{\{A\cap\Delta_n\neq\emptyset\}}$. Then the f_m are continuous, nonnegative functions with compact supports increasing to $\mathcal{P}_+(\Lambda)$. It follows that μ_n converges vaguely to μ if and only if for each m the weighted measures $f_m(A)\mu_n(\mathrm{d}A)$ converge weakly to $f_m(A)\mu(\mathrm{d}A)$. Now if $\sup_n\int\mu_n(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} < \infty$ for each $B \in \mathcal{P}_{\rm fin}(\Lambda)$, then by a diagonal argument each subsequence of the μ_n contains a further subsequence such that $f_m(A)\mu_n(\mathrm{d}A)$ converges weakly for each m, hence the μ_n converge vaguely. The linear span of the functions $B \mapsto 1_{\{A\cap B=\emptyset\}}$ with $A \in \mathcal{P}_{\rm fin}(\Lambda)$ forms an algebra that separates points, hence by the Stone-Weierstrass theorem is dense in the space of continuous functions on $\mathcal{P}(\Lambda)$. It follows that μ_n converges vaguely to μ if and only if $\int f_m(A)\mu_n(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}$ converges to $\int f_m(A)\mu(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}$ for each m and for each $B\in\mathcal{P}_{\rm fin}(\Lambda)$. Since $f_m(A)1_{\{A\cap B\neq\emptyset\}}=1_{\{A\cap\Delta_n\neq\emptyset\}}+1_{\{A\cap B\neq\emptyset\}}-1_{\{A\cap(B\cup\Delta_n)\neq\emptyset\}}$, this is in turn implied by the condition that $\int\mu_n(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}\to\int\mu(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}$ for each $B\in\mathcal{P}_{\rm fin}(\Lambda)$. ∎

The next lemma guarantees that expressions as in the left-hand side of (1.8) are well-defined and yield a homogeneous, locally finite measure on $\mathcal{P}_+(\Lambda)$

Lemma 3.3 (Evolution of locally finite measures) *If μ is a homogeneous, locally finite measure on $\mathcal{P}_+(\Lambda)$, then for each $t\geq 0$, the measure $\int\mu(\mathrm{d}A)\mathbb{P}[\eta^A_t\in\cdot\,]$ is homogeneous and locally finite on $\mathcal{P}_+(\Lambda)$. If μ_n are homogeneous, locally finite measures on $\mathcal{P}_+(\Lambda)$ converging vaguely to a limit μ, then*

$$\int\mu_n(\mathrm{d}A)\mathbb{P}[\eta^A_t\in\cdot\,]\underset{n\to\infty}{\Longrightarrow}\int\mu(\mathrm{d}A)\mathbb{P}[\eta^A_t\in\cdot\,]. \tag{3.14}$$

Proof We start by observing that any homogeneous, locally finite measure μ on $\mathcal{P}_+(\Lambda)$ satisfies

$$\int\mu(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}}\leq\sum_{i\in B}\int\mu(\mathrm{d}A)1_{\{i\in A\}}=|B|\int\mu(\mathrm{d}A)1_{\{0\in A\}}<\infty. \tag{3.15}$$

Using duality (see (2.4)), it follows that

$$\begin{aligned}&\int\mu(\mathrm{d}A)\mathbb{P}[\eta^A_t\in\mathrm{d}A]1_{\{A\cap B\neq\emptyset\}}=\int\mu(\mathrm{d}A)\mathbb{P}[\eta^A_t\cap B\neq\emptyset]=\int\mu(\mathrm{d}A)\mathbb{P}[A\cap\eta^{\dagger\,B}_t\neq\emptyset]\\&\quad=\int\mathbb{P}[\eta^{\dagger\,B}_t\in\mathrm{d}C]\int\mu(\mathrm{d}A)1_{\{A\cap C\neq\emptyset\}}\leq\int\mathbb{P}[\eta^{\dagger\,B}_t\in\mathrm{d}C]\,|C|\int\mu(\mathrm{d}A)1_{\{0\in A\}}\\&\quad=\mathbb{E}\big[|\eta^{\dagger\,B}_t|\big]\int\mu(\mathrm{d}A)1_{\{0\in A\}}<\infty\qquad\qquad(B\in\mathcal{P}_{\rm fin}(\Lambda)).\end{aligned} \tag{3.16}$$

By Lemma 3.1, it follows that the measure $\int\mu_n(\mathrm{d}A)\mathbb{P}[\eta^A_t\in\cdot\,]$ is locally finite. It is obviously homogeneous. Now if μ_n are homogeneous, locally finite measures on $\mathcal{P}_+(\Lambda)$ converging vaguely to a limit μ, then

$$\int\mu_n(\mathrm{d}A)\mathbb{P}[\eta^A_t\in\mathrm{d}A]1_{\{A\cap B\neq\emptyset\}}=\int\mathbb{P}[\eta^{\dagger\,B}_t\in\mathrm{d}C]\int\mu_n(\mathrm{d}A)1_{\{A\cap C\neq\emptyset\}}, \tag{3.17}$$

for each $B\in\mathcal{P}_{\rm fin}(\Lambda)$, and this quantity converges to the analogue quantity for μ by dominated convergence, using the estimate (3.15) and the fact that the $\int\mu_n(\mathrm{d}A)1_{\{0\in A\}}$ are uniformly bounded in n since they converge. Applying Lemma 3.2, we arrive at (3.14). ∎

Proof of Proposition 1.4 It suffices to prove, for each (Λ,a,δ)-contact process, the following three claims:

1. There exists a homogeneous, locally finite eigenmeasure of the (Λ,a,δ)-contact process, with eigenvalue $r=r(\Lambda,a,\delta)$.

2. If λ is the eigenvalue of a locally finite eigenmeasure of the (Λ,a,δ)-contact process, then $\lambda\leq r$.

3. The set $\mathcal{E}(\Lambda,a,\delta)$ is closed and bounded from below.

We start with claim 1. Define (by Lemma 3.3) homogeneous, locally finite measures μ_t on $\mathcal{P}_+(\Lambda)$ by

$$\mu_t := \sum_i \mathbb{P}[\eta_t^{\{i\}} \in \cdot\,] \qquad (t \geq 0). \tag{3.18}$$

Let $\hat{\mu}_\lambda$ be the Laplace transform of μ_t, i.e.,

$$\hat{\mu}_\lambda := \int_0^\infty \mu_t\, e^{-\lambda t}\mathrm{d}t \qquad (\lambda > r). \tag{3.19}$$

We claim that the measures $\hat{\mu}_\lambda$, properly renormalized, are relatively compact in the topology of vague convergence and that each subsequential limit as $\lambda \downarrow r$ is a homogeneous, locally finite eigenmeasure of the (Λ, a, δ)-contact process, with eigenvalue r.

Note that by duality (see (2.4)),

$$\int \mu_t(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} = \sum_i \mathbb{P}[\eta_t^{\{i\}} \cap B \neq \emptyset] = \sum_i \mathbb{P}[i \in \eta_t^{\dagger\,B}] = \mathbb{E}\big[|\eta_t^{\dagger\,B}|\big] = \pi_t^\dagger(B) \tag{3.20}$$

$(t \geq 0,\ B \in \mathcal{P}_{\rm fin}(\Lambda))$, where $\pi_t^\dagger(A)$ is defined in analogy with (3.1) for the $(\Lambda, a^\dagger, \delta)$-contact process. It follows that

$$\int \hat{\mu}_t(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} = \hat{\pi}_\lambda^\dagger(B), \tag{3.21}$$

where

$$\hat{\pi}_\lambda^\dagger(A) := \int_0^\infty \pi_t^\dagger(A)\, e^{-\lambda t}\mathrm{d}t \qquad (\lambda > r,\ A \in \mathcal{P}_{\rm fin}(\Lambda)). \tag{3.22}$$

By (3.5) and Theorem 1.2 (a), which has been proved in Section 3.1, for every $\varepsilon > 0$, there exists a $T_\varepsilon < \infty$ such that

$$e^{rt} \leq \pi_t^\dagger(\{0\}) \leq e^{(r+\varepsilon)t} \qquad (t \geq T_\varepsilon). \tag{3.23}$$

It follows from the upper bound in (3.23) that $\hat{\pi}_\lambda^\dagger(\{0\}) < \infty$ for all $\lambda > r$. Hence, by (3.21) and Lemma 3.1, the measures $\hat{\mu}_\lambda$ are locally finite for each $\lambda > r$. The lower bound in (3.23) and monotone convergence moreover show that

$$\lim_{\lambda\downarrow r} \hat{\pi}_\lambda^\dagger(\{0\}) = \lim_{\lambda\downarrow r} \int_0^\infty \pi_t^\dagger(\{0\})\, e^{-\lambda t}\mathrm{d}t = \int_0^\infty \pi_t^\dagger(\{0\})\, e^{-rt}\mathrm{d}t = \infty. \tag{3.24}$$

