Exploiting Functional Dependence in Bayesian Network Inference

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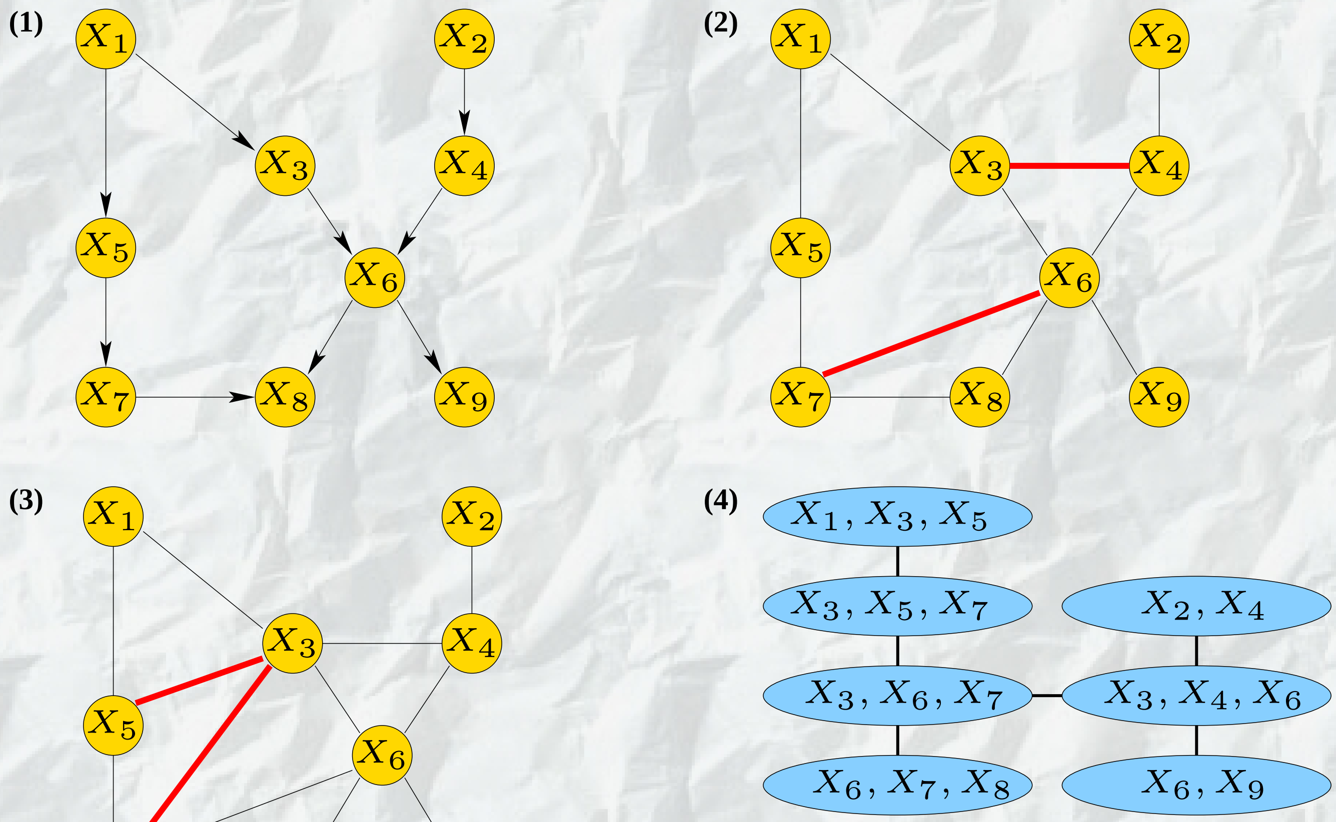
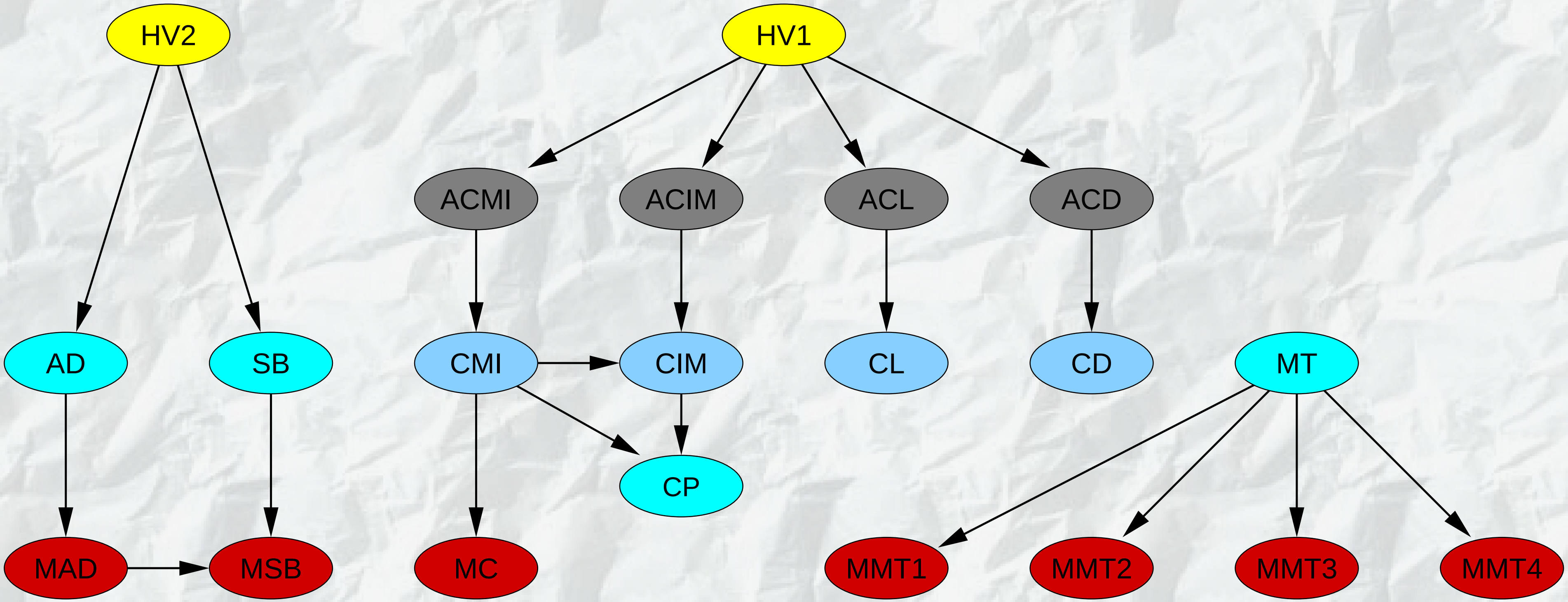
Bayesian networks: junction tree propagation



