Computationally efficient probabilistic inference with noisy threshold models based on a CP tensor decomposition

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- Noisy threshold models
- CP-decomposition of conditional probability tables

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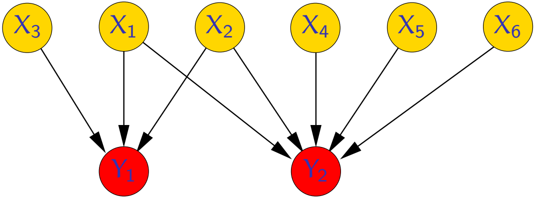

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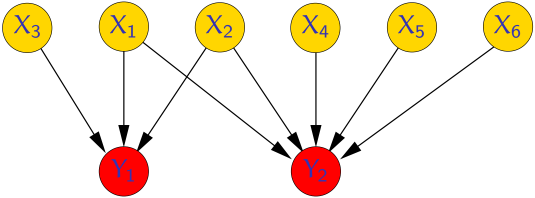
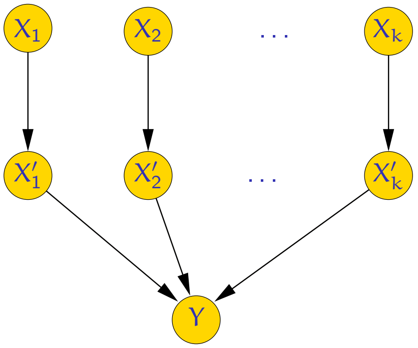


Definition (The inference task)

Given a subset of observations (e.g. Y_1 and Y_2) compute probabilities of diseases (e.g. $P(X_i|Y_1 = y_1, Y_2 = y_2)$, $i = 1, \ldots, 6$.

Noisy threshold - a generalization of noisy-or



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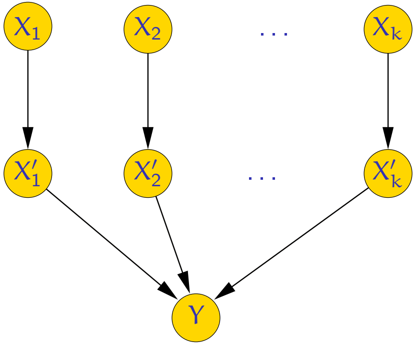

Y takes value 1 if at least ℓ out of k parents take value 1:

$$P(Y = 1 | X'_1 = x'_1, \ldots, X'_k = x'_k) = \begin{cases} 1 & \text{if } x'_1 + \ldots + x'_k \geqslant \ell \\ 0 & \text{otherwise.} \end{cases}$$

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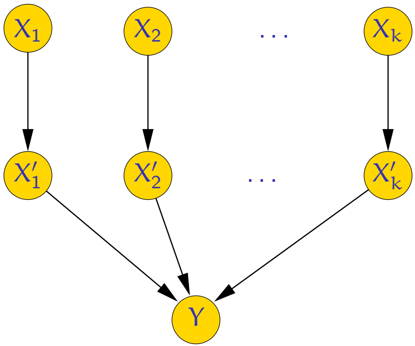

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Noise: for $i = 1, \ldots, k$

$$P(X'_i = 1 | X_i = x_i) = \begin{cases} 0 & \text{if } x_i = 0 \\ \pi_i & \text{otherwise.} \end{cases}$$

An example for $k = 4, \ell = 1$, and $\pi_i = 1, i = 1, \ldots, k$ - i.e., for deterministic OR function

$P(Y = 1 | X_1 = x_1, \ldots, X_4 = x_4)$

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$$= \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{pmatrix}$$

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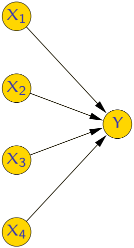
= & (1,1)^{\otimes k} - (1,0)^{\otimes k}
\end{aligned}
$$

Compilation of the threshold model for $\ell = 1$ - the standard approach

Lauritzen and Spiegelhalter (1988), Jensen et al. (1990), Shafer and Shenoy (1990)



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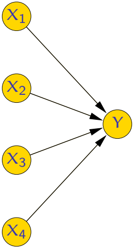

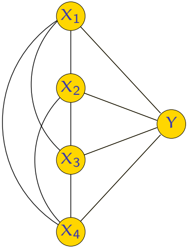

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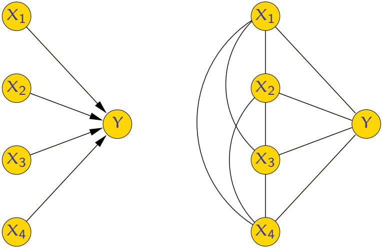

The total table size is $2^5 = 32$.

Compilation of the threshold model for $\ell = 1$ - after the suggested decomposition

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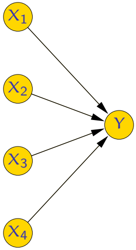

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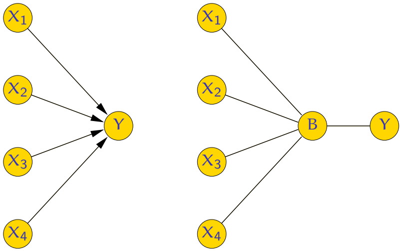

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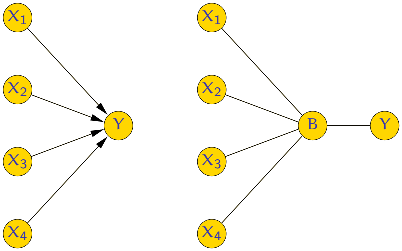

The total table size is $5 \cdot 2^2 = 20$.

Decomposition of $\mathbf{T}(\ell, k)$ into sum of tensor products

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Symmetric rank (srank) is the minimum number r such that

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where for $i = 1, \ldots, k$:

- $b_i \in \mathbb{R}$ and
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- This decomposition is called **Canonical Polyadic** (CP) or **CANDECOMP-PARAFAC** (CP) or **tensor rank-one**.

Theoretical results

Results in the proceedings:

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- An algorithm for CP-decomposition to $k-1$ factors. But we don't know yet if we can avoid complex numbers for some values of ℓ.

Experimental results

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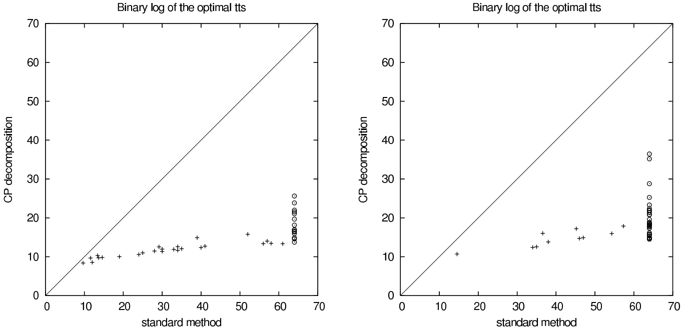

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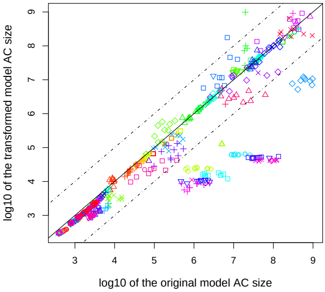

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- An algorithm for CP decomposition of these tensors.
- The CP tensor decomposition lead to a computational gain in the order of several magnitudes and made many intractable models manageable.

Acknowledgments

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- Gregory F. Cooper from University of Pittsburgh for the structural part of QMR-DT model.