Using imsets for learning Bayesian networks

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Conditional Independence

Example (CI model)

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- Seeing the length of hair of a person will tell us more about his/her gender and conversely. It means, the value of G is dependent on the value of H.
- Knowing more about the gender will focus our belief on his/her stature - S is dependent on G and (through G) also on H.
- Nevertheless, if we know the gender of a person then length of hair of that person gives us no extra clue on his/her stature - H is independent of S given G.

Conditional Independence Statements

Definition (CI statement)

Let A, B, C be pairwise disjoint subsets of a set of variables N. Then the statement "A is conditionally independent of B given C" is a CI statement (over N), written as $I(A, B, C)$.

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Example (CI statement)

In Example 1 we have indicated only one CI statement, $I(H, S, G)$. On the other hand, we have indicated two dependence statements, namely $\neg I(G, H) = \neg I(G, H, \emptyset)$ and $\neg I(S, G)$.

Conditional Independence (CI) model

Definition (CI in PDs)

Let P be a discrete probability distribution over N. Given any $A \subseteq N$, let

- $\mathbf{x}_A$ denote a configuration of values of variables $\mathbf{X}_A = \{X_i\}_{i \in A}$ and
- for $B \subseteq N \setminus A$ let $P(\mathbf{x}_A \mid \mathbf{x}_B)$ denote the conditional probability of $\mathbf{X}_A = \mathbf{x}_A$ given $\mathbf{X}_B = \mathbf{x}_B$.

The CI statement $I(A, B, C)$ is induced by probability distribution P over N if for all $\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C$ such that $P(\mathbf{x}_C) > 0$

$$P(\mathbf{x}_A, \mathbf{x}_B \mid \mathbf{x}_C) \;=\; P(\mathbf{x}_A \mid \mathbf{x}_C) \cdot P(\mathbf{x}_B \mid \mathbf{x}_C) \;.$$

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What CI-statements are represented by a DAG?

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Two nodes a and b in a DAG G are d-separated by a set C if for all paths between a and b there is a node c ($c \neq a$ and $c \neq b$) such that either:

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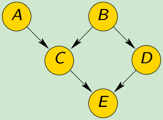

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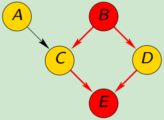

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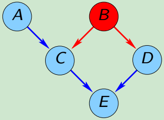
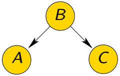
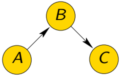


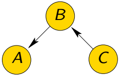
$$\neg I(A, D, \{B, E\})$$

Equivalence classes of Bayesian networks

What independence statements are represented by these three models?







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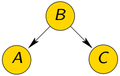

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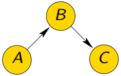

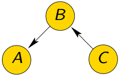

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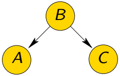

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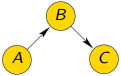

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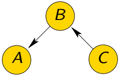

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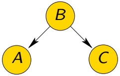

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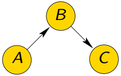

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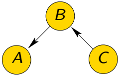

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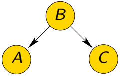

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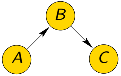

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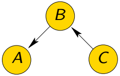

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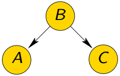

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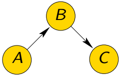

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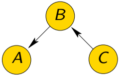

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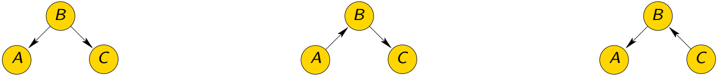

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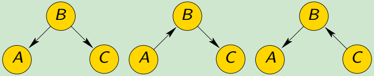

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An immorality in a DAG G is an induced subgraph of G for a set $\{A, B, C\}$, where A, B, C are distinct nodes of G such that there are edges $A \rightarrow C$ and $B \rightarrow C$ and there is no edge between A and B in G.

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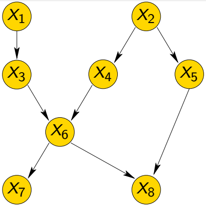

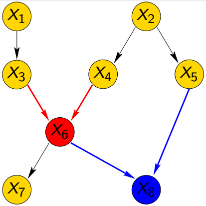

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An underlaying graph of a DAG is the undirected graph that has the same set of nodes and all directed edges $A \rightarrow B$ are replaced by undirected edges $A - B$.

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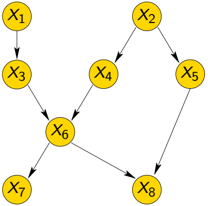

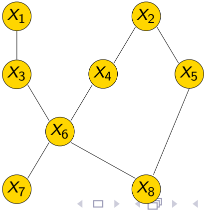

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Theorem

Bayesian networks belong to the same equivalence class iff they have the same underlaying graph and the same set of immoralities.

Essential graphs

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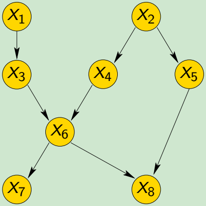

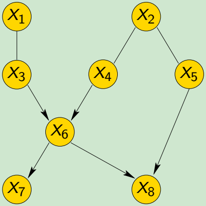

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Function $m : \mathcal{P}(N) \mapsto \mathbb{N}$ is sometimes called multiset. Thus, imset is an abbreviation from **I**nteger valued **M**ulti**SET**. Studený (2001)

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0	0	+1	0	-1	-1	0	+1

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Using the convention we will write

$$u = \delta_{\{b\}} - \delta_{\{a,b\}} - \delta_{\{b,c\}} + \delta_{\{a,b,c\}}$$

Elementary imset

Definition (Elementary imset)

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Standard imset is another uniquely determined representative of an equivalence class of Bayesian networks.

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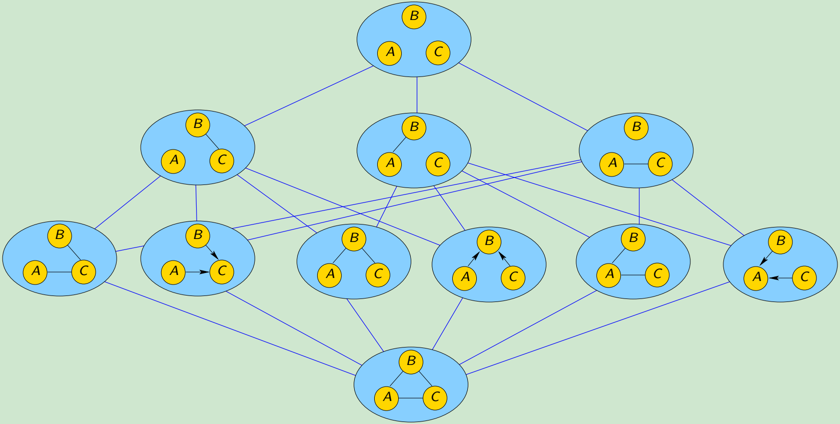
We say that $\mathcal{M}_L$ is an upper neighbor of $\mathcal{M}_K$ or, dually, that $\mathcal{M}_K$ is a lower neighbor of $\mathcal{M}_L$.