Set $\overline{\mu}_\lambda := \hat{\pi}^\dagger(\{0\})^{-1}\hat{\mu}_\lambda$. Then for each $\lambda > r$, the measure $\overline{\mu}_\lambda$ is homogenous, locally finite, and normalized such that $\int \overline{\mu}_\lambda(\mathrm{d}A)1_{\{0\in A\}} = 1$. Therefore, by Lemma 3.2 and the estimate (3.15), the measures $\overline{\mu}_\lambda$ are relatively compact in the topology of vague convergence as $\lambda \downarrow r$. Choose $\lambda_n \downarrow r$ such that $\overline{\mu}_{\lambda_n} \Rightarrow \overline{\mu}_r$ for some homogenous, locally finite $\overline{\mu}_r$. Then, filling in our definitions, using Lemma 3.3 and the Markov property of the contact process,

$$\begin{aligned}\int \overline{\mu}_r(\mathrm{d}A)\mathbb{P}[\eta_t^A \in \cdot\,] &= \lim_{n\to\infty} \hat{\pi}_{\lambda_n}^\dagger(\{0\})^{-1} \int_0^\infty e^{-\lambda_n s}\mathrm{d}s \sum_i \int \mathbb{P}[\eta_s^{\{i\}} \in \mathrm{d}A]\mathbb{P}[\eta_t^A \in \cdot\,] \\ &= e^{rt} \lim_{n\to\infty} \hat{\pi}_{\lambda_n}^\dagger(\{0\})^{-1} \int_0^\infty e^{-\lambda_n(s+t)}\mathrm{d}s \sum_i \mathbb{P}[\eta_{s+t}^{\{i\}} \in \cdot\,] \\ &= e^{rt}\Big(\overline{\mu}_r - \lim_{n\to\infty} \hat{\pi}_{\lambda_n}^\dagger(\{0\})^{-1} \int_0^t e^{-\lambda_n s}\mathrm{d}s \sum_i \mathbb{P}[\eta_s^{\{i\}} \in \cdot\,]\Big) = e^{rt}\overline{\mu}_r,\end{aligned} \tag{3.25}$$

where in the last step we have used (3.24). This shows that $\overline{\mu}_r$ is an eigenmeasure with eigenvalue r.

To prove claim 2, we observe that if μ is an eigenmeasure with eigenvalue λ, then by duality (see (2.4)) and (3.15),

$$\begin{aligned} e^{\lambda t}\int\mu(\mathrm{d}A)1_{\{0\in A\}} &= \int\mu(\mathrm{d}A)\mathbb{P}[0\in\eta_t^A] \\ &= \int\mu(\mathrm{d}A)\mathbb{P}[\eta_t^{\dagger\,\{0\}}\cap A\neq 0]\leq\mathbb{E}\big[|\eta_t^{\dagger\,\{0\}}|\big]\int\mu(\mathrm{d}A)1_{\{0\in A\}}. \end{aligned} \tag{3.26}$$

By (3.23), for each $\varepsilon>0$ we can choose t large enough such that $\mathbb{E}\big[|\eta_t^{\dagger\,\{0\}}|\big]\leq e^{(r+\varepsilon)t}$. Since $\int\mu(\mathrm{d}A)1_{\{0\in A\}}>0$, we may divide by it, yielding $e^{\lambda t}\leq e^{(r+\varepsilon)t}$, which implies $\lambda\leq r+\varepsilon$. Since $\varepsilon>0$ is arbitrary, it follows that $\lambda\leq r$.

To prove claim 3, finally, we observe that since we may estimate a contact process from below by a simple death process, for any homogeneous locally finite eigenmeasure μ with eigenvalue λ, one has

$$e^{\lambda t}\int\mu(\mathrm{d}A)1_{\{0\in A\}}=\int\mu(\mathrm{d}A)\mathbb{P}[0\in\eta_t^A]\geq e^{-\delta t}\int\mu(\mathrm{d}A)1_{\{0\in A\}}, \tag{3.27}$$

which shows that $\mathcal{E}(\Lambda,a,\delta)\subset[-\delta,\infty)$. To show that $\mathcal{E}(\Lambda,a,\delta)$ is closed, assume that $\lambda_n\in\mathcal{E}(\Lambda,a,\delta)$ and $\lambda_n\to\lambda$. Then we can find homogeneous, locally finite eigenmeasures μ_n with eigenvalues λ_n. Normalizing such that $\int\mu_n(\mathrm{d}A)1_{\{0\in A\}}=1$, using Lemma 3.2 and (3.15), we see that the sequence μ_n is relatively compact in the topology of vague convergence, hence has a subsequential limit μ, which by Lemma 3.3 is a homogeneous, locally finite eigenmeasure with eigenvalue λ. ∎

The proof of Proposition 1.4 yields a useful corollary.

Corollary 3.4 (Convergence to eigenmeasure) *Let μ_t be defined as in (3.18). Then the measures*

$$\hat{\pi}^\dagger(\{0\})^{-1}\int_0^\infty\mu_t\,e^{-\lambda t}\mathrm{d}t \tag{3.28}$$

are relatively compact as $\lambda\downarrow r$ in the topology of vague convergence, and each subsequential limit as $\lambda\downarrow r$ is a homogeneous, locally finite eigenmeasure of the (Λ,a,δ)-contact process, with eigenvalue $r(\Lambda,a,\delta)$.

Remark It seems intuitively plausible that the measures μ_t, suitably rescaled, should be increasing in t, hence by Corollary 3.4 should converge, as $t\to\infty$, to an eigenmeasure with eigenvalue r. Indeed, it seems plausible that these eigenmeasures are the 'lowest' possible eigenmeasures, in a suitable stochastic order. So far, however, I have not been able to make this rigorous. Should these conjectures be correct, then these eigenmeasures are quite similar to the 'second lowest invariant measure' from [SS97, SS99], by which they are inspired.

3.3 Proof of Theorem 1.5

We start with a preparatory lemma. We say that a function $f:\mathcal{P}_{\mathrm{fin}}(\Lambda)\to\mathbb{R}$ is *shift-invariant* if $f(iA)=f(A)$ for all $i\in\Lambda$, *monotone* if $A\subset B$ implies $f(A)\leq f(B)$, and *subadditive* if $f(A\cup B)\leq f(A)+f(B)$, for all $A,B\in\mathcal{P}_{\mathrm{fin}}(\Lambda)$. Recall the definition of the generator G of the (Λ,a,δ)-contact process from (2.11). We define $G^\dagger$ analogously, for the $(\Lambda,a^\dagger,\delta)$-contact process.

Lemma 3.5 (Eigenmeasures and harmonic functions) *If μ is a locally finite, homogeneous eigenmeasure of the (Λ, a, δ)-contact process with eigenvalue λ, then*

$$v(A) := \int \mu(\mathrm{d}B) 1_{\{A\cap B\neq\emptyset\}} \qquad (A \in \mathcal{P}_{\mathrm{fin}}(\Lambda)) \tag{3.29}$$

is a shift-invariant, monotone, subadditive function such that $v(\emptyset) = 0$, $v(A) > 0$ for any $\emptyset \neq A \in \mathcal{P}_{\mathrm{fin}}(\Lambda)$, $v \in \mathcal{S}(\mathcal{P}_{\mathrm{fin}}(\Lambda))$, and $G^\dagger v = \lambda v$.

Proof The function v is obviously shift-invariant, monotone, and satisfies $v(\emptyset) = 0$. Since μ is homogeneous and nonzero, $v(A) > 0$ for any $\emptyset \neq A \in \mathcal{P}_{\mathrm{fin}}(\Lambda)$. The function v is subadditive since $1_{\{(A\cup A')\cap B\neq\emptyset\}} \leq 1_{\{A\cap B\neq\emptyset\}} + 1_{\{A'\cap B\neq\emptyset\}}$. Subadditivity, shift-invariance, and $v(\emptyset) = 0$ imply that $v(A) \leq v(\{0\})|A|$, so certainly $v \in \mathcal{S}(\mathcal{P}_{\mathrm{fin}}(\Lambda))$.

To see that $G^\dagger v = \lambda v$, observe that by duality (see (2.4))

$$\begin{aligned}\mathbb{E}[v(\eta_t^{\dagger\, A})] &= \mathbb{P}[\eta_t^{\dagger\, A} \cap B \neq \emptyset] = \int \mu(\mathrm{d}B)\mathbb{P}[A \cap \eta_t^B \neq \emptyset] \\ &= \int \mu(\mathrm{d}B)\mathbb{P}[\eta_t^B \in \cdot\,] 1_{\{A\cap B\neq\emptyset\}} = e^{\lambda t} \int \mu(\mathrm{d}B) 1_{\{A\cap B\neq\emptyset\}} = e^{\lambda t} v(A).\end{aligned} \tag{3.30}$$

By (2.27) this implies that $G^\dagger v(A) = \lambda v(A)$. ∎

Recall the definition of the survival probability $\rho(A)$ from (2.6). Theorem 1.5 follows from the following, stronger result.

Proposition 3.6 (Shift invariant monotone harmonic functions) *Assume that the infection rates satisfy the irreducibility condition (1.7) and that the (Λ, a, δ)-contact process survives. Assume that $f : \mathcal{P}_{\mathrm{fin}}(\Lambda) \to \mathbb{R}$ is shift invariant, monotone, $f(\emptyset) = 0$, $f \in \mathcal{S}(\mathcal{P}_{\mathrm{fin}}(\Lambda))$, and $Gf = 0$. Then there exists a constant $c \geq 0$ such that $f = c\rho$.*

Before we prove this, we first show how this implies Theorem 1.5.

Proof of Theorem 1.5 Let μ be a homogeneous, locally finite eigenmeasure of the (Λ, a, δ)-contact process, and let $v(A) := \int \mu(\mathrm{d}B) 1_{\{A\cap B\neq\emptyset\}}$. By Lemma 3.5, v is shift invariant, monotone, $v(\emptyset) = 0$, $v \in \mathcal{S}(\mathcal{P}_{\mathrm{fin}}(\Lambda))$, and $G^\dagger v = 0$. By assumption, the upper invariant measure $\overline{\nu}$ of the (Λ, a, δ)-contact process is nontrivial, hence the $(\Lambda, a^\dagger, \delta)$-contact process survives, so by Proposition 3.6, $v = c\rho^\dagger$ for some $c \geq 0$, where $\rho^\dagger$ denotes the survival probability of the $(\Lambda, a^\dagger, \delta)$-contact process. By the characterization of the upper invariant measure in (2.9), it follows that $\mu = c\overline{\nu}$. ∎

In order to prove Proposition 3.6, we need one more lemma.