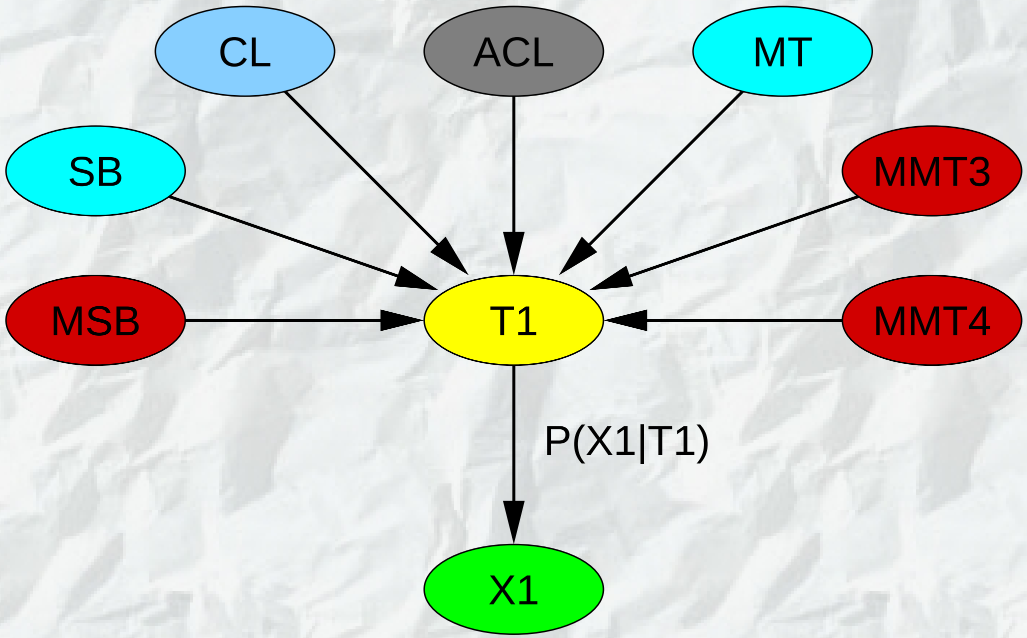
Original model of a student



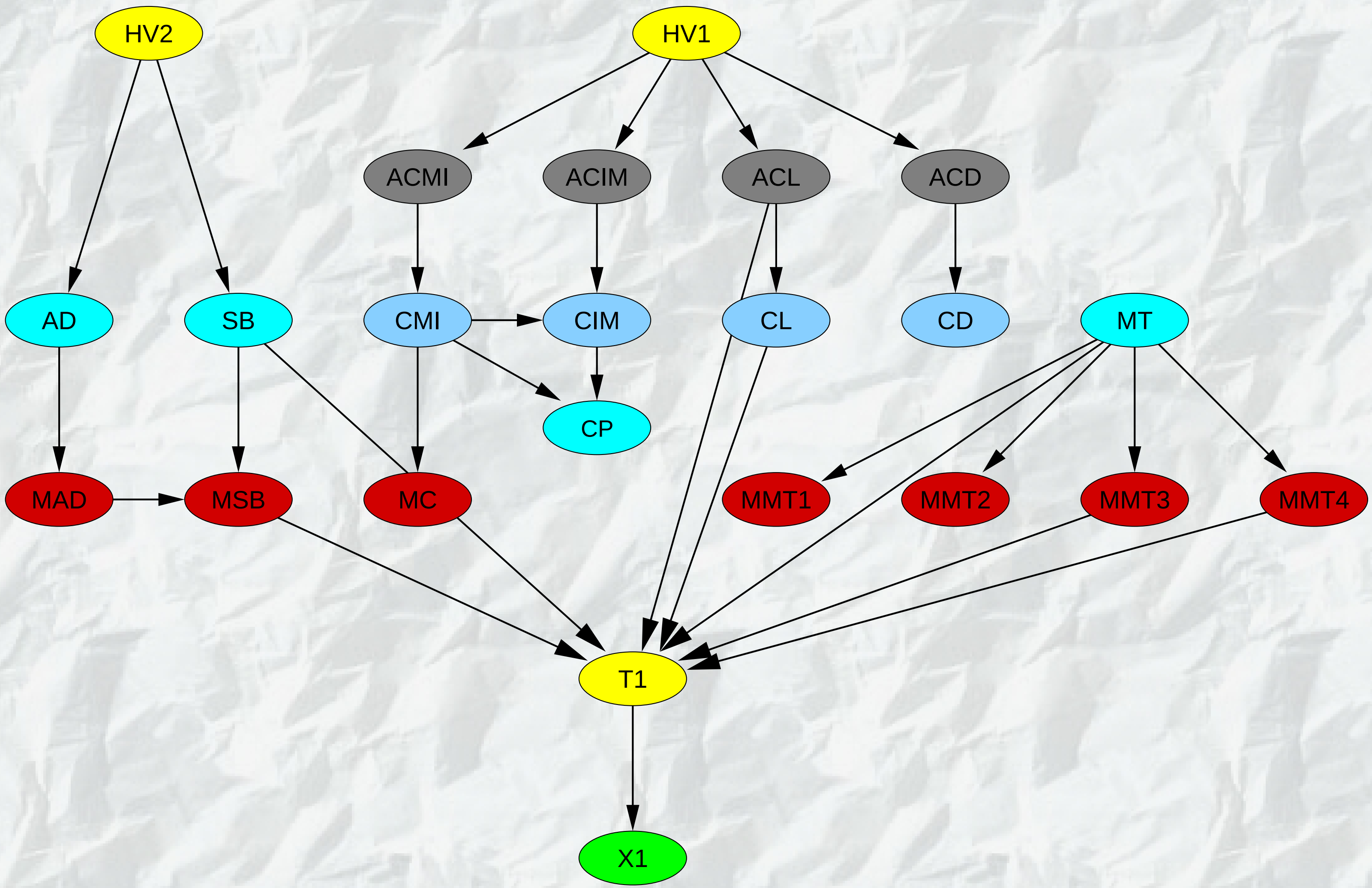
Evidence model of a task

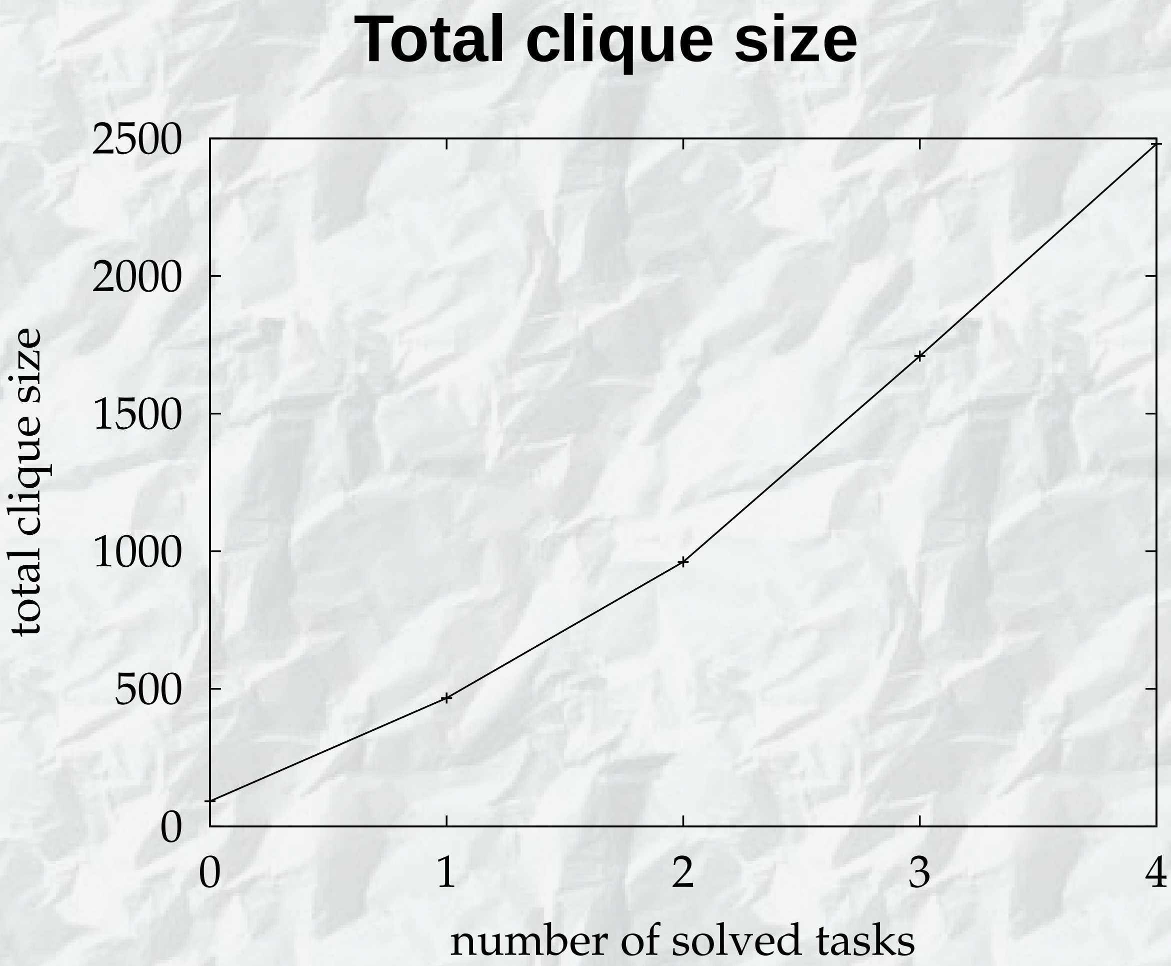
$$\left(\frac{3}{4}\cdot\frac{5}{6}\right)-\frac{1}{8}=\frac{15}{24}-\frac{1}{8}=\frac{5}{8}-\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