Search space for models of three variables

Example



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The likelihood of D given G is the probability of data D being generated from the Bayesian network model with the structure given by directed acyclic graph G and representing joint probability distribution P is

$$P(D|G) = \prod_{m=1}^{M} P(\mathbf{X} = \mathbf{x}^m)$$

Scores

Lemma (Maximum loglikelihood)

The maximum log-likelihood for a given Bayesian network with graph G is

$$MLL(G|D) = \sum_{i=1}^{N} \sum_{k=1}^{r(i)} \sum_{j=1}^{q(i,G)} N(i,j,k) \log \frac{N(i,j,k)}{N(i,j)}$$

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Definition (Bayesian Information Criterion)

$$BIC(G|D) = MLL(G|D) - \frac{\log M}{2} d(G)$$

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Theorem

In the limit of large datasets, if the CI-statements that hold in the dataset are exactly those of a Bayesian network then the algorithm terminates in the essential graph of this Bayesian network.

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- From each equivalence class no more than one model is generated.

Experiments with the imset version of GES

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The code is freely available from: http://www.utia.cz/vomlel/imset

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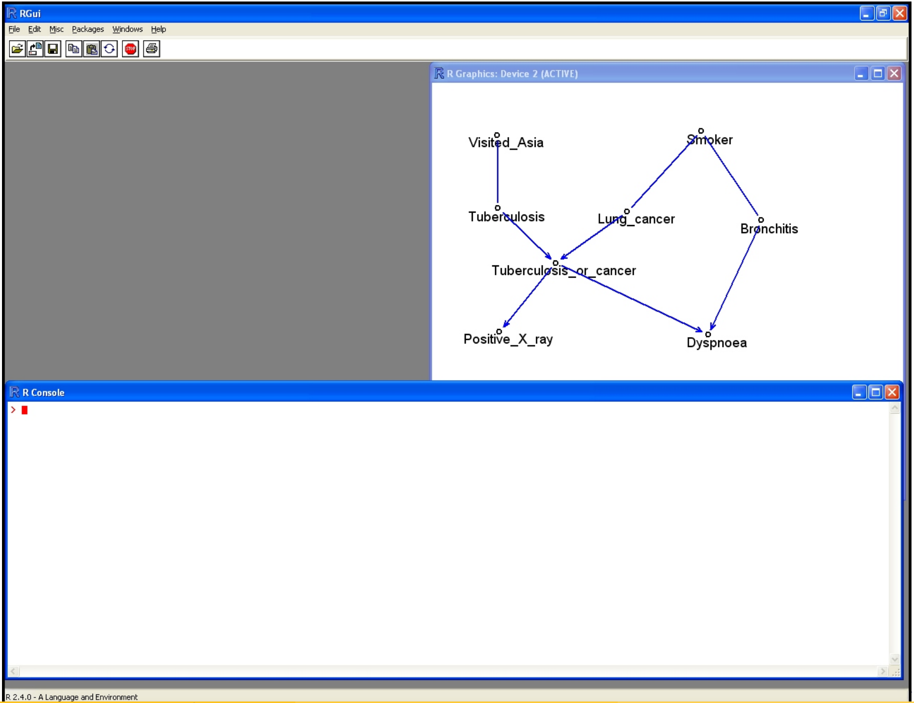
For the alarm model there are more than $2^{\frac{37 \cdot 36}{2}} = 2^{666}$ essential graphs.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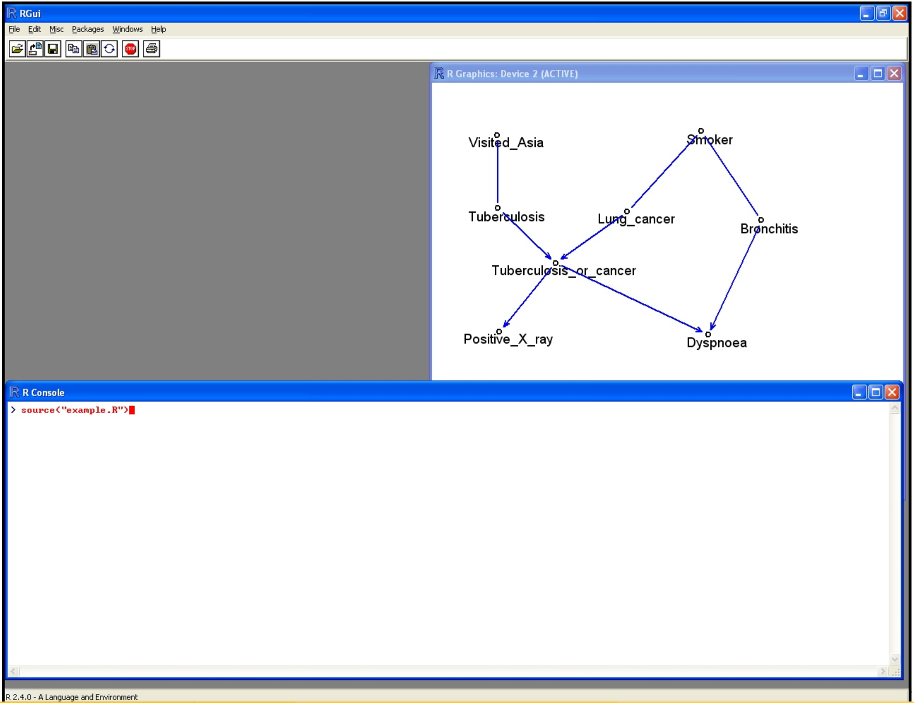
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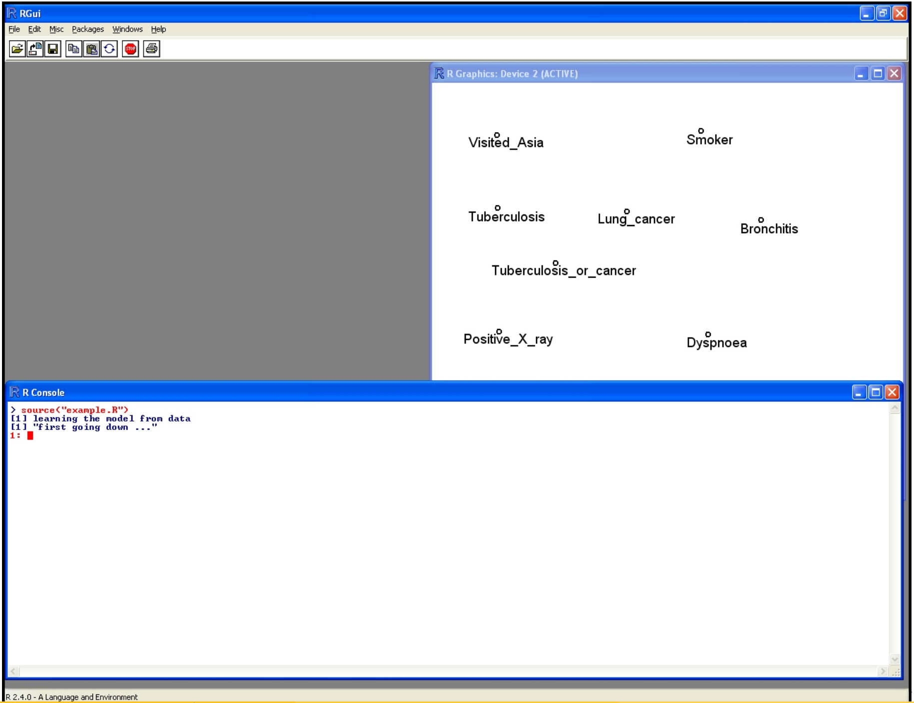
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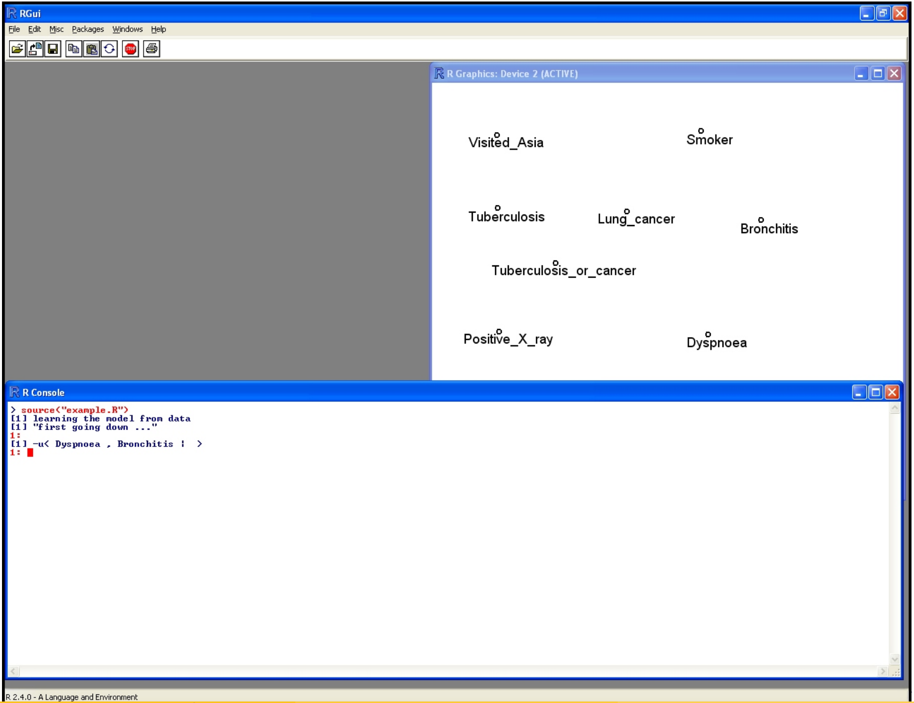
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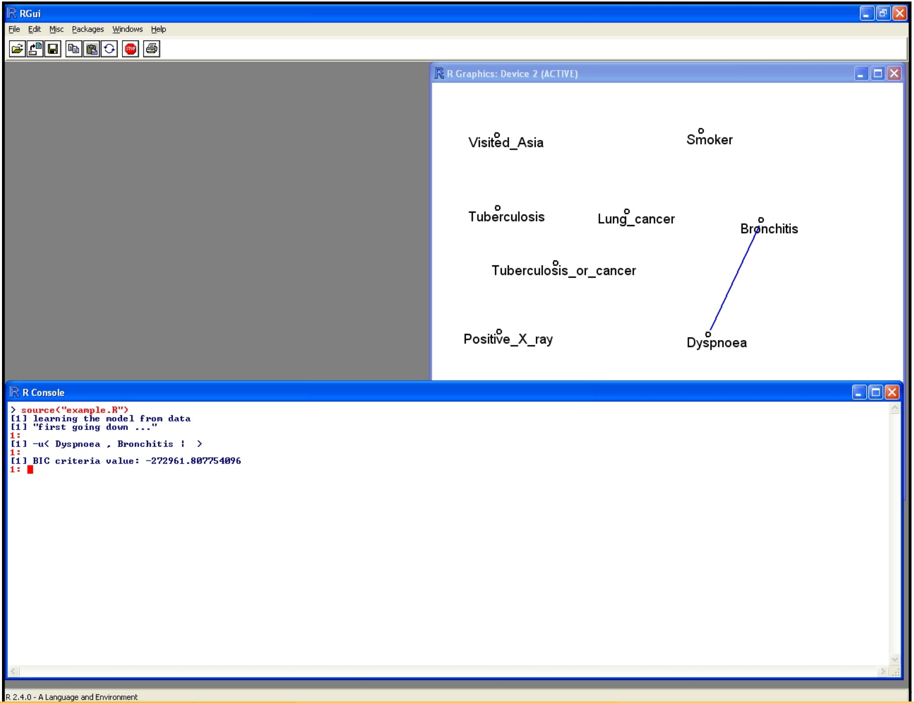