Lemma 3.7 (Eventual domination of finite configurations) *Assume that the infection rates satisfy the irreducibility condition (1.7) and that the (Λ, a, δ)-contact process survives. Then*

$$\lim_{t\to\infty} \mathbb{P}\big[\exists i \in \Lambda \text{ s.t. } \eta_t^A \geq iB \,\big|\, \eta_t^A \neq \emptyset\big] = 1 \qquad (A, B \in \mathcal{P}_{\mathrm{fin}}(\Lambda),\ A \neq \emptyset). \tag{3.31}$$

Formula (3.31) says that η exhibits a form of 'extinction versus unbounded growth'. More precisely, either η_t gets extinct or η_t is eventually larger than a random shift of any finite configuration. We remark that Lemma 3.7 is no longer true if the infection rates fail to satisfy the irreducibility condition (1.7).

Proof of Proposition 3.6 Since the (Λ, a, δ)-contact process solves the martingale problem for G, and $Gf = 0$, the process $f(\eta_t^A)$ is a martingale. In particular:

$$f(A) = \mathbb{E}[f(\eta_t^A)] \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda),\ t \geq 0). \tag{3.32}$$

Enumerate the elements of Λ an arbitrary way, and for $A, B \in \mathcal{P}_{\rm fin}(\Lambda)$, put

$$\hat{\imath}_{A,B} := \begin{cases} \min\{i \in \Lambda : A \geq iB\} & \text{if } \{i \in \Lambda : A \geq iB\} \text{ is nonempty,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.33}$$

Since f is monotone and shift invariant, we have, using Lemma 3.7,

$$\begin{aligned} f(A) &= \lim_{t\to\infty} \mathbb{E}[f(\eta_t^A)] \\ &\geq \limsup_{t\to\infty} \mathbb{E}[1_{\{\exists i \in \Lambda \text{ s.t. } \eta_t^A \geq iB\}} f(\hat{\imath}_{\eta_t^A,B} B)] \\ &= f(B) \limsup_{t\to\infty} \mathbb{P}[\exists i \in \Lambda \text{ s.t. } \eta_t^A \geq iB] \geq f(B)\rho(A) \qquad (A, B \in \mathcal{P}_{\rm fin}(\Lambda)). \end{aligned} \tag{3.34}$$

In particular, this shows that

$$f(B) \leq \frac{f(\{0\})}{\rho(\{0\})} < \infty \qquad (B \in \mathcal{P}_{\rm fin}(\Lambda)), \tag{3.35}$$

hence f is bounded. Now let $A_n, B_m \in \mathcal{P}_{\rm fin}(\Lambda)$ be sequences such that $\rho(A_n) \to 1$ and $\rho(B_n) \to 1$. Then, by (3.34),

$$\liminf_{n\to\infty} f(A_n) \geq \liminf_{n\to\infty} f(B_m)\rho(A_n) = f(B_m) \quad \forall m, \tag{3.36}$$

and therefore

$$\liminf_{n\to\infty} f(A_n) \geq \limsup_{m\to\infty} f(B_m). \tag{3.37}$$

This proves that the limit

$$\lim_{\rho(A_n)\to 1} f(A_n) =: f(\infty) \tag{3.38}$$

exists and does not depend on the choice of the sequence A_n with $\rho(A_n) \to 1$. By the Markov property and continuity of the conditional expectation with respect to increasing limits of σ-fields (see Complement 10(b) from [Loe63, Section 29] or [Loe78, Section 32]),

$$\rho(\eta_t^A) = \mathbb{P}\big[\eta_s^A \neq 0\ \forall s \geq 0 \,\big|\, \eta_t^A\big] \to 1_{\{\eta_s^A \neq 0\ \forall s \geq 0\}} \quad \text{a.s.} \quad \text{as } t \to \infty. \tag{3.39}$$

We conclude that

$$f(A) = \lim_{t\to\infty} \mathbb{E}[f(\eta_t^A)] = \rho(A) f(\infty) \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda)), \tag{3.40}$$

which shows that f is a scalar multiple of ρ. ∎

The proof of Lemma 3.7 depends on two preparatory lemmas.

Lemma 3.8 (Local creation of finite configurations) *For each $B \in \mathcal{P}_{\rm fin}(\Lambda)$ and $t > 0$, there exists a finite $\Delta \subset \Lambda$ and $j \in \Lambda$ such that*

$$\varepsilon := \mathbb{P}\big[\eta_t^{\{0\}} \supset jB \text{ and } \eta_s^{\{0\}} \subset \Delta\ \forall 0 \leq s \leq t\big] > 0. \tag{3.41}$$

Proof It follows from assumption (1.7) that there exists a site $j^{-1} \in \Lambda$ with $\mathbb{P}[\eta_t^{\{j^{-1}\}} \supset B] > 0$, and therefore $\mathbb{P}[\eta_t^{\{0\}} \supset jB] > 0$. Since $\bigcup_{0 \leq s \leq t} \eta_s^{\{0\}}$ is a.s. finite, we can choose a finite but large enough Δ such that (3.41) holds. ∎

Lemma 3.9 (Domination of finite configurations) *For each $B \in \mathcal{P}_{\rm fin}(\Lambda)$, $t > 0$, and $A_n \in \mathcal{P}_{\rm fin}(\Lambda)$ satisfying $\lim_{n\to\infty} |A_n| = \infty$, one has*

$$\lim_{n\to\infty} \mathbb{P}[\exists i \in \Lambda \ s.t. \ \eta_t^{A_n} \geq iB] = 1. \tag{3.42}$$

Proof Let Δ, j, and ε be as in Lemma 3.8. We can find $\tilde{A}_n \subset A_n$ such that $|\tilde{A}_n| \to \infty$ as $n \to \infty$, and for fixed n, the sets $(k\Delta)_{k\in\tilde{A}_n}$ are disjoint. It follows that

$$\begin{aligned}
&\mathbb{P}[\exists i \in \Lambda \text{ s.t. } \eta_t^{A_n} \geq iB] \\
&\quad \geq 1 - \prod_{k\in\tilde{A}_n} \big(1 - \mathbb{P}[\eta_t^{\{k\}} \supset kjB \text{ and } \eta_s^{\{k\}} \subset k\Delta \ \forall 0 \leq s \leq t]\big) \\
&\quad = 1 - (1-\varepsilon)^{|\tilde{A}_n|} \underset{n\to\infty}{\longrightarrow} 1,
\end{aligned} \tag{3.43}$$

where we have used (3.41) and the fact that events concerning the graphical representation in disjoint parts of space are independent. ∎

Proof of Lemma 3.7 If $\delta = 0$, then obviously $\lim_{t\to\infty} |\eta_t^A| = \infty$ a.s. If $\delta > 0$, then it is easy to see that $\inf\{\rho(A) : |A| \leq M\} < 1$ for all $M < \infty$. Therefore, by (3.39),

$$\eta_t^A = \emptyset \text{ for some } t \geq 0 \quad \text{or} \quad |\eta_t^A| \underset{t\to\infty}{\longrightarrow} \infty \qquad \text{a.s.} \tag{3.44}$$

Fix $\emptyset \neq B \in \mathcal{P}_{\rm fin}(\Lambda)$ and set $\psi_t(A) := P[\exists i \in \Lambda \text{ s.t. } \eta_t^A \geq iB] \quad (A \in \mathcal{P}_{\rm fin}(\Lambda),\ t \geq 0)$. Then, for each $t > 0$,

$$\lim_{T\to\infty} P[\exists i \in \Lambda \text{ s.t. } \eta_T^A \supset iB] = \lim_{T\to\infty} E[\psi_t(\eta_{T-t}^A)] = \rho(A), \tag{3.45}$$

where we have used Lemma 3.9 and (3.44). ∎

4 Proof of the main results

4.1 Exponentially growing processes

In this section, we prove Theorem 1.2 (d). Indeed, we prove the following, more detailed result. Recall that by Theorem 1.2 (a) and Proposition 1.4, there exists a homogeneous, locally finite eigenmeasure for the $(\Lambda, a^\dagger, \delta)$-contact process, with eigenvalue $r = r(\Lambda, a^\dagger, \delta) = r(\Lambda, a, \delta)$.

Proposition 4.1 (Exponential growth) *Let Λ be a finite or countably infinite group, let $a = (a(i,j))_{i,j\in\Lambda}$ be infection rates satisfying (1.1) and let $\delta \geq 0$. Let μ be any homogeneous, locally finite eigenmeasure of the $(\Lambda, a^\dagger, \delta)$-contact process, with eigenvalue r, and let the function v be defined in terms of μ as in Lemma 3.5. If $r > 0$, then, for each $A \in \mathcal{P}_{\rm fin}(\Lambda)$, the limit*

$$W_A := \lim_{t\to\infty} e^{-rt} v(\eta_t^A) \tag{4.1}$$

exists a.s., and satisfies $\mathbb{E}[W_A] = v(A)$. If the infection rates satisfy (1.7), then moreover

$$\mathbb{P}[W_A > 0] = \mathbb{P}[\eta_t^A \neq 0 \ \forall t \geq 0]. \tag{4.2}$$

Proof Our proof follows a strategy that is familiar from the theory of supercritical branching processes. Using duality (see (2.4)) and the fact that μ is an eigenmeasure, we see that

$$\begin{aligned}\mathbb{E}[v(\eta_t^A)] &= \int \mu(\mathrm{d}B)\mathbb{P}[\eta_t^A \cap B \neq \emptyset] = \int \mu(\mathrm{d}B)\mathbb{P}[A \cap \eta_t^{\dagger B} \neq \emptyset] \\ &= e^{rt}\int \mu(\mathrm{d}B)\mathbb{P}[A \cap B \neq \emptyset] = e^{rt}v(A),\end{aligned} \tag{4.3}$$

so, by the Markov property of $(\eta_t^A)_{t\geq 0}$, the process $(e^{-rt}v(\eta_t^A))_{t\geq 0}$ is a martingale. Every nonnegative martingale converges, so for each $A \in \mathcal{P}_{\rm fin}(\Lambda)$, there exists a random variable W_A such that (4.1) holds.

