Tensor rank-one decomposition of probability tables

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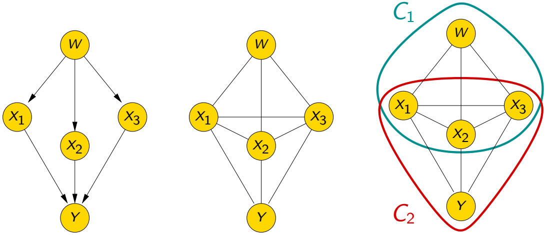
Probabilistic inference with Bayesian networks (the junction tree method)

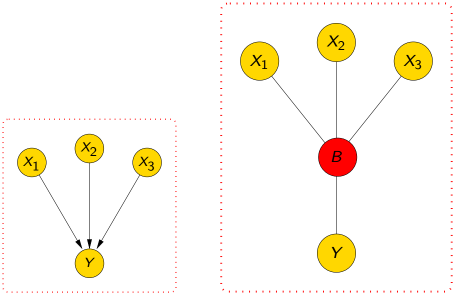


$$2^4 + 2^4 = 32$$

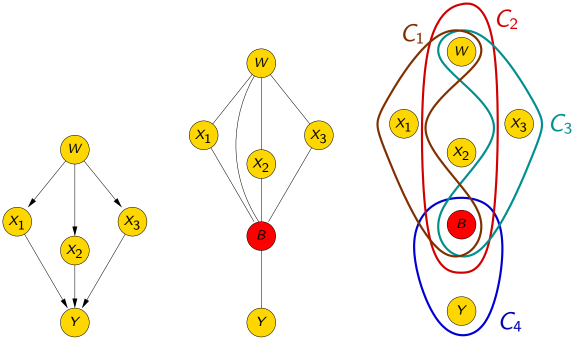
Decomposition of a conditional probability table

We say that $\psi : \mathcal{X}_1 \times \ldots \times \mathcal{X}_m \times \mathcal{Y} \mapsto \mathbb{R}$ can be factorized by use of variable B if there exist potentials $\xi : \mathcal{Y} \times \mathcal{B} \mapsto \mathbb{R}$ and $\varphi_i : \mathcal{X}_i \times \mathcal{B} \mapsto \mathbb{R}$, for $i = 1, \ldots, m$ such that for all $(x_1, \ldots, x_m, y) \in \mathcal{X}_1 \times \ldots \mathcal{X}_m \times \mathcal{Y}$, we have

$$\psi(x_1, \ldots, x_m, y) = \sum_{b \in \mathcal{B}} \xi(y, b) \cdot \prod_{i=1}^{m} \varphi_i(x_i, b)$$

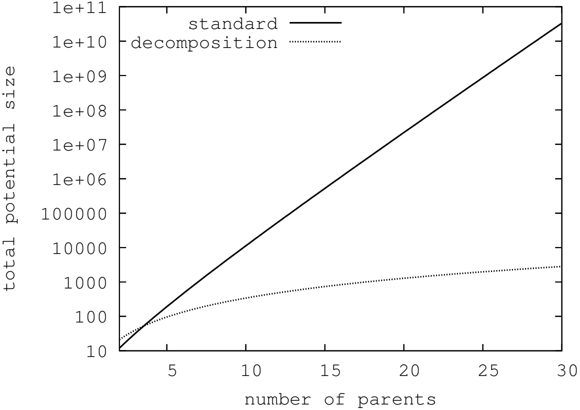


Probabilistic inference after the decomposition



If B has two states:
$2^3 + 2^3 + 2^3 + 2^2 = 28$
which is less than 32.

The savings with respect to number of parents



Comparison of the total size of junction tree.

Correspondence to tensor rank-one decomposition

A decomposition using the auxiliary variable B

$$\psi(x_1, \ldots, x_m, y) \quad = \quad \sum_{b \in \mathcal{B}} \xi(y, b) \cdot \prod_{i=1}^{m} \varphi_i(x_i, b)$$

that has the minimal number of states of B

is in fact a (minimal) tensor rank-one decomposition of tensor ψ.

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Tensor rank

Let $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$. A n-th order tensor $\varphi : \mathcal{X} \mapsto \mathbb{R}$ has rank 1 if it is the outer product of n vectors:

$$\varphi(x_1, \ldots, x_n) \quad = \quad \varphi_1(x_1) \cdot \ldots \cdot \varphi_n(x_n)$$

An example of a 3rd-order tensor that has rank 1:

$$\begin{pmatrix} (1,2) & (2,4) \\ (2,4) & (4,8) \end{pmatrix} \quad = \quad (1,2) \otimes (1,2) \otimes (1,2)$$

Rank r of a tensor ψ is the minimal number of rank-one tensors necessary to add in order to yield ψ.

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Example of a rank-2 tensor

$$\begin{pmatrix} (1,2) & (2,4) \\ (2,4) & (4,9) \end{pmatrix} = \begin{pmatrix} (1,2) & (2,4) \\ (2,4) & (4,8) \end{pmatrix} + \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (0,1) \end{pmatrix}$$

$$= (1,2) \otimes (1,2) \otimes (1,2) + (0,1) \otimes (0,1) \otimes (0,1)$$

$$\neq (a,b) \otimes (c,d) \otimes (e,f) \text{ for any } a,b,c,d,e,f \in \mathbb{R}.$$

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&= (1,2) \otimes (1,2) \otimes (1,2) \; + \; (0,1) \otimes (0,1) \otimes (0,1) \\
&\neq (a,b) \otimes (c,d) \otimes (e,f) \;\; \text{for any } a,b,c,d,e,f \in \mathbb{R}.
\end{aligned}$$

Tensors of our special interest

Definition (Indicator function)

$$I(expr) \quad = \quad \begin{cases} 1 & \text{if } expr \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

Definition (Tensor ψ_f that represent a functional dependence)

Given a function $f : \mathcal{X} \mapsto \mathcal{Y}$ the tensor defined for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ as

$$\psi_f(x, y) = I(y = f(x))$$

is a tensor ψ_f that represent a functional dependence of one variable Y on variables $X_1, \ldots, X_m$.

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Example of a tensor representing functional dependence

- $\mathcal{X}_i = \{0, 1\}$ for $i = 1, 2$,
- $f(x_1, x_2) = x_1 + x_2$, and
- $\mathcal{Y} = \{0, 1, 2\}$.

$$
\begin{aligned}
\psi_f(x_1, x_2, y) &= I(y = x_1 + x_2) \\
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Results

- $\mathcal{X}_i = [0, a_i]$ for $i = 1, \ldots, m$
- function $f : \mathcal{X} \mapsto \mathcal{Y}$ and
- $\psi_f : \mathcal{X} \times \mathcal{Y} \mapsto \{0, 1\}$ be a tensor representing the functional dependence given by f.

Lemma (Rank lower bound)

$rank(\psi_f) \geq |\mathcal{Y}|$.

Lemma (Minimimum, maximum)

If $f(x) = \min\{x_1, \ldots, x_m\}$ *or* $\max\{x_1, \ldots, x_m\}$ *then* $rank(\psi_f) = |\mathcal{Y}|$.

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Conclusions

- Standard methods for inference in Bayesian networks may become intractable if a variable has many parents.
- Tensor rank-one decomposition open new possibilities for more compact inference structures.
- We have shown how (minimal) tensor rank-one decompositions can be found for some special tensors representing functional dependence.
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