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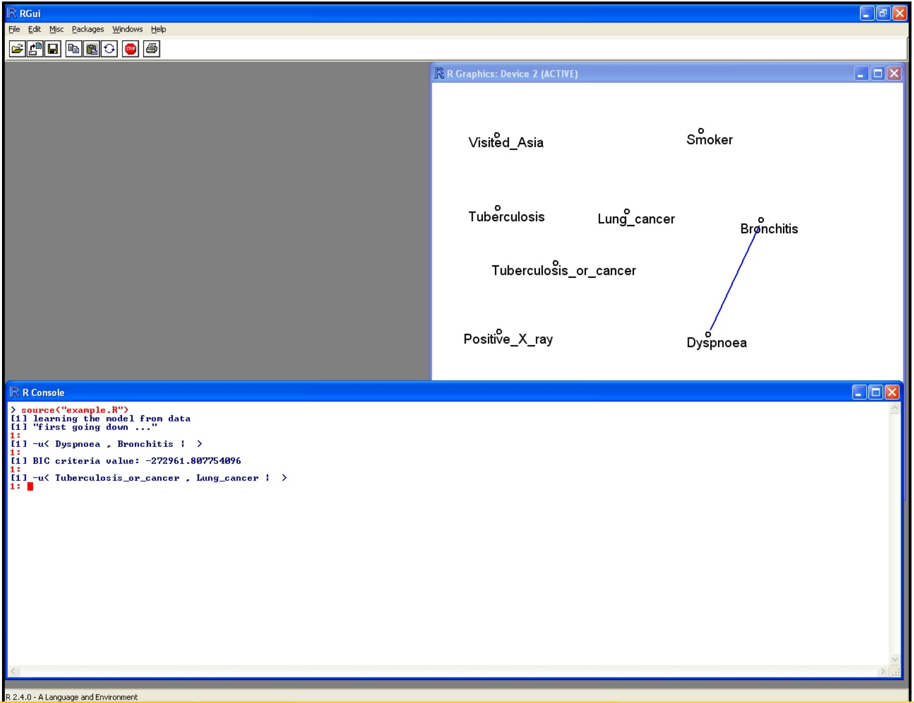
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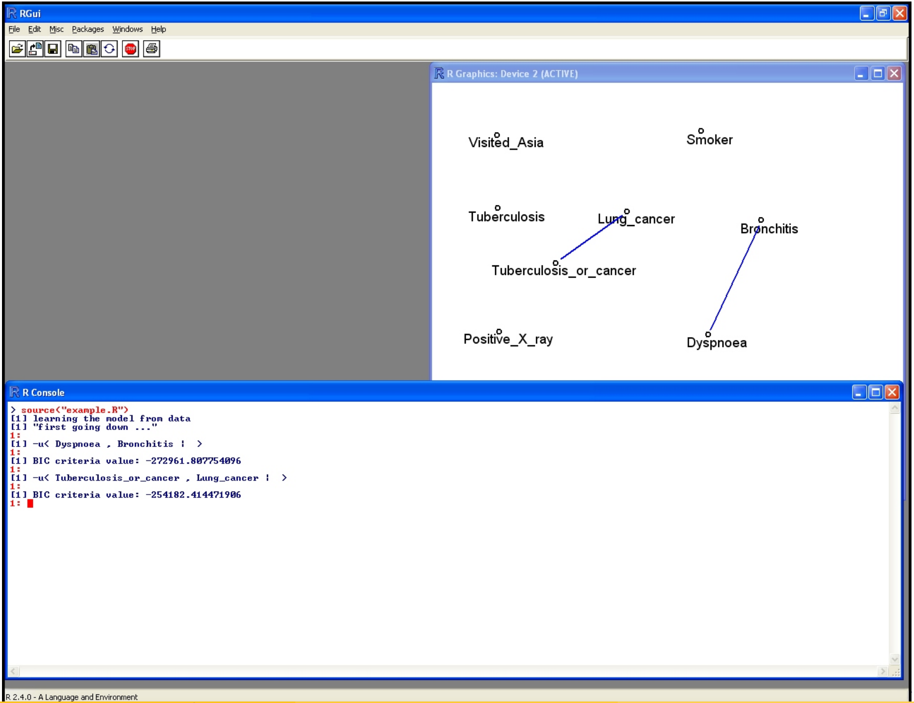
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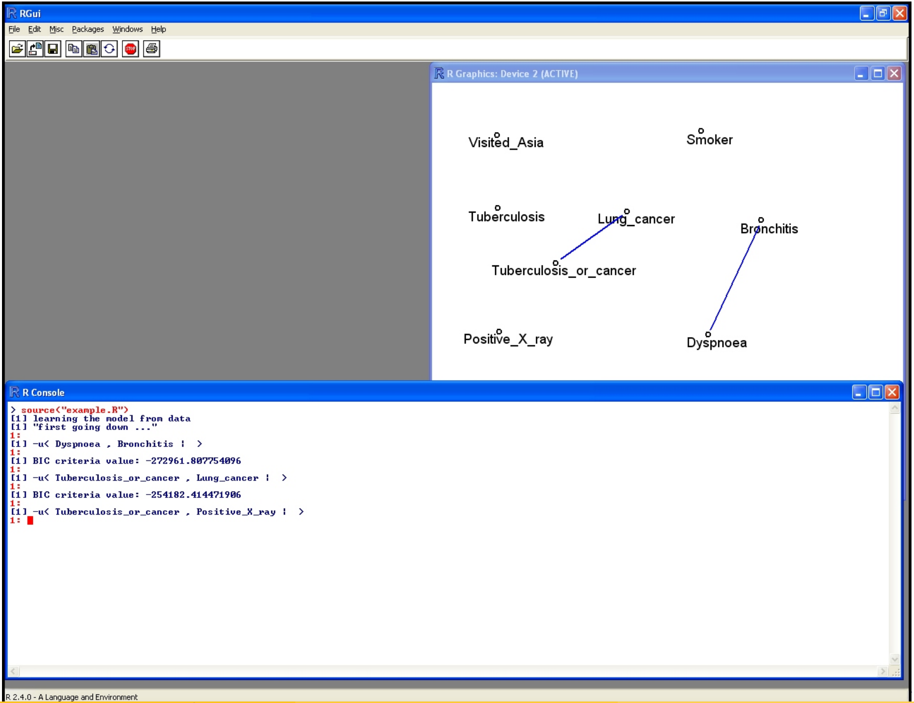
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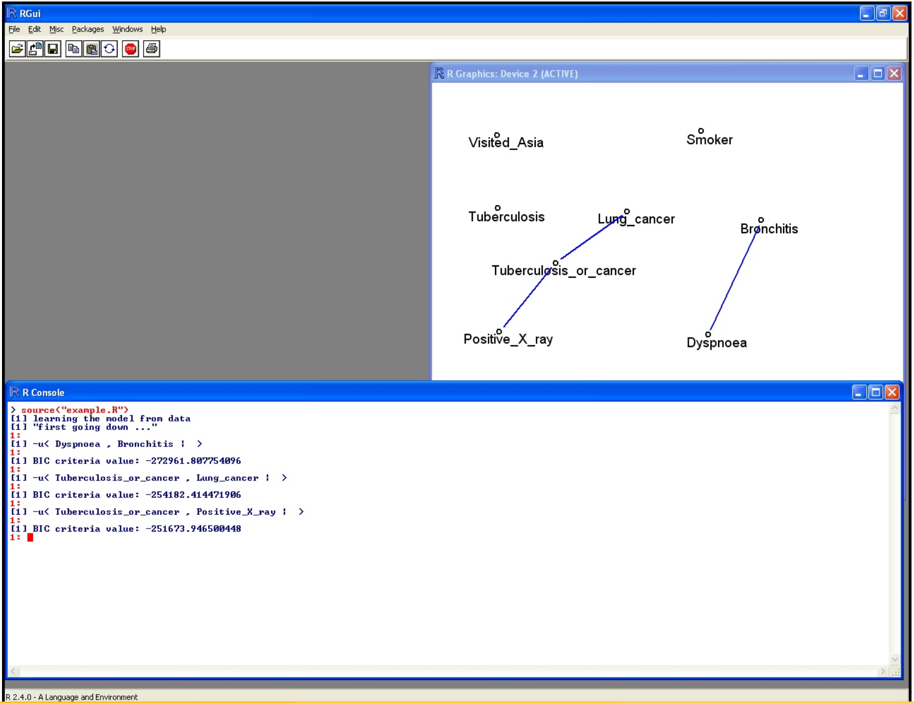
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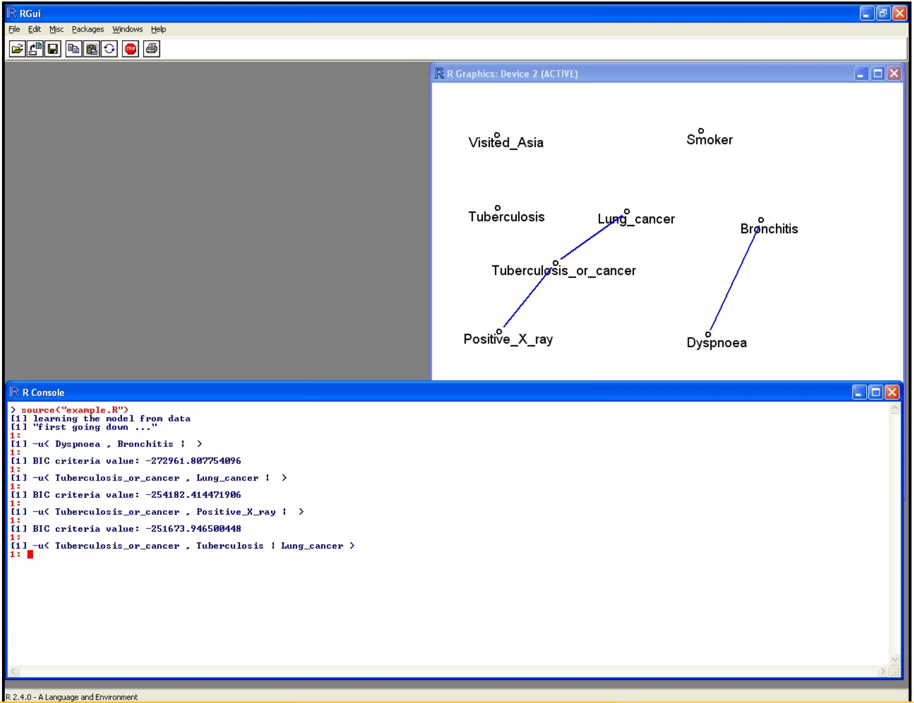
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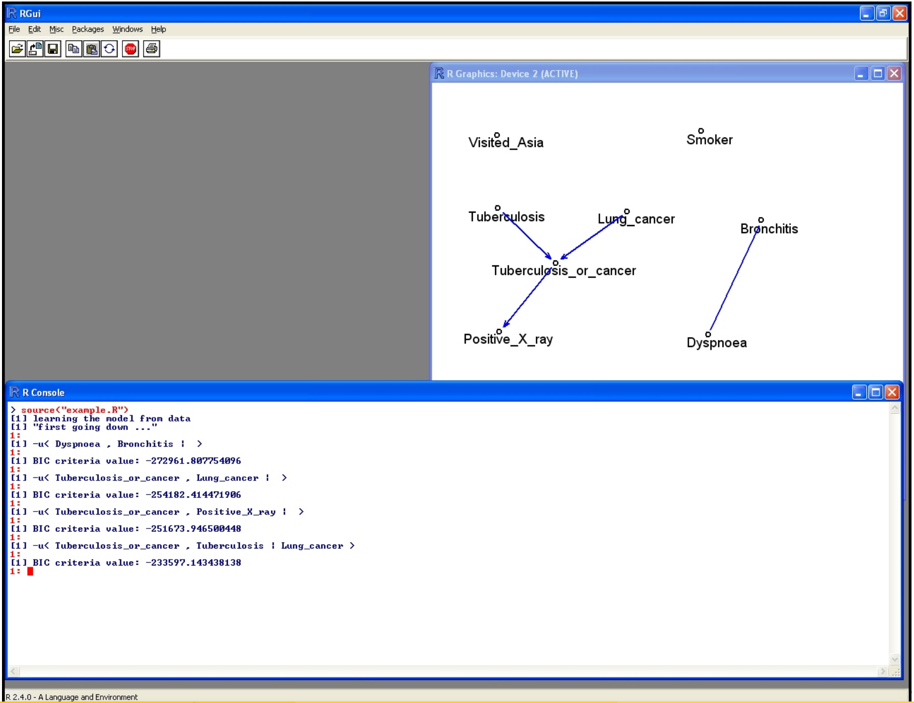
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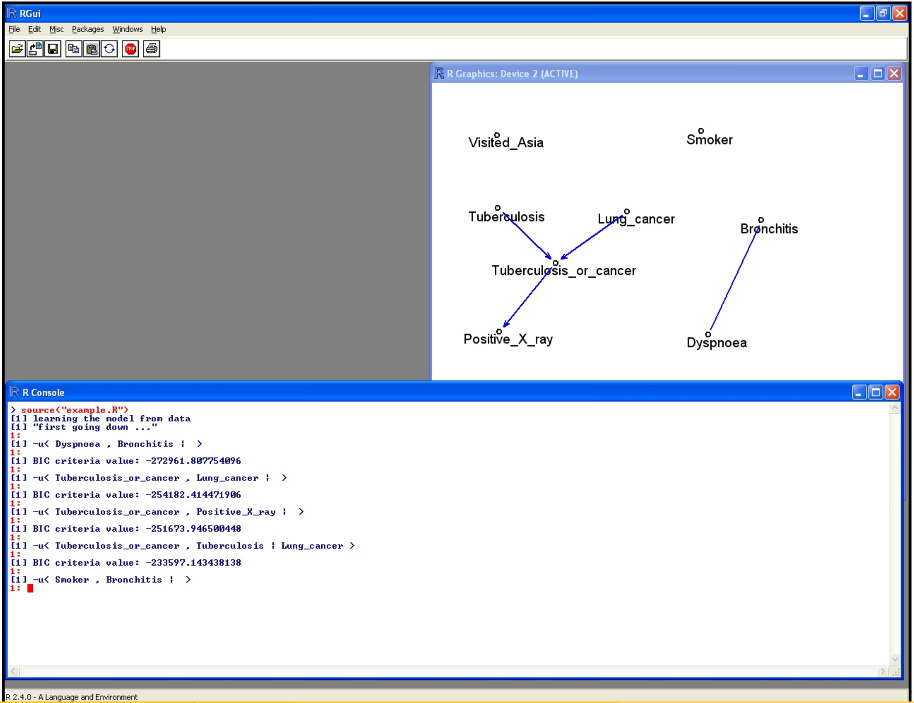
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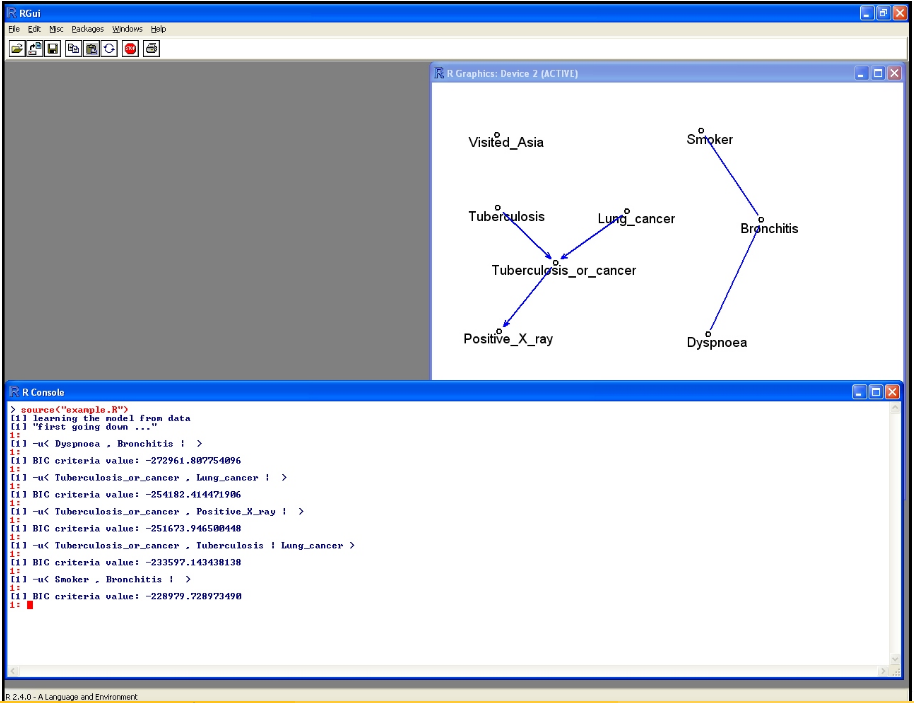
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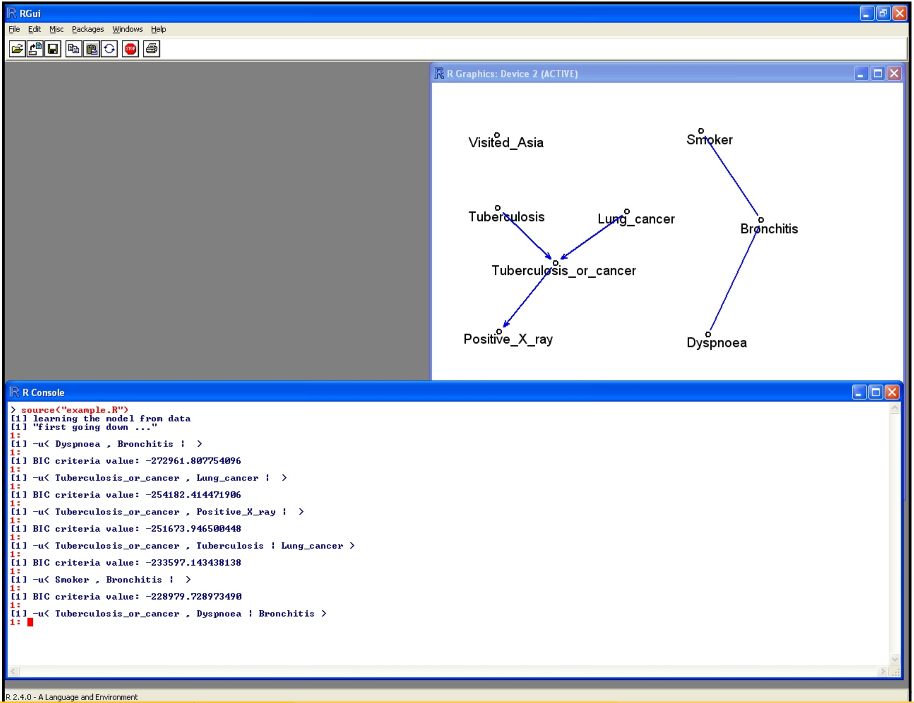
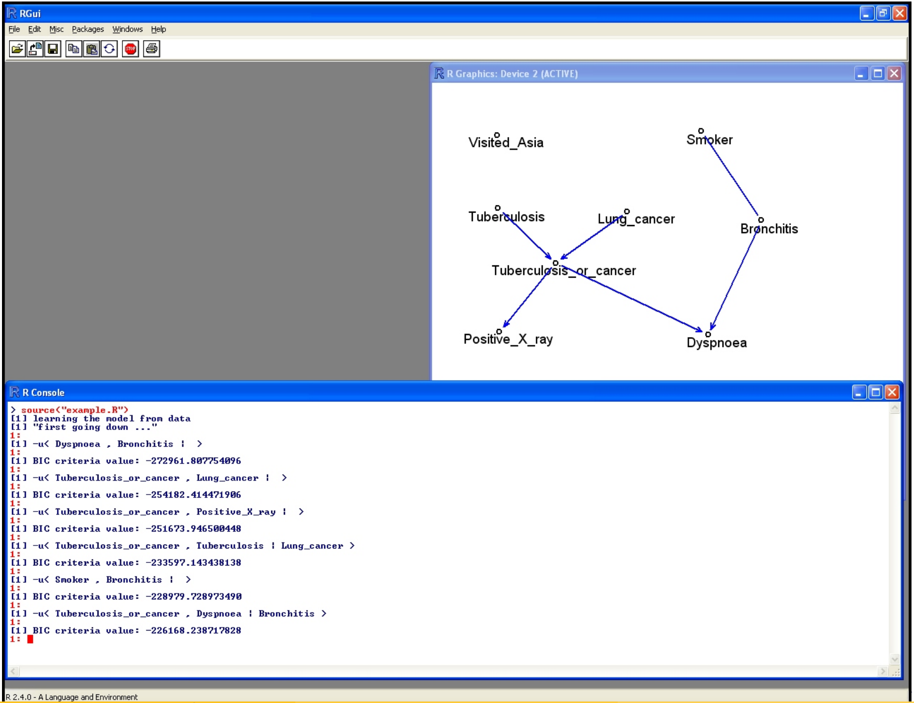
R 2.4.0 - A Language and Environment