$$T_1 \quad \Leftrightarrow \quad MT \,\&\, CL \,\&\, ACL \,\&\, SB \,\&\, \neg MMT3 \,\&\, \neg MMT4 \,\&\, \neg MSB$$



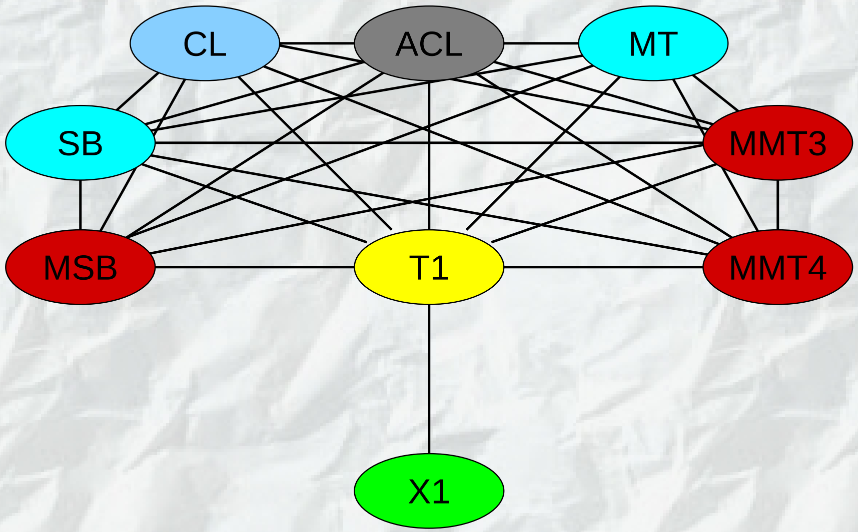
Original model + one evidence model



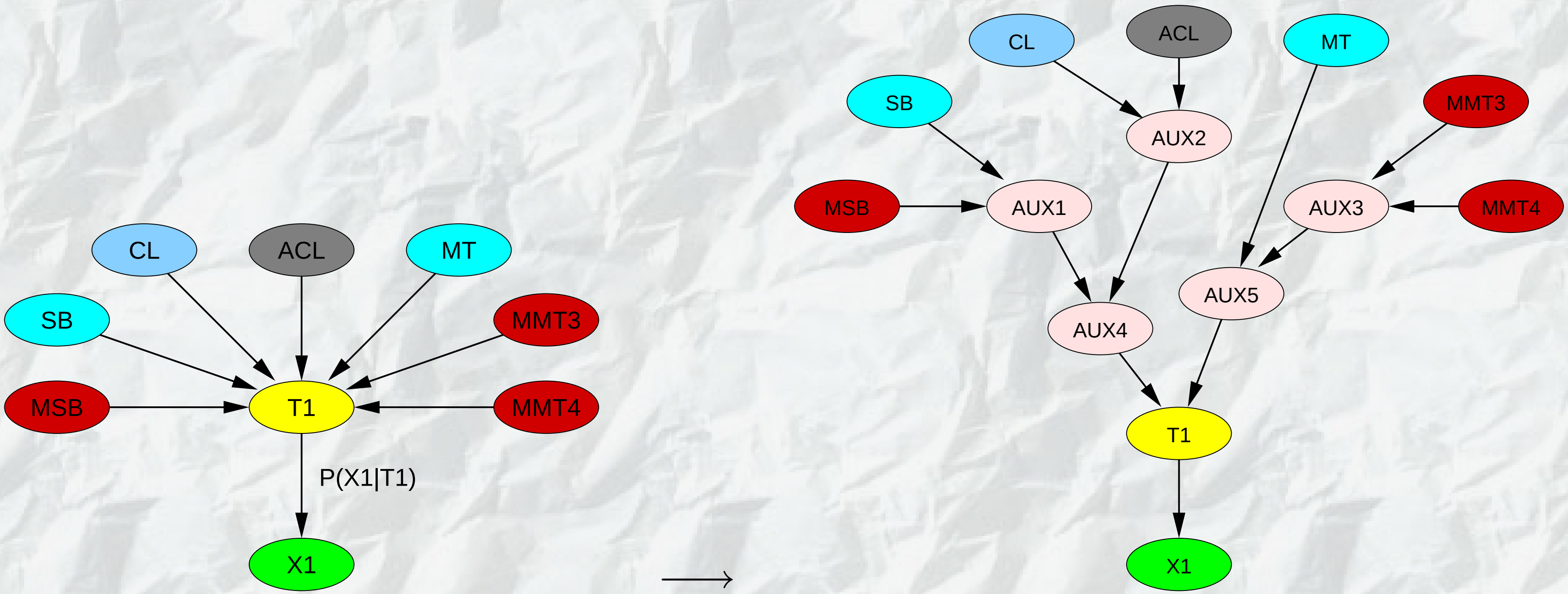

Total clique size
total clique size
number of solved tasks
0
500
1000
1500
2000
2500
0
1
2
3
4

This is the problem ...