To prove that $\mathbb{E}[W_A] = v(A)$, it suffices to show that the random variables $\{e^{-rt}v(\eta_t^A) : t \geq 0\}$ are uniformly integrable. By Proposition 2.2, the variance of $v(\eta_t^A)$ is given by

$$\mathrm{Var}(v(\eta_t^A)) = 2\int_0^t \mathbb{E}\big[\Gamma(S_s v, S_s v)(\eta_{t-s}^A)\big]\mathrm{d}s, \tag{4.4}$$

where S_t and Γ are defined in (2.17) and (2.19). Formula (4.3) tells us that $S_t v = e^{rt}v$, so

$$\mathrm{Var}(v(\eta_t^A)) = 2\int_0^t \mathbb{E}\big[\Gamma(e^{rs}v, e^{rs}v)(\eta_{t-s}^A)\big]\mathrm{d}s = 2\int_0^t \mathbb{E}\big[\Gamma(v,v)(\eta_{t-s}^A)\big]e^{2rs}\mathrm{d}s. \tag{4.5}$$

It is not hard to see that for any $f, g \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ and $A \in \mathcal{P}_{\rm fin}(\Lambda)$,

$$\begin{aligned}\Gamma(f,g)(A) = &\tfrac{1}{2}\sum_{ij} a(i,j)1_{\{i\in A\}}1_{\{j\notin B\}}\big(f(A\cup\{j\}) - f(A)\big)\big(g(A\cup\{j\}) - g(A)\big) \\ &+ \tfrac{1}{2}\delta\sum_i 1_{\{i\in A\}}\big(f(A\backslash\{i\}) - f(A)\big)\big(g(A\backslash\{i\}) - g(A)\big).\end{aligned} \tag{4.6}$$

Without loss of generality we can normalize v such that $v(\{0\}) = 1$. Then, by monotonicity and subadditivity (see Lemma 3.5), $0 \leq v(A \cup \{j\}) - v(A) \leq 1$ for all j, A, and therefore

$$\Gamma(v,v)(A) \leq \tfrac{1}{2}\sum_{ij} a(i,j)1_{\{i\in A\}}1_{\{j\notin B\}} + \tfrac{1}{2}\delta\sum_i 1_{\{i\in A\}} \leq \tfrac{1}{2}\big(|a| + \delta\big)|A|. \tag{4.7}$$

Inserting this into (4.5) yields

$$\begin{aligned}\mathrm{Var}(e^{-rt}v(\eta_t^A)) &\leq \big(|a| + \delta\big)e^{-2rt}\int_0^t \mathbb{E}\big[|\eta_{t-s}^A|\big]e^{2rs}\mathrm{d}s \\ &\leq \big(|a| + \delta\big)\int_0^t \mathbb{E}\big[|\eta_{t-s}^A|\big]e^{-2r(t-s)}\mathrm{d}s.\end{aligned} \tag{4.8}$$

Since r is the exponential growth rate of η^A and $r > 0$, we can find $K < \infty$ such that $\mathbb{E}\big[|\eta_t^A|\big] \leq Ke^{\frac{3}{2}rt}$ $(t \geq 0)$. It follows that

$$\mathrm{Var}(e^{-rt}v(\eta_t^A)) \leq (|a| + \delta)K\int_0^\infty e^{-\frac{1}{2}rs}\mathrm{d}s < \infty \qquad (t \geq 0), \tag{4.9}$$

which proves the required uniform integrability.

Set $f(A) := \mathbb{P}[W_A > 0]$ and recall that $\rho(A) := \mathbb{P}[\eta_t^A \neq 0 \ \forall t \geq 0]$. Obviously $f \leq \rho$. We have just shown that $\mathbb{E}[W_A > 0] > 0$ if $A \neq \emptyset$, so $\rho(A) \geq f(A) > 0$ for each $A \neq \emptyset$. Assuming that the infection rates satisfy (1.7), we claim that $f = \rho$. We observe that

$$f(\eta_t^A) = \mathbb{P}\big[\lim_{s\to\infty} e^{-rs}\eta_s^A > 0 \,\big|\, \eta_t^A\big]. \tag{4.10}$$

In particular, this shows that $(f(\eta_t^A))_{t\geq 0}$ is a martingale, hence $Gf = 0$. It is easy to see that f is shift-invariant, monotone, bounded, and satisfies $f(\emptyset) = 0$, so applying Proposition 3.6, we see that $f = c\rho$ for some $c \geq 0$. Since $f \leq \rho$, we have $c \leq 1$.

By continuity of the conditional expectation with respect to increasing limits of σ-fields (compare (3.39)), the right-hand side of (4.10) converges a.s. to the indicator function of the event that $W_A > 0$. Since this event has positive probability, the event $\lim_{t\to\infty} f(\eta_t^A) = 1$ has positive probability. In particular, this shows that for each $\varepsilon > 0$ there exists a finite set B with $f(B) \geq 1 - \varepsilon$. This is possible only if the constant c in the equation $f = c\rho$ satisfies $c \geq 1$. ∎

4.2 Subexponential lattices

Proof of Theorem 1.2 (e) Consider a branching process on Λ, started with one particle in the origin, where a particle at i produces a new particle at j with rate $a(i,j)$, and each particle dies with rate δ. Let $B_t(i)$ denote the number of particles at site $i \in \Lambda$ and time $t \geq 0$. It is not hard to see that $\eta^{\{0\}}$ and B may be coupled such that

$$1_{\eta_t^{\{0\}}} \leq B_t \qquad (t \geq 0). \tag{4.11}$$

Let $(\xi_t)_{t\geq 0}$ be a random walk on Λ that jumps from i to j with rate $a(i,j)$, started in $\xi_0 = 0$. Then it is not hard to see that (compare [Lig99, Proposition I.1.21])

$$\mathbb{E}[B_t(i)] = \mathbb{P}[\xi_t = i]e^{(|a|-\delta)t} \qquad (i \in \Lambda,\ t \geq 0). \tag{4.12}$$

Let $h > 0$ be a constant, to determined later. It follows from (4.11) and (4.12) that

$$\begin{aligned} \mathbb{E}\big[|\eta_t^{\{0\}}|\big] &\leq \sum_i \big(1 \wedge \mathbb{P}[\xi_t = i]e^{(|a|-\delta)t}\big) \\ &= |\{i \in \Lambda : |i| \leq ht\}| + \mathbb{P}[|\xi_t| > ht]e^{(|a|-\delta)t} \qquad (t \geq 0). \end{aligned} \tag{4.13}$$

Let $(Y_i)_{i\geq 1}$ be i.i.d. $\mathbb{N}$-valued random variables with $\mathbb{P}[Y_i = k] = \frac{1}{|a|}\sum_{j:\ |j|=k} a(0,j)$ $(k \geq 0)$, let N be a Poisson-distributed random variable with mean $|a|$, independent of the $(Y_i)_{i\geq 1}$, and let $(X_m)_{m\geq 1}$ be i.i.d. random variables with law $\mathbb{P}[X_m \in \cdot\,] = \mathbb{P}[\sum_{i=1}^N Y_i \in \cdot\,]$. Since the random walk ξ makes jumps whose sizes are distributed in the same way as the Y_i, and the number of jumps per unit of time is Poisson distributed with mean $|a|$, it follows that

$$\mathbb{P}[|\xi_t| > ht] \leq \mathbb{P}\Big[\frac{1}{\lceil t\rceil}\sum_{m=1}^{\lceil t\rceil} X_m > h\frac{t}{\lceil t\rceil}\Big] \qquad (t > 0), \tag{4.14}$$

where $\lceil t\rceil$ denotes t rounded up to the next integer. By our assumptions,

$$\mathbb{E}\big[e^{\varepsilon X_m}\big] = \mathbb{E}\big[e^{\varepsilon\sum_{i=1}^N Y_k}\big] = e^{-|a|}\sum_{n=0}^\infty \frac{|a|^n}{n!}\mathbb{E}\big[e^{\varepsilon Y_1}\big]^n = e^{-|a|(1-\mathbb{E}[e^{\varepsilon Y_1}])} < \infty, \tag{4.15}$$

for some $\varepsilon > 0$. Therefore, by [DZ98, Theorem 2.2.3 and Lemma 2.2.20], for each $R > 0$ there exists a $h > 0$ and $K < \infty$ such that

$$\mathbb{P}\Big[\frac{1}{n}\sum_{m=1}^n X_m > h\Big] \leq Ke^{-nR} \qquad (n \geq 1). \tag{4.16}$$

Choosing h such that (4.16) holds for some $R > |a| - \delta$ yields, by (4.14)

$$\lim_{t\to\infty} \mathbb{P}\big[|\xi_t| > ht\big] e^{(|a|-\delta)t} = 0. \tag{4.17}$$

