Tensor rank-one decomposition of noisy-or models

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Additive decomposition of a probability table

We say that a conditional probability table $P(Y|X_1, \ldots X_n)$ can be decomposed by use of an auxiliary variable B if there exist tables $\xi(B, Y)$ and $\varphi_i(B, X_i)$, for $i = 1, \ldots, n$ such that

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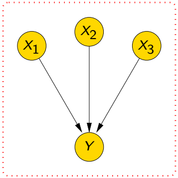

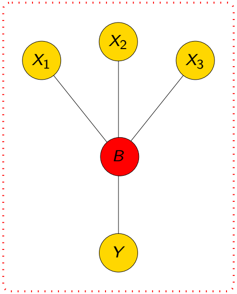

Correspondence to tensor rank-one decomposition

A decomposition using the auxiliary variable B

$$P(Y|X_1,\ldots,X_n) \;=\; \sum_B \xi(Y,B)\cdot\prod_{i=1}^{n}\varphi_i(B,X_i)$$

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is in fact a (minimal) tensor rank-one decomposition of $P(Y|X_1,\ldots,X_n)$.

Tensor rank

Definition (Tensor)

Let $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$, where $\mathcal{X}_i \subset \mathbb{N}$. $\psi : \mathcal{X} \mapsto \mathbb{R}$ is tensor of order n.

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The minimal number of rank-one tensors necessary to add in order to yield a tensor ψ is called rank of the tensor ψ.

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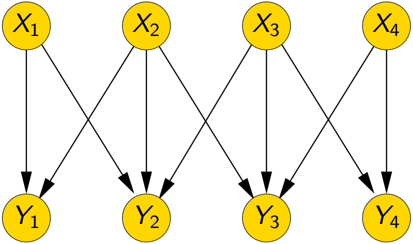
$$= (1,2) \otimes (1,2) \otimes (1,2) + (0,1) \otimes (0,1) \otimes (0,1)$$

$$\neq (a,b) \otimes (c,d) \otimes (e,f) \text{ for any } a,b,c,d,e,f \in \mathbb{R}.$$

BN2O model



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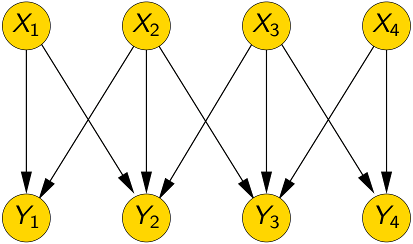

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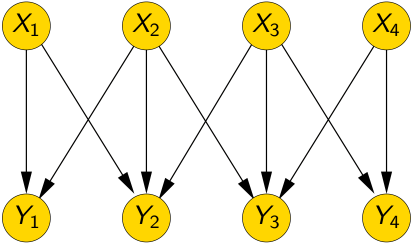

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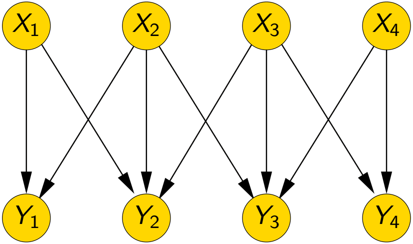

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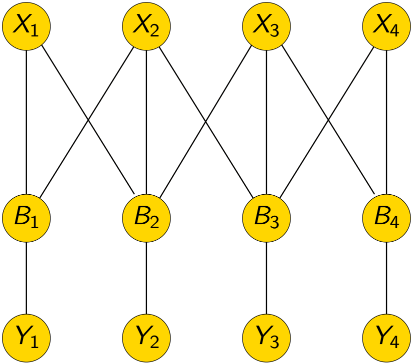
- Let $p_{i,j}$ be the inhibition probability for the parent node X_j of Y_i.
- $P(Y_i = 0 | X_{pa(Y_i)} = x_{pa(Y_i)}) \quad = \quad \prod_{j \in pa(Y_i)}^{n} p_{i,j}^{x_j}$

BN2O model after rank-one decomposition



BN2O model after rank-one decomposition (Díez and Galán, 2003)

$$P(Y_i = y_i | X_1 = x_1, \ldots, X_n = x_n) \quad = \quad \sum_{b_i=0}^{1} \xi(b_i, y_i) \cdot \prod_{j=1}^{n} \varphi_j(b_i, x_j)$$

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Arithmetic circuits

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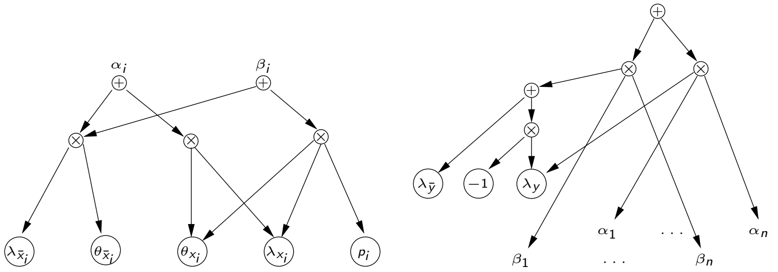
Definition (Arithmetic circuit (AC))

Arithmetic circuit is a rooted, directed acyclic graph whose leaf nodes correspond to circuit inputs and whose other nodes are labeled with multiplication and addition operations. The root node corresponds to circuit output.



α_i
β_i
$\lambda_{\bar{x}_i}$
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θ_{x_i}
λ_{x_i}
p_i
$\lambda_{\bar{y}}$
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λ_y
α_1
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α_n
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- An AC may be used to represent the computations determined by a junction tree,
- An AC may also represent more efficient computations, if they exist due to specific properties of the initial BN (e.g., determinism, context specific independence).

Construction of an AC

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We apply Darwiche's method to the BN2O transformed using the tensor rank-one decomposition.

Comparisons of AC sizes - created by Ace running on

`aligator.utia.cas.cz`: 8x AMD Opteron 8220, 64GB RAM

Median of the ratio o/t of the AC sizes for the original o and transformed model t computed from 10 models with randomly generated structure of the type x-y-e

x-y-e	o/t
6-40-160	1.49
10-15-60	1.24
10-20-80	1.19
20-30-150	1.03
20-40-200	1.01
30-15-150	474.86
50-50-200	1.11

x-y-e	o/t
10-1-10	1.47
10-15-90	1.21
10-30-90	1.13
20-30-300	0.98
30-10-100	303.48
50-20-80	1.90

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Models for which it was possible to construct an AC for the transformed model only (program run out of memory for the original model):

30-20-200 30-25-250 50-20-200

Transformed vs. original model AC size in the $\log_{10}$ scale