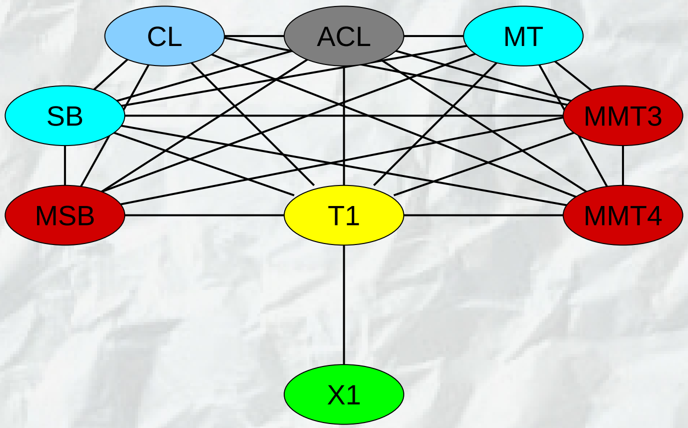
... all parents get inter-connected.



Parent divorcing (Olesen et al., 1989)



Parent divorcing



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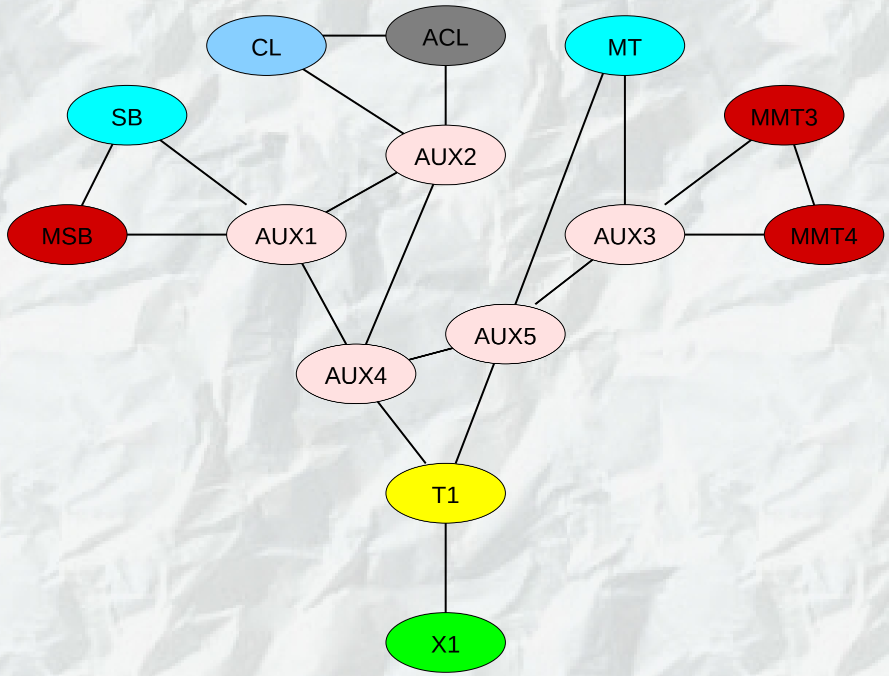
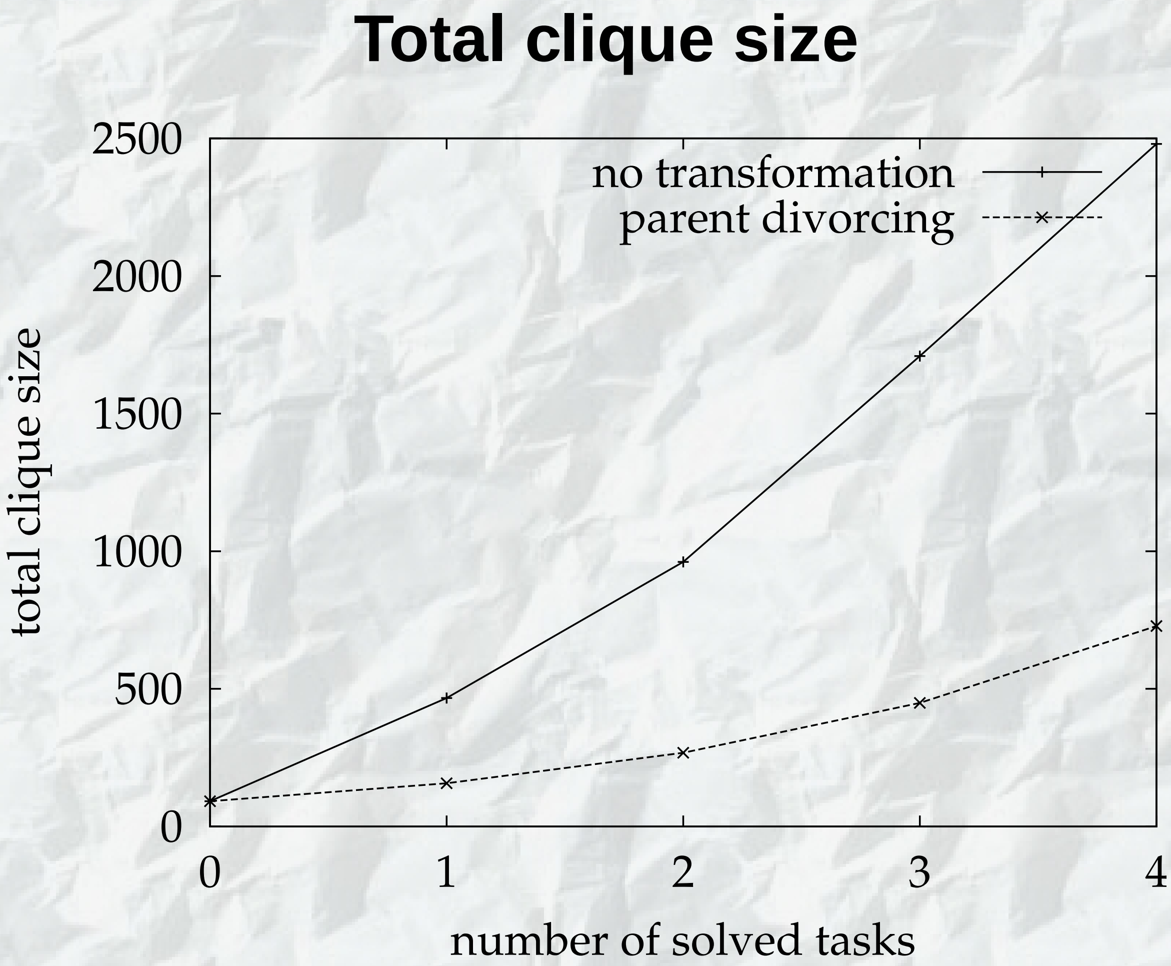
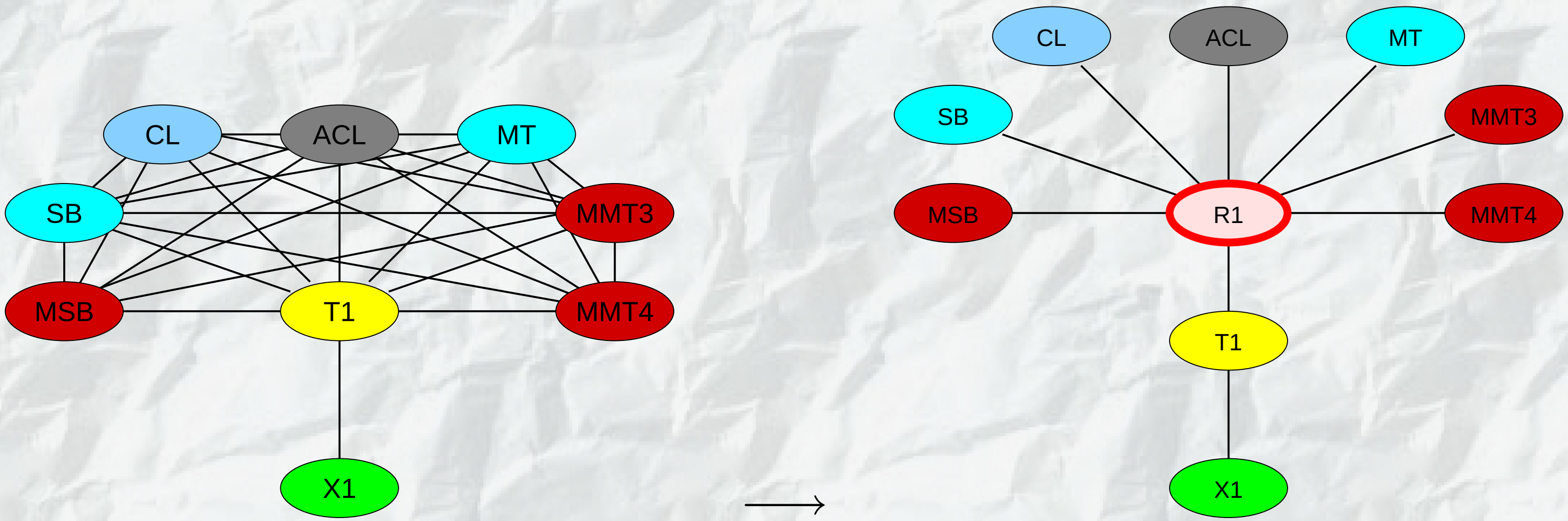