Inserting this into (4.13) we find that the exponential growth rate $r = r(\Lambda, a, \delta)$ satisfies

$$r \leq \limsup_{t\to\infty} \tfrac{1}{t} \log |\{i \in \Lambda : |i| \leq ht\}| = 0, \tag{4.18}$$

where we have used that the group Λ has subexponential growth. ∎

4.3 Nonamenable lattices

In this section, we prove Theorem 1.2 (f) and Corollary 1.3. We start by introducing some notation. If R is a nonnegative real random variable, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $0 < \mathbb{E}[R] < \infty$, then we define the *size-biased law* $\overline{\mathbb{P}}_R$ associated with R by

$$\overline{\mathbb{P}}_R(\mathcal{A}) := \frac{\mathbb{E}[1_{\mathcal{A}} R]}{\mathbb{E}[R]} \qquad (\mathcal{A} \in \mathcal{F}). \tag{4.19}$$

If Δ is a $\mathcal{P}_{\rm fin}(\Lambda)$-valued random variable, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $0 < \mathbb{E}[|\Delta|] < \infty$, then we define a probability law $\hat{\mathbb{P}} = \hat{\mathbb{P}}_\Delta$ on the product space $\Omega \times \Lambda$ by

$$\hat{\mathbb{P}}_\Delta(\mathcal{A} \times \{i\}) := \frac{\mathbb{P}(\{i \in \Delta\} \cap \mathcal{A})}{\mathbb{E}[|\Delta|]} \qquad (\mathcal{A} \in \mathcal{F},\ i \in \Lambda). \tag{4.20}$$

We call $\hat{\mathbb{P}}_\Delta$ the *Campbell law* associated with Δ. It is not hard to see that the projection of $\mathbb{P}_\Delta$ onto Ω is the size-biased law $\overline{\mathbb{P}}_{|\Delta|}$. Moreover, if we let $\iota(\omega, i) := i$ denote the projection from $\Omega \times \Lambda$ to Λ and we use the symbol Δ to denote (also) the random variable on $\Omega \times \Lambda$ defined by $\Delta(\omega, i) := \Delta(\omega)$, then

$$\hat{\mathbb{P}}_\Delta\big[\iota = i \,\big|\, \Delta\big] = \tfrac{1}{|\Delta|} 1_\Delta(i), \tag{4.21}$$

i.e., conditional on Δ, the site ι is chosen with equal probabilities from all sites in Δ. We may view ι as a 'typical' element of Δ. Campbell laws are closely related to the more widely known Palm laws; both play an important role in the theory of branching processes.

The next lemma relates Campbell laws to things we have been considering so far. Note that if μ is a locally finite measure on $\mathcal{P}_+(\Lambda)$ and $\int 1_{\{0\in A\}} \mu(\mathrm{d}A) > 0$, then the conditional law

$$\mu(\mathrm{d}A \,|\, 0 \in A) := \frac{1_{\{0\in A\}} \mu(\mathrm{d}A)}{\int 1_{\{0\in B\}} \mu(\mathrm{d}B)} \tag{4.22}$$

is a well-defined probability law.

Lemma 4.2 (Campbell law) *Let η be a (Λ, a, δ)-contact process. For each $t \geq 0$, let μ_t be defined as in (3.18) and let $\hat{\mathbb{P}}_t := \hat{\mathbb{P}}_{\eta_t^{\{0\}}}$ be the Campbell law associated with $\eta_t^{\{0\}}$. Then*

$$\mu_t(\mathrm{d}A \,|\, 0 \in A) = \hat{\mathbb{P}}_t\big[\iota^{-1} \eta_t^{\{0\}} \in \mathrm{d}A\big] \tag{4.23}$$

Proof This follows by writing

$$
\begin{aligned}
&\hat{\mathbb{P}}_t\big[\iota^{-1}\eta_t^{\{0\}} \in \mathrm{d}A\big] = \sum_i \hat{\mathbb{P}}_t\big[i^{-1}\eta_t^{\{0\}} \in \mathrm{d}A,\ \iota = i\big] \\
&= \sum_i \frac{\mathbb{P}\big[i^{-1}\eta_t^{\{0\}} \in \mathrm{d}A,\ i \in \eta_t^{\{0\}}\big]}{\mathbb{E}[|\eta_t^{\{0\}}|]} = \frac{\sum_i \mathbb{P}\big[i^{-1}\eta_t^{\{0\}} \in \mathrm{d}A,\ i \in \eta_t^{\{0\}}\big]}{\sum_i \mathbb{P}[i \in \eta_t^{\{0\}}]} \\
&= \frac{\sum_i \mathbb{P}\big[i^{-1}\eta_t^{\{0\}} \in \mathrm{d}A,\ 0 \in i^{-1}\eta_t^{\{0\}}\big]}{\sum_i \mathbb{P}[0 \in i^{-1}\eta_t^{\{0\}}]} = \frac{\sum_i \mathbb{P}\big[\eta_t^{\{i^{-1}\}} \in \mathrm{d}A,\ 0 \in \eta_t^{\{i^{-1}\}}\big]}{\sum_i \mathbb{P}[0 \in \eta_t^{\{i^{-1}\}}]} \\
&= \frac{\sum_j \mathbb{P}\big[\eta_t^{\{j\}} \in \mathrm{d}A,\ 0 \in \eta_t^{\{j\}}\big]}{\sum_j \mathbb{P}[0 \in \eta_t^{\{j\}}]} = \frac{1_{\{0\in A\}} \sum_j \mathbb{P}\big[\eta_t^{\{j\}} \in \mathrm{d}A\big]}{\int 1_{\{0\in B\}} \sum_j \mathbb{P}\big[\eta_t^{\{j\}} \in \mathrm{d}B\big]} \\
&= \frac{1_{\{0\in A\}}\mu_t(\mathrm{d}A)}{\int 1_{\{0\in B\}}\mu_t(\mathrm{d}B)} = \mu_t(\mathrm{d}A\,|\,0 \in A).
\end{aligned} \tag{4.24}
$$

∎

The next proposition is a direct consequence of Theorem 1.5 and Corollary 3.4.

Proposition 4.3 (Convergence of Campbell laws) *Assume that the (Λ, a, δ)-contact process has a nontrivial upper invariant measure $\overline{\nu}$, that its exponential growth rate $r(\Lambda, a, \delta)$ is zero, and that the infection rates satisfy (1.7). Let $\eta^{\{0\}}$ be the (Λ, a, δ)-contact process started in $\{0\}$ and for $\gamma \geq 0$, let τ_γ be an exponentially distributed random variable with mean γ, independent of $\eta^{\{0\}}$. For each $\gamma \geq 0$, let $\hat{\mathbb{P}}_\gamma = \hat{\mathbb{P}}_{\eta^{\{0\}}_{\tau_\gamma}}$ be the Campbell law associated with $\eta^{\{0\}}_{\tau_\gamma}$. Then*

$$
\lim_{\gamma\to\infty} \hat{\mathbb{P}}_\gamma\big[\iota^{-1}\eta^{\{0\}}_{\tau_\gamma} \in \mathrm{d}A\big] \underset{\gamma\to\infty}{\Longrightarrow} \overline{\nu}(\mathrm{d}A\,|\,0 \in A), \tag{4.25}
$$

where $\Rightarrow$ denotes weak convergence of probability measures.

Proof For $\lambda > 0$, let $\hat{\mu}_\lambda$ denote the Laplace transform of μ_t, defined in (3.19). In analogy with Lemma 4.2, it is straightforward to check that

$$
\hat{\mu}_\lambda(\mathrm{d}A\,|\,0 \in A) = \hat{\mathbb{P}}_{1/\lambda}\big[\iota^{-1}\eta^{\{0\}}_{\tau_{1/\lambda}} \in \mathrm{d}A\big] \qquad (\lambda > 0). \tag{4.26}
$$

By Theorem 1.5 and Corollary 3.4, the measures $\hat{\mu}_\lambda$, suitably rescaled, converge vaguely to $\overline{\nu}$ as $\lambda \downarrow 0$. By (4.26), this implies the weak convergence in (4.25). ∎

The next proposition shows that if the assumptions of Proposition 4.3 are satisfied, then Λ must be amenable.

Proposition 4.4 (Campbell laws and amenability) *Let Λ be a countable group and let Δ_n be random finite subsets of Λ such that $0 < \mathbb{E}[|\Delta_n|] < \infty$ for each n. Assume that there exists a nontrivial, homogeneous probability measure ν on $\mathcal{P}(\Lambda)$ such that the Campbell laws $\hat{\mathbb{P}}_n = \hat{\mathbb{P}}_{\Delta_n}$ associated with Δ_n satisfy*

$$
\hat{\mathbb{P}}_n\big[\iota^{-1}\Delta_n \in \mathrm{d}A\big] \underset{n\to\infty}{\Longrightarrow} \nu(\mathrm{d}A\,|\,0 \in A). \tag{4.27}
$$

Then Λ must be amenable.

We postpone the proof of Proposition 4.4 till the next section.