RGui
File Edit Misc Packages Windows Help
R Graphics: Device 2 (ACTIVE)
Visited_Asia
Smoker
Tuberculosis
Lung_cancer
Bronchitis
Tuberculosis_or_cancer
Positive_X_ray
Dyspnoea
R Console
> source("example.R")
[1] learning the model from data
[1] "first going down ..."
1:
[1] -u< Dyspnoea , Bronchitis | >
1:
[1] BIC criteria value: -272961.807754096
1:
[1] -u< Tuberculosis_or_cancer , Lung_cancer | >
1:
[1] BIC criteria value: -254182.414471906
1:
[1] -u< Tuberculosis_or_cancer , Positive_X_ray | >
1:
[1] BIC criteria value: -251673.946500448
1:
[1] -u< Tuberculosis_or_cancer , Tuberculosis | Lung_cancer >
1:
[1] BIC criteria value: -233597.143438138
1:
[1] -u< Smoker , Bronchitis | >
1:
[1] BIC criteria value: -228979.728973490
1:
[1] -u< Tuberculosis_or_cancer , Dyspnoea | Bronchitis >
1:
R 2.4.0 - A Language and Environment


RGui
File Edit Misc Packages Windows Help
R Graphics: Device 2 (ACTIVE)
Visited_Asia
Smoker
Tuberculosis
Lung_cancer
Bronchitis
Tuberculosis_or_cancer
Positive_X_ray
Dyspnoea
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[1] -u< Tuberculosis_or_cancer , Positive_X_ray | >
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[1] BIC criteria value: -251673.946500448
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[1] -u< Tuberculosis_or_cancer , Tuberculosis | Lung_cancer >
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1:
[1] -u< Smoker , Bronchitis | >
1:
[1] BIC criteria value: -228979.728973490
1:
[1] -u< Tuberculosis_or_cancer , Dyspnoea | Bronchitis >
1:
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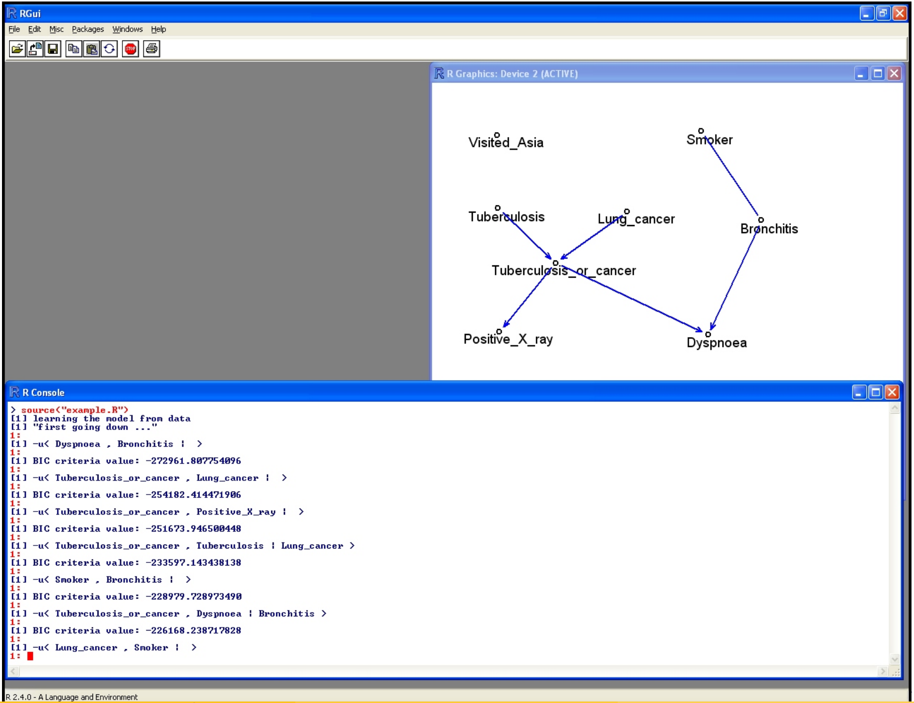
1:
R 2.4.0 - A Language and Environment