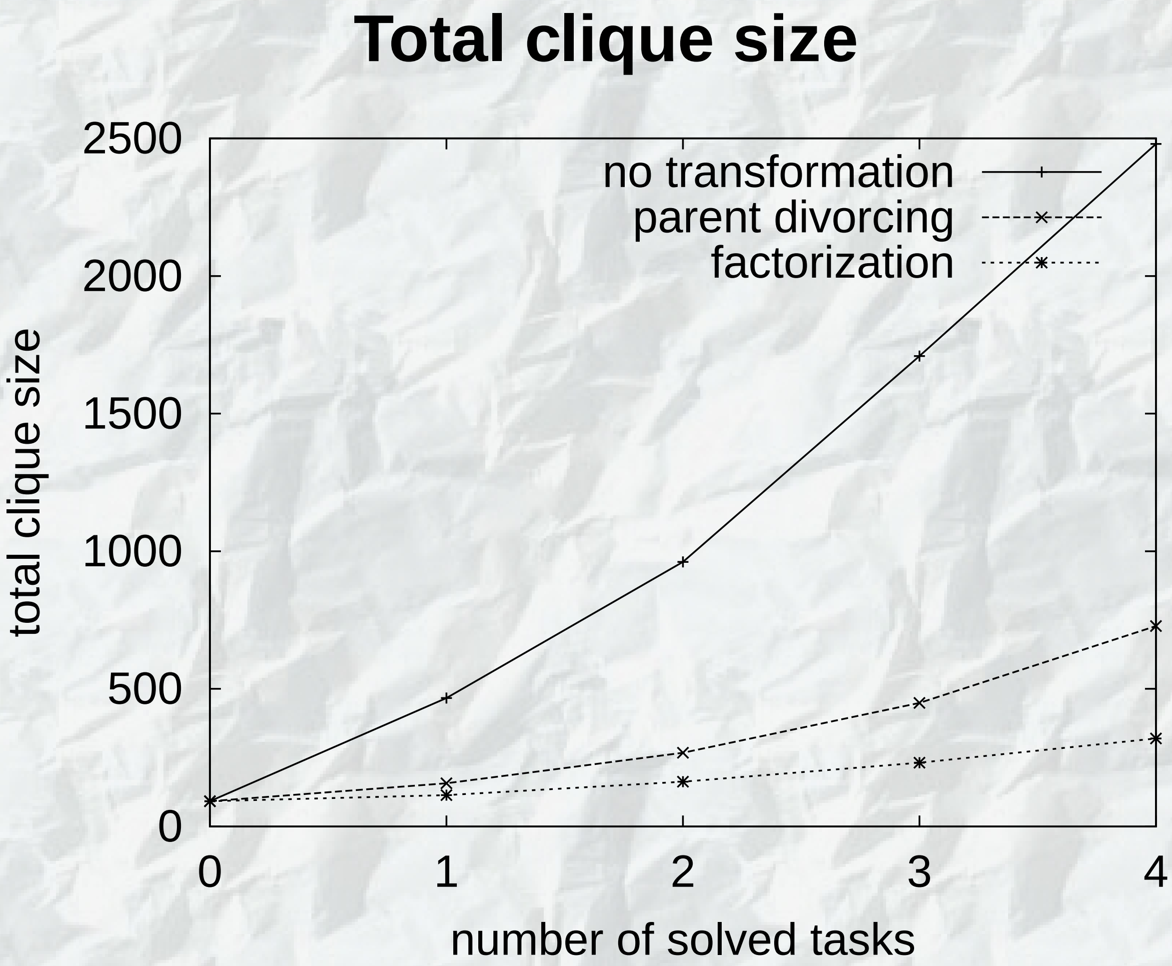


Total clique size
total clique size
2500
2000
1500
1000
500
0
0
1
2
3
4
number of solved tasks
no transformation
parent divorcing