Proof of Theorem 1.2 (f) Assume (1.7). Assume that the (Λ, a, δ)-contact process survives and that its exponential growth rate $r(\Lambda, a, \delta)$ is zero. Then the $(\Lambda, a^\dagger, \delta)$-contact process has a notrivial upper invariant law, and, by Theorem 1.2 (a), $r(\Lambda, a^\dagger, \delta) = 0$. Therefore, by Propositions 4.3 and 4.4, Λ must be amenable. ∎

Proof of Corollary 1.3 Let $\mathcal{S} := \{\delta \geq 0 : \text{the } (\Lambda, a, \delta)\text{-contact process survives}\}$. Note that $\mathcal{S}$ is nonempty since $0 \in \mathcal{S}$. By Theorem 1.2 (d) and (f), $\mathcal{S} = \{\delta \geq 0 : r(\Lambda, a, \delta) > 0\}$. By Theorem 1.2 (b), the function $\delta \to r(\Lambda, a, \delta)$ is continuous, hence $\mathcal{S}$ is an open subset of $[0, \infty)$. Hence, by monotonicity, $\mathcal{S} = [0, \delta_{\rm c})$, where $\delta_{\rm c} := \sup\{\delta \geq 0 : \text{the } (\Lambda, a, \delta)\text{-contact process survives}\}$ satisfies $\delta_{\rm c} > 0$. ∎

4.4 Campbell laws and amenability

In this section, we prove Proposition 4.4. Intuitively, the result is quite natural. Let Δ_n and ν be as in Proposition 4.4 and let Δ be a random variable with law ν. Then formula (4.27) says that for large n, the set Δ_n looks like a random finite piece cut out of the spatially homogeneous configuration Δ, such that most points in this piece are far from the boundary. This contradicts nonamenability, since in any finite subset of a nonamenable group, a positive fraction of the points must lie near the boundary. Making this intuition rigorous requires a bit of work, though.

We start by introducing some notation We let $L_\infty(\Lambda)$ be the space of all bounded real functions $f : \Lambda \to \mathbb{R}$, equipped with the supremumnorm $\|f\|_\infty := \sup_i |f(i)|$, and let $L_1(\Lambda)$ be the space of all functions $x : \Lambda \to \mathbb{R}$ such that $\sum_i |x(i)| < \infty$, equipped with the norm $|x| := \sum_i |x(i)|$. We define *left* and *right translation operators* $T^{\rm l}_i$ and $T^{\rm r}_i$ by

$$T^{\rm l}_i f(j) := f(i^{-1}j) \quad \text{and} \quad T^{\rm r}_i f(j) := f(ji^{-1}) \qquad (i, j \in \Lambda,\ f \in L_\infty(\Lambda)). \tag{4.28}$$

Note that $T^{\rm l}_i 1_A = 1_{iA}$ and $T^{\rm r}_i 1_A = 1_{Ai}$.

If X is a nonnegative $L_1(\Lambda)$-valued random variable random, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $0 < \mathbb{E}[|X|] < \infty$, then in analogy with (4.20), we define the *Campbell law* $\hat{\mathbb{P}}_X$ associated with X by

$$\hat{\mathbb{P}}_X(\mathcal{A} \times \{i\}) := \frac{\mathbb{E}[X(i) 1_{\mathcal{A}}]}{\mathbb{E}[|X|]} \qquad (\mathcal{A} \in \mathcal{F},\ i \in \Lambda). \tag{4.29}$$

As before, we let $\iota(\omega, i) := i$ denote the projection from $\omega \times \Lambda$ to Λ. Recall the definition of size-biased laws from (4.19). Proposition 4.4 is implied by the following, somewhat more general result. Below, we say that a probability law on $L_\infty(\Lambda)$ is nontrivial if it gives zero probability to the zero function.

Proposition 4.5 (Campbell laws and amenability) *Let Λ be a countable group. Let X_n be nonnegative $L_1(\Lambda)$-valued random variables such that $0 < \mathbb{E}[|X_n|] < \infty$ for each n. Let X be a nonnegative $L_\infty(\Lambda)$-valued random variable such that its law $\mathbb{P}[X \in \cdot\,]$ is nontrivial and homogeneous. Assume that the Campbell laws associated with the X_n and the size-biased law associated with $X(0)$ satisfy*

$$\hat{\mathbb{P}}_{X_n}\big[T^{\rm l}_{\iota^{-1}} X_n \in \cdot\,\big] \underset{n\to\infty}{\Longrightarrow} \overline{\mathbb{P}}_{X(0)}\big[X \in \cdot\,\big]. \tag{4.30}$$

Then Λ must be amenable.

Before we prove Proposition 4.5 we need some preliminary results. The next lemma is very similar to Lemma 4.2.

Lemma 4.6 (Campbell laws and infinite measures) *Let X be a nonnegative $L_1(\Lambda)$-valued random variable, defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, such that $0<\mathbb{E}[|X|]<\infty$. Define a locally finite, homogeneous measure μ on $\mathcal{P}_+(\Lambda)$ by*

$$\mu := \sum_i \mathbb{P}[T^{\mathrm{l}}_i X \in \cdot\,]. \tag{4.31}$$

Then

$$\hat{\mathbb{P}}_X[T^{\mathrm{l}}_{\iota^{-1}} X \in \mathrm{d}x] = \frac{x(0)\mu(\mathrm{d}x)}{\int y(0)\mu(\mathrm{d}y)}. \tag{4.32}$$

Proof This follows by writing

$$\begin{aligned}
\hat{\mathbb{P}}_X[T^{\mathrm{l}}_{\iota^{-1}} X \in \mathrm{d}x] &= \sum_i \hat{\mathbb{P}}_X[T^{\mathrm{l}}_{i^{-1}} X \in \mathrm{d}x,\ \iota = i]\\
&= \sum_i \frac{\mathbb{E}[1_{\{T^{\mathrm{l}}_{i^{-1}}X\in\mathrm{d}x\}}X(i)]}{\mathbb{E}[|X|]} = \frac{\sum_i \mathbb{E}[1_{\{T^{\mathrm{l}}_{i^{-1}}X\in\mathrm{d}x\}}X(i)]}{\sum_i \mathbb{E}[X(i)]}\\
&= \frac{\sum_i \mathbb{E}[1_{\{T^{\mathrm{l}}_{i^{-1}}X\in\mathrm{d}x\}}T^{\mathrm{l}}_{i^{-1}}X(0)]}{\sum_i \mathbb{E}[T^{\mathrm{l}}_{i^{-1}}X(0)]} = \frac{\sum_j \mathbb{E}[1_{\{T^{\mathrm{l}}_{j}X\in\mathrm{d}x\}}T^{\mathrm{l}}_{j}X(0)]}{\sum_j \mathbb{E}[T^{\mathrm{l}}_{j}X(0)]} = \frac{x(0)\mu(\mathrm{d}x)}{\int y(0)\mu(\mathrm{d}y)}.
\end{aligned} \tag{4.33}$$

∎

By definition, a *mean* on Λ is a continuous linear map $m: L_\infty(\Lambda)\to\mathbb{R}$ such that $m(f)\geq 0$ for all $f\geq 0$ and $m(1)=1$. We say that m is *right-invariant* if $m(T^{\mathrm{r}}_i f)=m(f)$ for each $f\in L_\infty(\Lambda)$ and $i\in\Lambda$. The following fact is well-known.

Proposition 4.7 (Amenability) *A countable group Λ is amenable in the sense of (1.3) if and only if there exists a right-invariant mean on Λ.*

Proof See [Pat88, Definition 0.2 and Theorem 4.10]. ∎

Say that $\pi:\Lambda\to\mathbb{R}$ is a probability distribution if $\pi\geq 0$ and $\sum_i \pi(i)=1$. For any probability distribution π on Λ and $j\in\Lambda$, set

$$m_j(\pi) := \sum_i \big|\pi(ij)-\pi(i)\big|. \tag{4.34}$$

If π is a probability distribution on Λ and $f\in L_\infty(\Lambda)$, then we write $\pi(f):=\sum_i \pi(i)f(i)$.

Lemma 4.8 (Approximation of invariant mean) *Let Λ be a countable group. Assume that there exist probability distributions π_n on Λ such that $\lim_{n\to\infty} m_j(\pi_n)=0$ for all $j\in\Lambda$. Then Λ is amenable.*

Proof If π is a probability distribution on Λ, then $f\mapsto\pi(f)$ is a continuous linear form on $L_\infty(\Lambda)$ such that $\pi(f)\geq 0$ for all $f\geq 0$ and $\pi(1)=1$. By the Banach-Alaoglu theorem, the sequence π_n is relatively compact in the weak-$*$ topology. It is easy to see that each weak-$*$ cluster point m is a mean. Since for any $f\in L_\infty(\Lambda)$,

$$\begin{aligned}
|\pi_n(T^{\mathrm{r}}_j f)-\pi_n(f)| &= \big|\sum_i \pi_n(i)f(ij^{-1}) - \sum_i \pi_n(i)f(i)\big|\\
&= \big|\sum_i \pi_n(ij)f(i) - \sum_i \pi_n(i)f(i)\big|\\
&\leq \sum_i \big|\pi_n(ij)-\pi_n(i)\big|\,|f(i)| \leq \|f\|_\infty m_j(\pi_n) \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned} \tag{4.35}$$

each weak-$*$ cluster point m of the π_n is right-invariant. ∎

If $x, y \in L_\infty(\Lambda)$ and at least one of them is in $L_1(\Lambda)$, then we define the convolution of x and y by

$$x * y(i) := \sum_j x(ij^{-1})y(j) = \sum_j x(j)y(j^{-1}i). \tag{4.36}$$

The convolution is linear in x and in y, associative, and satisfies $1_{\{i\}}1_{\{j\}} = 1_{\{ij\}}$. Moreover, $1_{\{i\}} * x = T^{\rm l}_i x$ and $x * 1_{\{i\}} = T^{\rm r}_i x$.