RGui
File Edit Misc Packages Windows Help
R Graphics: Device 2 (ACTIVE)
Visited_Asia
Smoker
Tuberculosis
Lung_cancer
Bronchitis
Tuberculosis_or_cancer
Positive_X_ray
Dyspnoea
R Console
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[1] -u< Tuberculosis_or_cancer , Positive_X_ray | >
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[1] BIC criteria value: -251673.946500448
1:
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1:
[1] BIC criteria value: -233597.143438138
1:
[1] -u< Smoker , Bronchitis | >
1:
[1] BIC criteria value: -228979.728973490
1:
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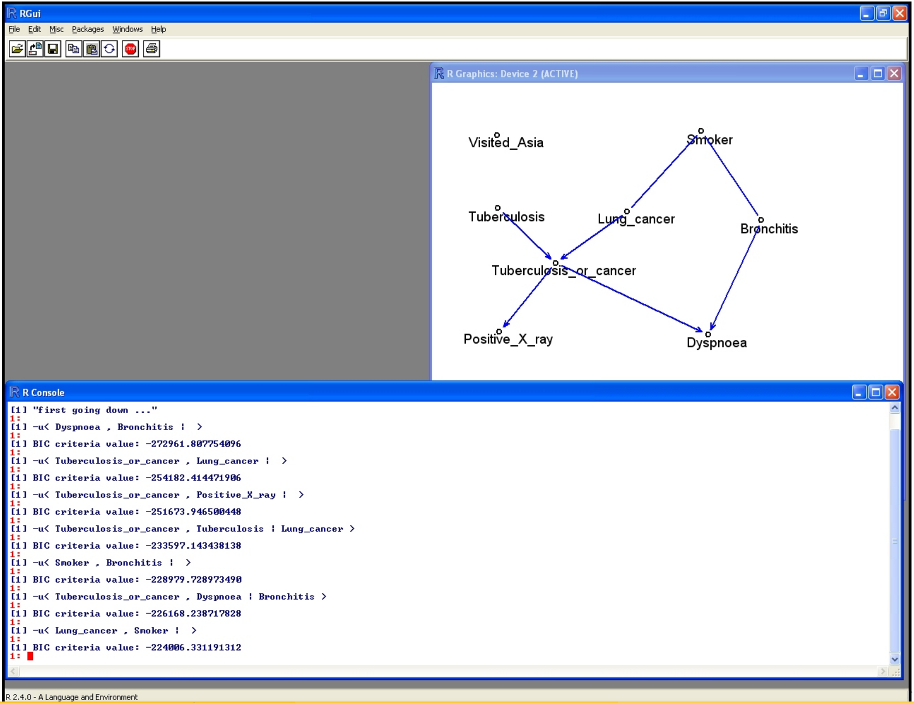
[1] -u< Lung_cancer , Smoker | >
1:
R 2.4.0 - A Language and Environment