Factorized evidence model



$$\psi(T1, MSB, SB, CL, ACL, MT, MMT3, MMT4) = \sum_{R1} \left(\begin{array}{l} \varphi(T1, R1) \cdot \varphi(R1, MSB) \cdot \varphi(R1, SB) \cdot \varphi(R1, CL) \cdot \\ \varphi(R1, ACL) \cdot \varphi(R1, MT) \cdot \varphi(R1, MMT3) \cdot \varphi(R1, MMT4) \end{array} \right)$$

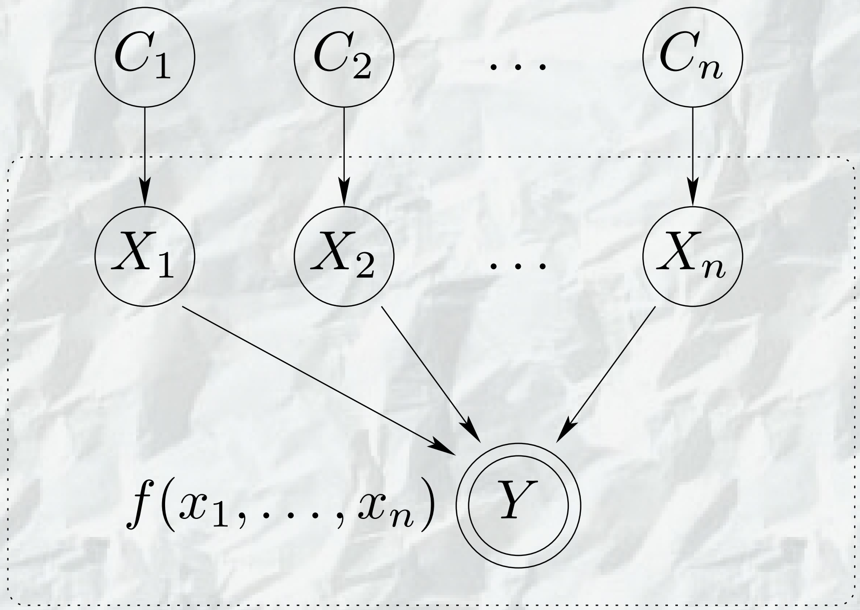

Total clique size
no transformation
parent divorcing
factorization
total clique size
0
500
1000
1500
2000
2500
0
1
2
3
4
number of solved tasks

Functional dependence

Y is functionally dependent on $X_1, \ldots, X_n$ if

$$P(Y = y, X_1 = x_1, \ldots, X_n = x_n) = \begin{cases} 1 & \text{if } y = f(x_1, \ldots, x_n) \\ 0 & \text{otherwise.} \end{cases}$$

Example - a model with independence of causal influence



Factorization of MAX

$$y = f(x_1, x_2) \; = \; \max\,\{x_1, x_2\}$$

$$P(Y|X_1, X_2) = \sum_R h(Y, R) \cdot g_1(X_1, R) \cdot g_2(X_2, R)$$

	0			+1			+2		
	0	+1	+2	0	+1	+2	0	+1	+2
0	1								
+1		1		1	1				
+2			1			1	1	1	1

$=$

$\sum_{r_1, r_2, r_3}($

	r_1	r_2	r_3
0	1		
+1	-1	1	
+2		-1	1

$\times$

	r_1	r_2	r_3
0	1	1	1
+1		1	1
+2			1

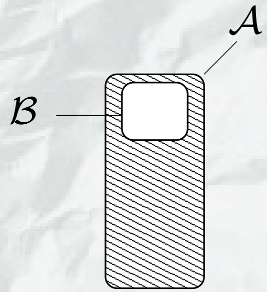
$\times$

	r_1	r_2	r_3
0	1	1	1
+1		1	1
+2			1

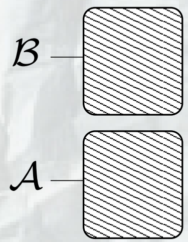
$)$

Proper difference and disjunctive union

- If $\mathcal{A} \supseteq \mathcal{B}$ then proper difference of $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A} \ominus \mathcal{B} = \{\mathbf{x} \in \mathcal{A} \wedge \mathbf{x} \notin \mathcal{B}\}$.



- If $\mathcal{A} \cap \mathcal{B} = \emptyset$ then disjunctive union of $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A} \oplus \mathcal{B} = \{\mathbf{x} \in \mathcal{A} \vee \mathbf{x} \in \mathcal{B}\}$.

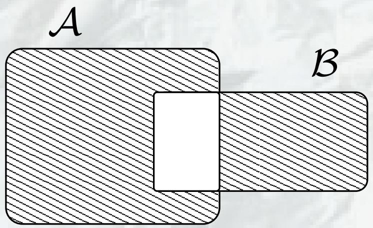


Special case of symmetric difference

If $\mathcal{A} \supseteq \mathcal{B}$ or $\mathcal{A} \cap \mathcal{B} = \emptyset$ then previous two operations can be replaced by a single operation of *symetric difference* of $\mathcal{A}$ and $\mathcal{B}$ defined as

$$\{\mathbf{x} \in \mathcal{A} \wedge \mathbf{x} \notin \mathcal{B}\} \ \cup \ \{\mathbf{x} \in \mathcal{B} \wedge \mathbf{x} \notin \mathcal{A}\} \ .$$

Example when symmetric difference can not be applied:

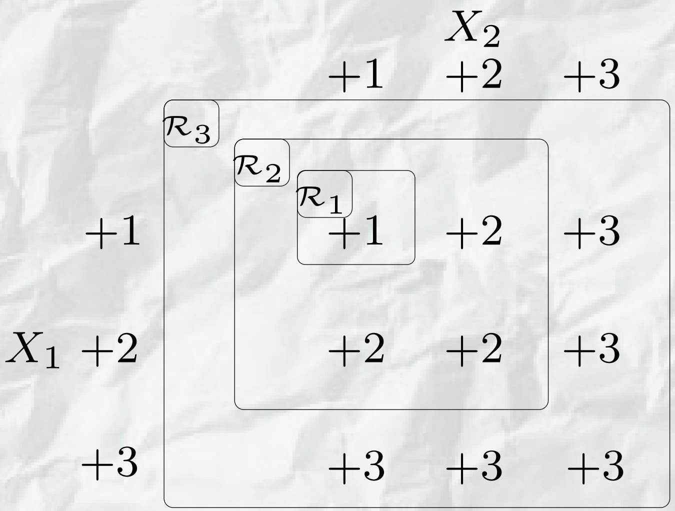


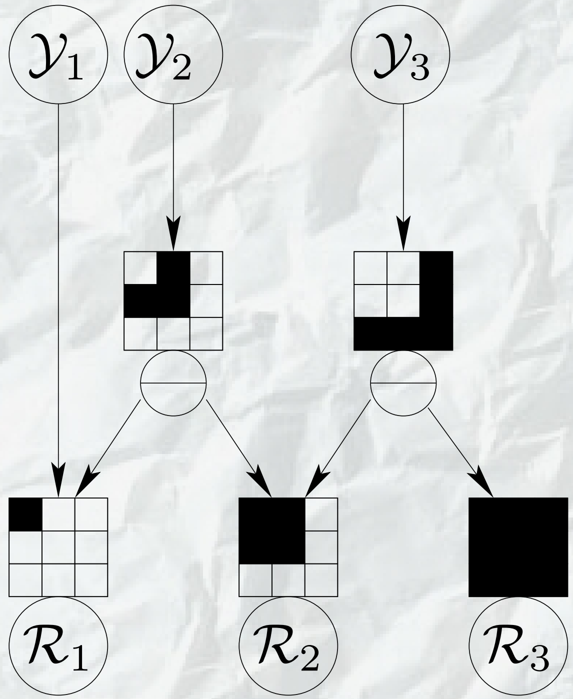
Minimal generator of rectangles (MGR)