Lemma 4.9 (Smoothing of homogeneous laws) *On any countable group Λ, there exists probability laws π_n with the following property. Let X be a nonnegative $L_\infty(\Lambda)$-valued random variable whose law is homogeneous and nontrivial. Then*

$$\mathbb{P}\big[(X * \pi_n) \in \cdot\,\big] \underset{\mathbb{P}\to\infty}{\Longrightarrow} [Z \in \cdot\,], \tag{4.37}$$

where Z is an $L_\infty(\Lambda)$-valued random variable such that $Z(j) = Z(0) > 0$ for all $j \in \Lambda$ a.s.

We defer the proof of Lemma 4.9 to Appendix A.

Proof of Proposition 4.5 Define homogeneous, locally finite measures μ_n on $\mathcal{P}_+(\Lambda)$ by

$$\mu_n := \sum_i \mathbb{P}[T^{\rm l}_i X_n \in \cdot\,], \tag{4.38}$$

and let $\nu := \mathbb{P}[X \in \cdot\,]$ denote the law of X. By Lemma 4.6, (4.30) is equivalent to the statement that the μ_n, suitably rescaled, converge vaguely to ν. Indeed, (4.30) is equivalent to

$$\mathbb{E}[|X_n|]^{-1} \sum_i \mathbb{P}[T^{\rm l}_i X_n \in \cdot\,] \underset{n\to\infty}{\Longrightarrow} \mathbb{E}[|X(0)|]^{-1}\mathbb{P}[X \in \cdot\,]. \tag{4.39}$$

By Lemma 4.9 we can find (deterministic) probability laws π_m such that

$$\mathbb{P}[(X * \pi_m) \in \cdot\,] \underset{m\to\infty}{\Longrightarrow} \mathbb{P}[Z \in \cdot\,], \tag{4.40}$$

for some $L_\infty(\Lambda)$-valued random variable Z that is a.s. positive and constant. It follows from (4.39) that

$$\mathbb{E}[|X_n|]^{-1} \sum_i \mathbb{P}\big[\big((T^{\rm l}_i X_n) * \pi_m\big) \in \cdot\,\big] \underset{n\to\infty}{\Longrightarrow} \mathbb{E}[|X(0)|]^{-1}\mathbb{P}[(X * \pi_m) \in \cdot\,] \tag{4.41}$$

for each fixed m, so by a diagonal argument, we can find $m(n)$ such that

$$\mathbb{E}[|X_n|]^{-1} \sum_i \mathbb{P}[(T^{\rm l}_i X_n) * \pi_{m(n)} \in \cdot\,] \underset{n\to\infty}{\Longrightarrow} \mathbb{E}[|X(0)|]^{-1}\mathbb{P}[Z \in \cdot\,]. \tag{4.42}$$

It is not hard to see that $\mathbb{E}[(X * \pi_m)(0)] = \mathbb{E}[X(0)]$ for each m, hence $\mathbb{E}[|X(0)|] = \mathbb{E}[|Z(0)|]$. Moreover, $|X_n * \pi_m| = |X_n|$ and $T^{\rm l}_i(X_n * \pi_m) = (T^{\rm l}_i X_n) * \pi_m$ for each m, so (4.42) may be rewritten as

$$\mathbb{E}[|X_n * \pi_{m(n)}|]^{-1} \sum_i \mathbb{P}\big[\big(T^{\rm l}_i(X_n * \pi_{m(n)})\big) \in \cdot\,\big] \underset{n\to\infty}{\Longrightarrow} \mathbb{E}[|Z(0)|]^{-1}\mathbb{P}[Z \in \cdot\,]. \tag{4.43}$$

Set $Y_n := X_n * \pi_{m(n)}$. Then, by Lemma 4.6, (4.43) is equivalent to

$$\hat{\mathbb{P}}_{Y_n}\big[T^{1}_{\iota^{-1}}Y_n \in \cdot\,\big] \underset{n\to\infty}{\Longrightarrow} \overline{\mathbb{P}}_{Z(0)}[Z \in \cdot\,]. \tag{4.44}$$

It follows that

$$\hat{\mathbb{P}}_{Y_n}\Big[\Big|1 - \frac{Y_n(\iota j)}{Y_n(\iota)}\Big| \leq \varepsilon\Big] \underset{m\to\infty}{\longrightarrow} 1 \qquad (\varepsilon > 0,\ j \in \Lambda). \tag{4.45}$$

Define random probability distributions Π_n by $\Pi_n := \frac{1}{|Y_n|}Y_n$. Note that $\hat{\mathbb{P}}_{Y_n}[0 < |Y_n| < \infty] = 0$, so a.s. with respect to $\hat{\mathbb{P}}_n$, the Π_n are well-defined probability distributions. We claim that (4.45) implies that

$$\hat{\mathbb{E}}_{Y_n}\big[m_j(\Pi_n)\big] \underset{n\to\infty}{\longrightarrow} 0 \qquad (j \in \Lambda), \tag{4.46}$$

where $\hat{\mathbb{E}}_{Y_n}$ denotes expectation with respect to $\hat{\mathbb{P}}_{Y_n}$ and $m_j(\Pi_n)$ is defined in (4.34). To prove (4.46), we observe that for any probability distribution π on Λ,

$$\begin{aligned}
m_j(\pi) &= \sum_i \big|\pi(ij) - \pi(i)\big| \\
&= \sum_i 1_{\{\pi(ij)<\pi(i)\}}\pi(i)\Big|\frac{\pi(ij)}{\pi(i)} - 1\Big| + \sum_i 1_{\{\pi(i)<\pi(ij)\}}\pi(ij)\Big|\frac{\pi(i)}{\pi(ij)} - 1\Big| \\
&= \sum_i 1_{\{\pi(ij)<\pi(i)\}}\pi(i)\Big|\frac{\pi(ij)}{\pi(i)} - 1\Big| + \sum_i 1_{\{\pi(ij^{-1})<\pi(i)\}}\pi(i)\Big|\frac{\pi(ij^{-1})}{\pi(i)} - 1\Big| \\
&= \sum_i \pi(i)\Big[\Big(\Big(1 - \frac{\pi(ij)}{\pi(i)}\Big) \vee 0\Big) + \Big(\Big(1 - \frac{\pi(ij^{-1})}{\pi(i)}\Big) \vee 0\Big)\Big].
\end{aligned} \tag{4.47}$$

Since $\Pi_n = \frac{1}{|Y_n|}Y_n$, it follows that

$$\begin{aligned}
\hat{\mathbb{E}}_{Y_n}\big[m_j(\Pi_n)\big] &= \hat{\mathbb{E}}_{Y_n}\Big[\tfrac{1}{|Y_n|}\sum_i Y_n(i)\Big(\Big(1 - \frac{Y_n(ij)}{Y_n(i)}\Big) \vee 0\Big)\Big] \\
&\quad + \hat{\mathbb{E}}_{Y_n}\Big[\tfrac{1}{|Y_n|}\sum_i Y_n(i)\Big(\Big(1 - \frac{Y_n(ij^{-1})}{Y_n(i)}\Big) \vee 0\Big)\Big] \\
&= \hat{\mathbb{E}}_{Y_n}\Big[\Big(\Big(1 - \frac{Y_n(\iota j)}{Y_n(\iota)}\Big) \vee 0\Big)\Big] + \hat{\mathbb{E}}_{Y_n}\Big[\Big(\Big(1 - \frac{Y_n(\iota j^{-1})}{Y_n(\iota)}\Big) \vee 0\Big)\Big],
\end{aligned} \tag{4.48}$$

where we have used that $\hat{\mathbb{P}}_{Y_n}\big[\iota = i \,\big|\, Y_n\big] = \frac{1}{|Y_n|}Y_n(i)$ a.s. Combining (4.48) with (4.45), we arrive at (4.46).

By (4.46), the $[0,2]^\Lambda$-valued random variables $(m_j(\Pi_n))_{j\in\Lambda}$ converge in distribution to zero, as $n \to \infty$. Hence, by Skorohod's representation theorem, we can couple the random variables Π_n in such a way that $m_j(\Pi_n) \to 0$ for all $j \in \Lambda$ a.s. By Lemma 4.8, this implies that Λ is amenable. ∎

A A spatial ergodic theorem

In this appendix, we prove Lemma 4.9. The main ingredient in the proof is a simple L^2-ergodic theorem for arbitrary countable groups. This result is not very difficult, but unfortunately, I do not know a reference for it.

Let $(E,\mathcal{B})$ be a measurable space and let Λ be a countable group. Equip the product space $E^\Lambda := \{x = (x(i))_{i\in\Lambda} : x(i) \in E\ \forall i \in \Lambda\}$ with the product-σ-field $\mathcal{B}^\Lambda$. Define (left) translation operators $T_i : E^\Lambda \to E^\Lambda$ by

$$T_i x(j) := x(i^{-1}j) \qquad (i,j \in \Lambda) \tag{A.1}$$

(compare (4.28)), and let $\mathcal{T} := \{A \in \mathcal{B}^\Lambda : T_i^{-1}(A) = A\ \forall i \in \Lambda\}$ denote the σ-field of left-invariant events. We say that a probability law ν on $(E^\Lambda, \mathcal{B}^\Lambda)$ is homogeneous if $\nu \circ T_i^{-1} = \nu$ for each $i \in \Lambda$. We let $L^2(\nu)$ the L^2-space of with respect to ν square integrable real functions, equipped with the L^2-norm $\|f\|_{\nu,2} := (\int |f|^2 \mathrm{d}\nu)^{\frac{1}{2}}$.

Recall the definition of the convolution from (4.36), and recall that $|x| := \sum_i |x(i)|$.