RGui
File Edit Misc Packages Windows Help
R Graphics: Device 2 (ACTIVE)
Visited_Asia
Smoker
Tuberculosis
Lung_cancer
Bronchitis
Tuberculosis_or_cancer
Positive_X_ray
Dyspnoea
R Console
[1] "first going down ..."
1:
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[1] BIC criteria value: -272961.807754096
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[1] -u< Tuberculosis_or_cancer , Lung_cancer | >
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[1] -u< Tuberculosis_or_cancer , Positive_X_ray | >
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1:
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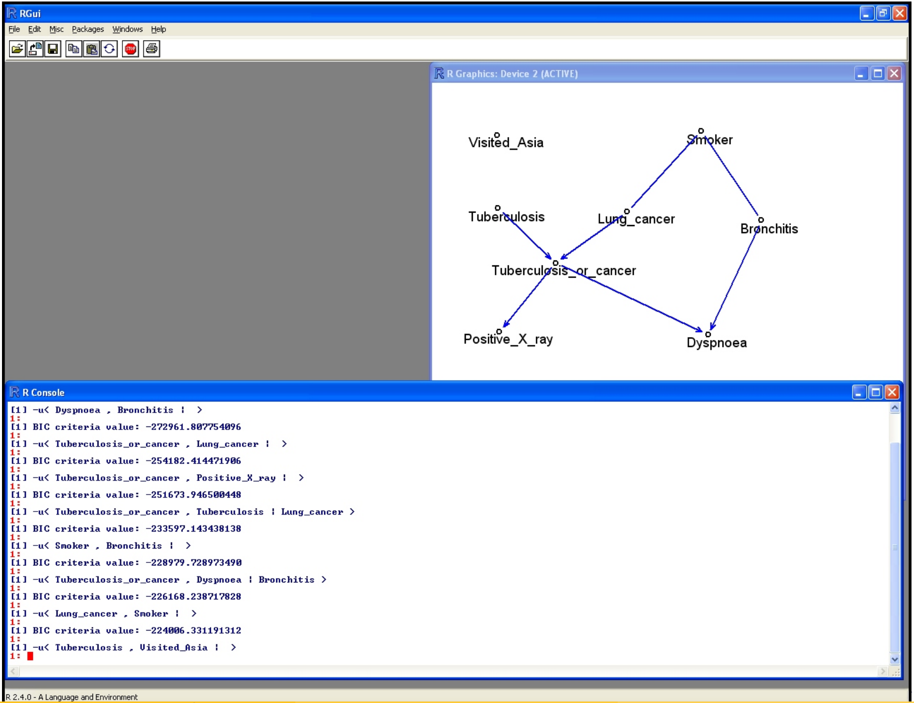
1:
[1] BIC criteria value: -224006.331191312
1:
R 2.4.0 - A Language and Environment