For a given partition $\mathcal{Y}_1, \dots, \mathcal{Y}_q$ of $\mathcal{X} = \times_{i=1}^{n} \mathcal{X}_i$
find a set $\{\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_k\}$ of minimal cardinality such that:

- for $j = 1, \dots, k$ set $\mathcal{R}_j$ is a rectangle,
  i.e. $\mathcal{R}_j = \times_{i=1}^{n} \mathcal{D}_i$, $\emptyset \neq \mathcal{D}_i \subseteq \mathcal{X}_i$,
- each element $\mathcal{Y}_\ell, \ell = 1, 2, \dots q$ of the partition can be generated from the generator $\{\mathcal{R}_1, \dots, \mathcal{R}_k\}$ using operations $\ominus$ and $\oplus$.

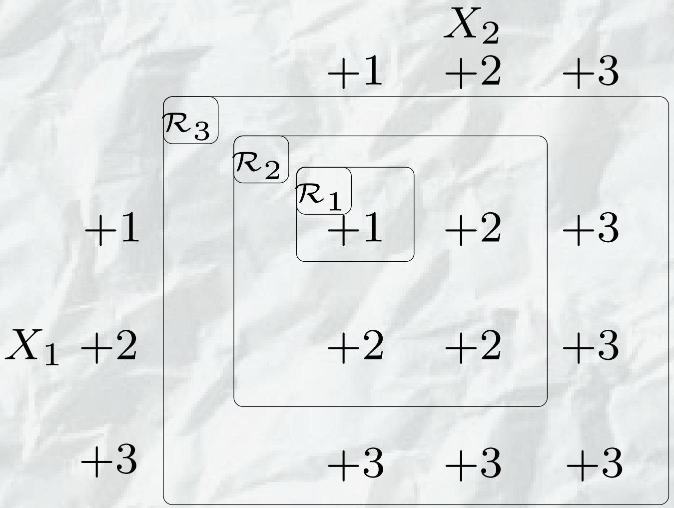
Factorization of MAX: problem reformulation





$$
\begin{aligned}
\mathcal{Y}_1 &= \{\ (+1,+1)\ \} \quad = \mathcal{R}_1 \\
\mathcal{Y}_2 &= \{\ (+1,+2),(+2,+1),(+2,+2)\ \} \quad = \mathcal{R}_2 \ominus \mathcal{R}_1 \\
\mathcal{Y}_3 &= \{\ (+1,+3),(+2,+3),(+3,+1),(+3,+2)(+3,+3\}) \quad = \ \mathcal{R}_3 \ominus \mathcal{R}_2
\end{aligned}
$$

Correspondence of MGR and factorization for MAX



$$\mathcal{Y}_1 = \mathcal{R}_1$$
$$\mathcal{Y}_2 = \mathcal{R}_2 \ominus \mathcal{R}_1$$
$$\mathcal{Y}_3 = \mathcal{R}_3 \ominus \mathcal{R}_2$$

Hidden variable R has one state for each rectangle

$h(Y, R)$

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$
$+1$	$+1$	0	0
$+2$	-1	$+1$	0
$+3$	0	-1	$+1$