Proposition A.1 (L^2-ergodic theorem) *Let Λ be a countable group and let Q be a probability law on Λ such that $\{i : Q(i) > 0\}$ generates Λ. Then:*

(a) *There exist probability distributions π_n on Λ such that*

$$\lim_{n\to\infty} |\pi_n - Q * \pi_n| = 0. \tag{A.2}$$

(b) *If ν is a homogeneous probability measure on E^Λ, $f \in L^2(\nu)$, and π_n are probability distributions on Λ such that (A.2) holds, then*

$$\sum_i \pi_n(i)\, f \circ T_i \underset{n\to\infty}{\longrightarrow} E[f|\mathcal{T}] \quad \text{in } L^2(\nu), \tag{A.3}$$

where $E[f|\mathcal{T}]$ denotes the conditional expectation (with respect to ν) of f given $\mathcal{T}$.

Proof To prove part (a), choose any probability distribution $\tilde\pi_0$ on Λ, define inductively $\tilde\pi_{n+1} := Q * \pi_n$ $(n \geq 0)$, and set $\pi_n := \frac{1}{n}\sum_{k=0}^{n-1} \tilde\pi_n$. Then $|\pi_n - Q * \pi_n| \leq \frac{1}{n}(|\tilde\pi_0| + |\tilde\pi_n|) \to 0$, as required.

The proof of part (b) would be standard if the probability distributions π_n would satisfy $\lim_{n\to\infty} |\pi_n - 1_{\{i\}} * \pi_n| = 0$ for all $i \in \Lambda$. If Λ is nonamenable, however, such π_n do not exist. To see that (A.2) is all we need, we need a little argument.

Define $S : L^2(\nu) \to L^2(\nu)$ by

$$Sf := \sum_i Q(i) f \circ T_i. \tag{A.4}$$

Note that since ν is homogeneous,

$$\|Sf\|_{\nu,2} \leq \sum_i Q(i) \|f \circ T_i\|_{\nu,2} = \|f\|_{\nu,2}, \tag{A.5}$$

i.e., S is a contraction. Set

$$\begin{aligned} H &:= \{f \in L^2(\nu) : f \circ T_i = f\ \forall i \in \Lambda\}, \\ H' &:= \{f \in L^2(\nu) : Sf = f\}. \end{aligned} \tag{A.6}$$

We claim that $H = H'$. The inclusion $H \subset H'$ is obvious, so assume that $f \in H'$. Let $Y = (Y(i))_{i\in\Lambda}$ be an E^Λ-valued random variable with law $\mathcal{L}(Y) = \nu$. Fix a point $k \in \Lambda$ and let $\xi = (\xi_n)_{n\geq 0}$ be a random walk on Λ, independent of Y, with transition probabilities

$$\mathbb{P}[\xi_{n+1} = j | \xi_n = i] = P(i,j) =: Q(ji^{-1}) \qquad (n \geq 0,\ i,j \in \Lambda), \tag{A.7}$$

and initial state $\xi_0 = k$. Define an E-valued process $Z = (Z_n)_{n\geq 0}$ by

$$Z_n := f \circ T_{\xi_n}(Y) \qquad (n \geq 0). \tag{A.8}$$

Note that, since ν is homogeneous,

$$\begin{aligned} E\big[|Z_n|^2\big] &= \sum_i P^n(k,i) \int \nu(\mathrm{d}y)\, |f \circ T_i(y)|^2 \\ &= \sum_i P^n(k,i) \|f\|_{\nu,2} = \|f\|_{\nu,2} \qquad (n \geq 0), \end{aligned} \tag{A.9}$$

where $P^n(i,j)$ denote the n-step transition probabilities of ξ. Let $\mathcal{F}_n := \sigma(Y, \xi_m : m \leq n)$ $(n \geq 0)$ be the σ-field generated by Y and the process ξ up to time $n \geq 0$. Then, since $f \in H'$,

$$\begin{aligned} \mathbb{E}[Z_{n+1}|\mathcal{F}_n] &= \sum_j P(\xi_n, j) f \circ T_j(Y) = \sum_i P(\xi_n, i\xi_n) f \circ T_{i\xi_n} \\ &= \sum_i Q(i) f \circ T_i \circ T_{\xi_n} = f \circ T_{\xi_n} = Z_n, \end{aligned} \tag{A.10}$$

where we have used that $T_i \circ T_j = T_{ij}$. Formulas (A.9) and (A.10) show that Z is a square integrable martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$. It follows that

$$\mathbb{E}[|Z_{n+1} - Z_n|^2] = \mathbb{E}[|Z_{n+1}|^2] - \mathbb{E}[|Z_n|^2] \qquad (n \geq 0). \tag{A.11}$$

By (A.9), $\mathbb{E}[|Z_n|^2]$ does not depend on n (in fact, Z is stationary) and therefore $Z_n = Z_0$ a.s. for all $n \geq 0$. Since $\{i \in \Lambda : Q(i) > 0\}$ generates Λ and the starting point k is arbitrary, this is possible only if $f = f \circ T_i$ a.s. for all $i \in \Lambda$. This proves that $H = H'$.

The rest of the proof is standard (compare the proof of Von Neumann's mean ergodic theorem in [Kre85, Theorem 1.1.4]). By [Kre85, Lemma 1.1.3] and the fact that $H = H'$, the orthogonal complement $H^\perp$ of H is the closure of the space $\mathrm{span}\{h - Sh : h \in L^2(\nu)\}$. If π_n are probability distributions on Λ satisfying (A.2) and $f = h - Sh$, then

$$\begin{aligned} \big\| \sum_i \pi_n(i) f \circ T_i \big\|_{\nu,2} &= \big\| \sum_i \pi_n(i) h \circ T_i - \sum_i \pi_n(i) \big(\sum_j Q(j) h \circ T_j \big) \circ T_i \big\|_{\nu,2} \\ &= \big\| \sum_i \pi_n(i) h \circ T_i - \sum_{ij} \pi_n(i) Q(j) h \circ T_{ji} \big\|_{\nu,2} \\ &= \big\| \sum_i \pi_n(i) h \circ T_i - \sum_k \big(\sum_i \pi_n(i) Q(ki^{-1}) \big) h \circ T_k \big\|_{\nu,2} \\ &= \big\| \sum_i \big(\pi_n(i) - Q * \pi_n(i) \big) h \circ T_i \big\|_{\nu,2} \\ &\leq \sum_i |\pi_n(i) - Q * \pi_n(i)|\, \|h \circ T_i\|_{\nu,2} \\ &= |\pi_n - Q * \pi_n|\, \|h\|_{\nu,2} \to 0. \end{aligned} \tag{A.12}$$

Therefore, by approximation, $\| \sum_i \pi_n(i) f \circ T_i \|_{\nu,2} \to 0$ for all $f \in H^\perp$. On the other hand, $\sum_i \pi_n(i) f \circ T_i = f$ for all $f \in H$. It follows that $\sum_i \pi_n(i) f \circ T_i$ converges in $L^2(\nu)$ to the orthogonal projection of f on H, which equals the conditional expectation of f given $\mathcal{T}$. ∎

Proof of Lemma 4.9 Without loss of generality we may assume that X is $[0,1]^\Lambda$-valued. Let $\mathcal{T}$ be the σ-field of left-invariant, measurable subsets of $[0,1]^\Lambda$ and let $X^{-1}(\mathcal{T})$ be the σ-field of events of the form $\{X \in A\}$ with $A \in \mathcal{T}$. For $i \in \Lambda$, define $f_i : [0,1]^\Lambda \to \mathbb{R}$ by $f(x) := x(i)$. Choose π_n as in Proposition A.1. Since

$$(X * \pi_n)(i) = \sum_j X(ij^{-1})\pi_n(j) = \sum_j \pi_n(j)(T_j X)(i) = \sum_j \pi_n(j) f_i \circ T_j(X), \tag{A.13}$$

Proposition A.1 tells us that

$$\mathbb{E}\big[\big|(X * \pi_n)(i) - Z(i)\big|^2\big] \underset{n\to\infty}{\longrightarrow} 0 \quad \text{where} \quad Z(i) := \mathbb{E}[X(i)|X^{-1}(\mathcal{T})]. \tag{A.14}$$

To see that $Z(j) = Z(0)$ a.s., by the definition of the conditional expectation, it suffices to check that for any $A \in \mathcal{T}$,

$$\mathbb{E}[Z(0)1_{\{X\in A\}}] = \mathbb{E}[X(0)1_{\{X\in A\}}] = \mathbb{E}[(T_{i^{-1}}X)(0)1_{\{T_{i^{-1}}X\in A\}}] = \mathbb{E}[X(i)1_{\{X\in A\}}], \tag{A.15}$$

where we have used that $A \in \mathcal{T}$ and the law of X is homogeneous. To see that $Z(0) > 0$ a.s., assume that conversely $\mathbb{P}[Z(0) = 0] > 0$. Then, since $Z(0) = Z(j)$ a.s. and $\{Z(j) = 0\} \in X^{-1}(\mathcal{T})$, one has $\mathbb{E}[X(j)1_{\{Z(0)=0\}}] = \mathbb{E}[X(j)1_{\{Z(j)=0\}}] = \mathbb{E}[Z(j)1_{\{Z(j)=0\}}] = 0$, hence $\mathbb{P}[X(j) = 0|Z(0) = 0] = 1$ for all $j \in \Lambda$. This implies that $\mathbb{P}[X(j) = 0\ \forall j \in \Lambda|Z(0) = 0] = 1$, so $\mathbb{P}[X(j) = 0\ \forall j \in \Lambda] \geq \mathbb{P}[Z(0) = 0] > 0$, which contradicts the assumption that $\mathbb{P}[X \in \cdot\,]$ is nontrivial. ∎

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