RGui
File Edit Misc Packages Windows Help
R Graphics: Device 2 (ACTIVE)
Visited_Asia
Smoker
Tuberculosis
Lung_cancer
Bronchitis
Tuberculosis_or_cancer
Positive_X_ray
Dyspnoea
R Console
[1] -u< Dyspnoea , Bronchitis | >
1:
[1] BIC criteria value: -272961.807754096
1:
[1] -u< Tuberculosis_or_cancer , Lung_cancer | >
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[1] BIC criteria value: -254182.414471906
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[1] -u< Tuberculosis_or_cancer , Positive_X_ray | >
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[1] BIC criteria value: -251673.946500448
1:
[1] -u< Tuberculosis_or_cancer , Tuberculosis | Lung_cancer >
1:
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[1] -u< Smoker , Bronchitis | >
1:
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1:
[1] -u< Tuberculosis_or_cancer , Dyspnoea | Bronchitis >
1:
[1] BIC criteria value: -226168.238717828
1:
[1] -u< Lung_cancer , Smoker | >
1:
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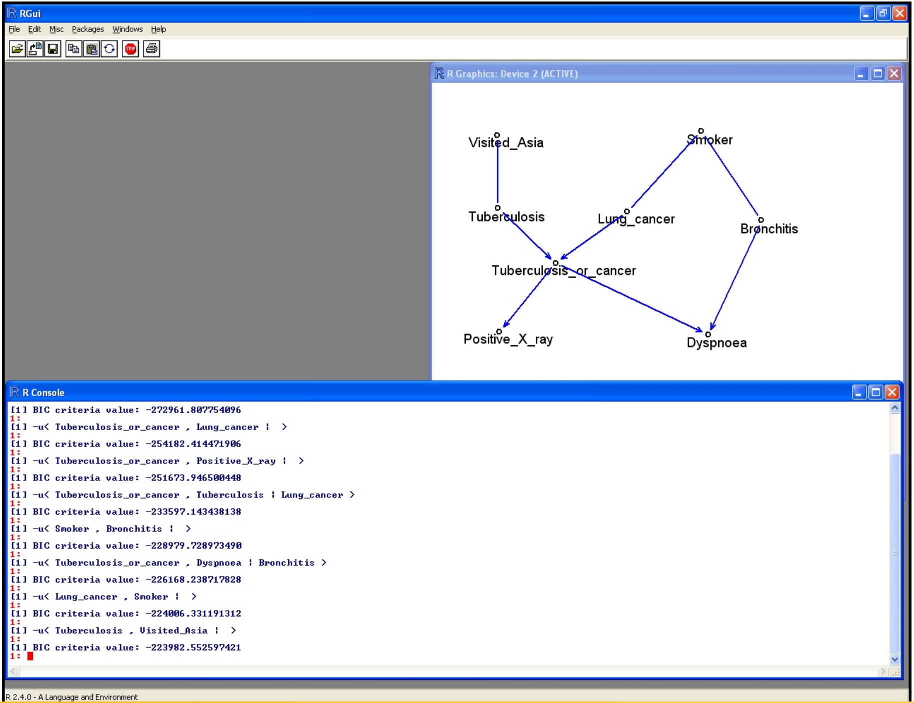
1:
[1] -u< Tuberculosis , Visited_Asia | >
1:
R 2.4.0 - A Language and Environment


RGui
File Edit Misc Packages Windows Help
R Graphics: Device 2 (ACTIVE)
Visited_Asia
Smoker
Tuberculosis
Lung_cancer
Bronchitis
Tuberculosis_or_cancer
Positive_X_ray
Dyspnoea
R Console
[1] BIC criteria value: -272961.807754096
1:
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[1] -u< Tuberculosis_or_cancer , Tuberculosis | Lung_cancer >
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[1] -u< Smoker , Bronchitis | >
1:
[1] BIC criteria value: -228979.728973490
1:
[1] -u< Tuberculosis_or_cancer , Dyspnoea | Bronchitis >
1:
[1] BIC criteria value: -226168.238717828
1:
[1] -u< Lung_cancer , Smoker | >
1:
[1] BIC criteria value: -224006.331191312
1:
[1] -u< Tuberculosis , Visited_Asia | >
1:
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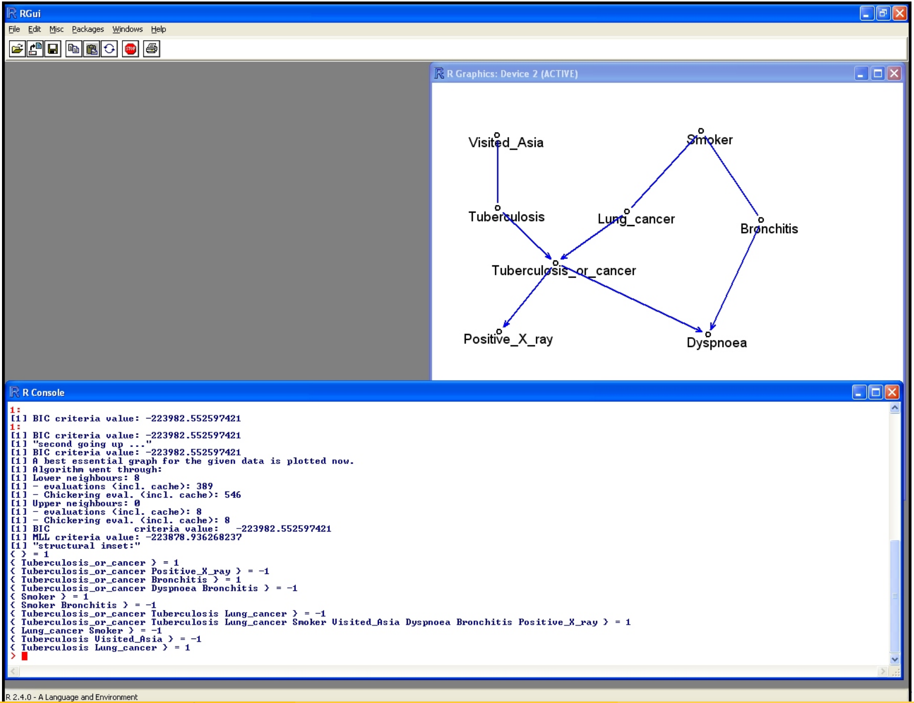
1:
R 2.4.0 - A Language and Environment


RGui
File Edit Misc Packages Windows Help
R Graphics: Device 2 (ACTIVE)
Visited_Asia
Smoker
Tuberculosis
Lung_cancer
Bronchitis
Tuberculosis_or_cancer
Positive_X_ray
Dyspnoea
R Console
1:
[1] BIC criteria value: -223982.552597421
1:
[1] BIC criteria value: -223982.552597421
[1] "second going up ..."
[1] BIC criteria value: -223982.552597421
[1] A best essential graph for the given data is plotted now.
[1] Algorithm went through:
[1] Lower neighbours: 8
[1] - evaluations (incl. cache): 389
[1] - Chickering eval. (incl. cache): 546
[1] Upper neighbours: 0
[1] - evaluations (incl. cache): 8
[1] - Chickering eval. (incl. cache): 8
[1] BIC criteria value: -223982.552597421
[1] MLL criteria value: -223878.936268237
[1] "structural imset:"
{ } = 1
{ Tuberculosis_or_cancer } = 1
{ Tuberculosis_or_cancer Positive_X_ray } = -1
{ Tuberculosis_or_cancer Bronchitis } = 1
{ Tuberculosis_or_cancer Dyspnoea Bronchitis } = -1
{ Smoker } = 1
{ Smoker Bronchitis } = -1
{ Tuberculosis_or_cancer Tuberculosis Lung_cancer } = -1
{ Tuberculosis_or_cancer Tuberculosis Lung_cancer Smoker Visited_Asia Dyspnoea Bronchitis Positive_X_ray } = 1
{ Lung_cancer Smoker } = -1
{ Tuberculosis Visited_Asia } = -1
{ Tuberculosis Lung_cancer } = 1
>
R 2.4.0 - A Language and Environment