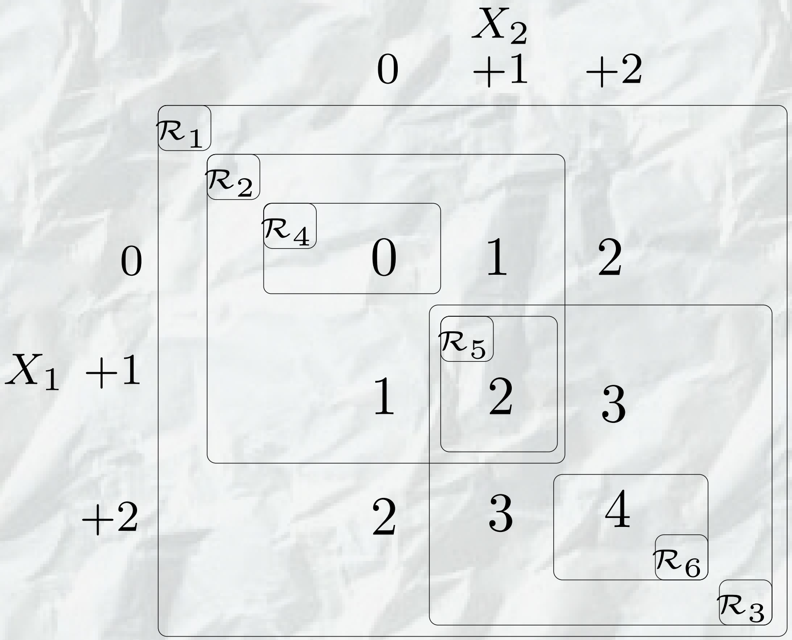
$g_1(X_1, R)$

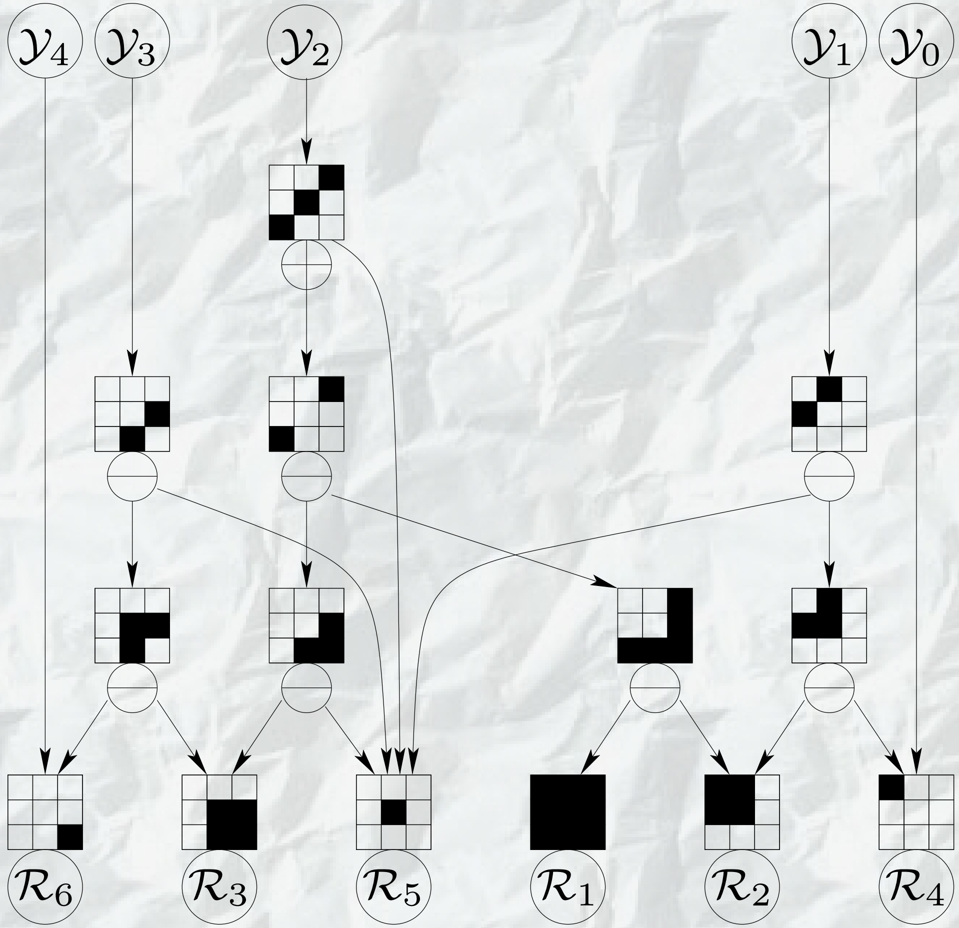
	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$
$+1$	1	1	1
$+2$	0	1	1
$+3$	0	0	1

$g_2(X_2, R)$

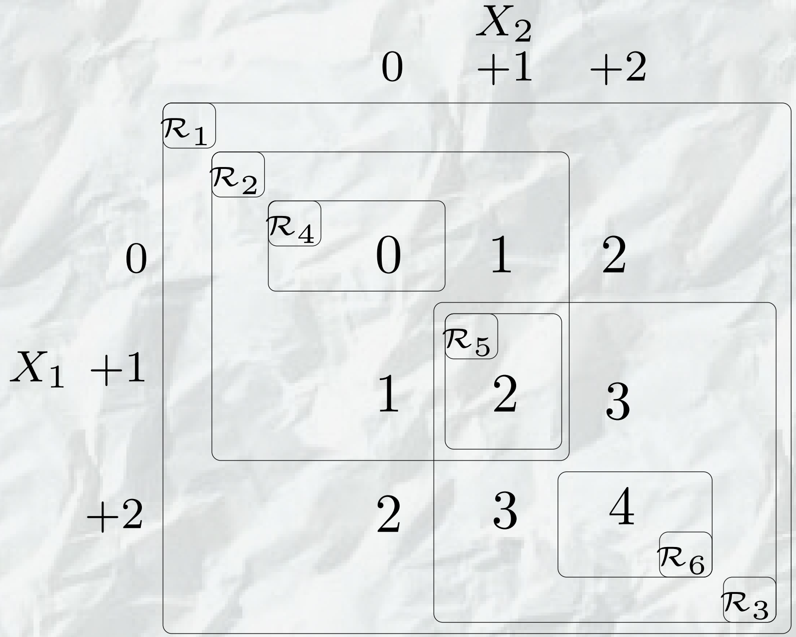
	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$
$+1$	1	1	1
$+2$	0	1	1
$+3$	0	0	1

Minimal generator of rectangles for ADD





Correspondence of MGR and factorization for ADD



$$
\begin{aligned}
\mathcal{Y}_1 &= \mathcal{R}_4 \\
\mathcal{Y}_2 &= \mathcal{R}_2 \ominus \mathcal{R}_4 \ominus \mathcal{R}_5 \\
\mathcal{Y}_3 &= (\mathcal{R}_1 \ominus (\mathcal{R}_2 \ominus \mathcal{R}_5)) \ominus (\mathcal{R}_3 \ominus \mathcal{R}_5) \\
\mathcal{Y}_4 &= \mathcal{R}_3 \ominus \mathcal{R}_5 \ominus \mathcal{R}_6 \\
\mathcal{Y}_5 &= \mathcal{R}_6 \ ,
\end{aligned}
$$

Hidden variable B has one state for each rectangle

$h(Y, R)$

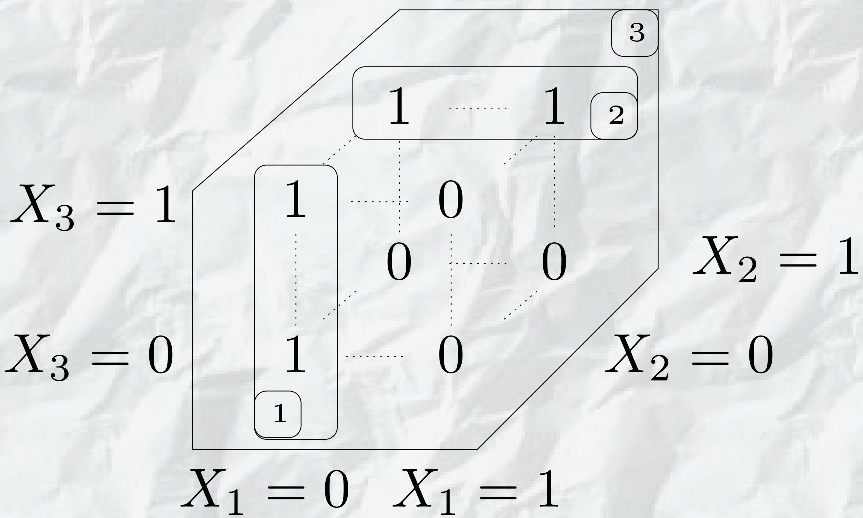
	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$
0	0	0	0	+1	0	0
+1	0	+1	0	−1	−1	0
+2	+1	−1	−1	0	+2	0
+3	0	0	+1	0	−1	−1
+4	0	0	0	0	0	+1

$g_1(X_1, R), \quad g_2(X_2, R)$

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$
0	1	1	0	1	0	0
+1	1	1	1	0	1	0
+2	1	0	1	0	0	1

A Boolean function

$$
\begin{aligned}
Y &= (X_1 \vee X_2) \Rightarrow (X_2 \wedge X_3) \\
&= (\neg X_1 \wedge \neg X_2) \vee (X_2 \wedge X_3)
\end{aligned}
$$



$$
\begin{aligned}
\mathcal{Y}_0 &= \mathcal{R}_3 \ominus (\mathcal{R}_2 \oplus \mathcal{R}_1) \\
\mathcal{Y}_1 &= \mathcal{R}_2 \oplus \mathcal{R}_1
\end{aligned}
$$

Open questions and problems

- A general algorithm that for a given function finds its minimal generator.
- Generalization of the method so that tables $g_1(X_1, R), \ldots, g_n(X_n, R)$ need not contain zeroes and ones only
- A method for approximation of any probability table by a product of two-dimensional tables.
- Links to communication complexity.
- ...

...and it is time to end.