Exploiting Functional Dependence in Bayesian Network Inference

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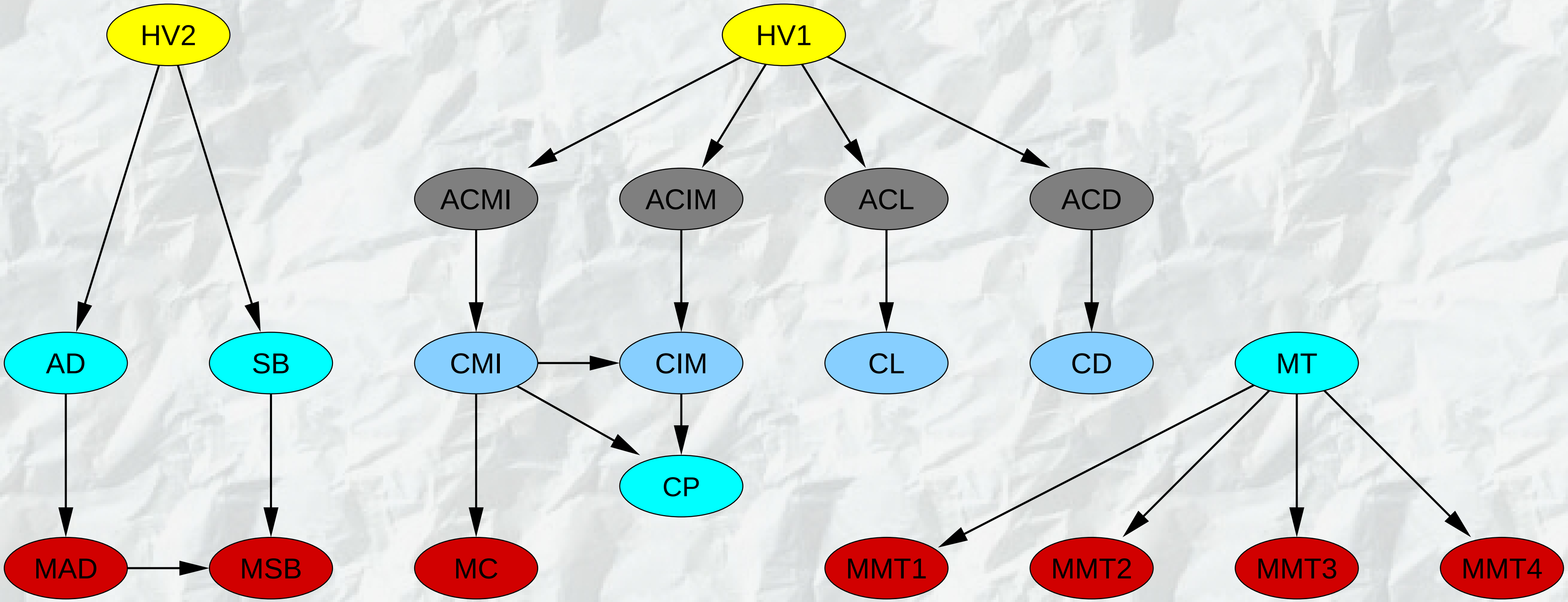
This presentation is available from:
http://www.utia.cas.cz/vomlel/

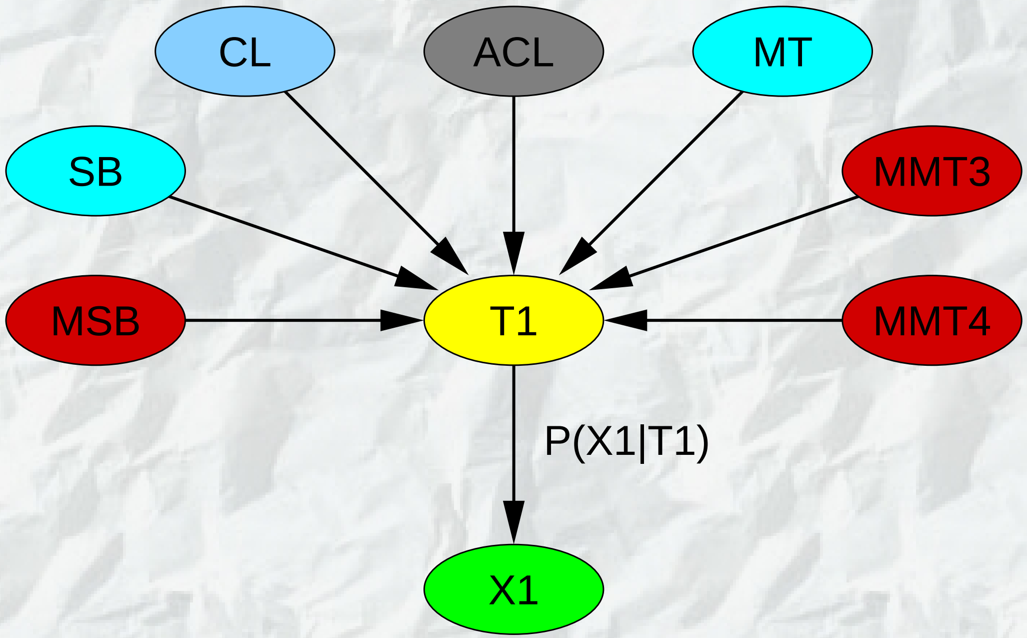
Original model



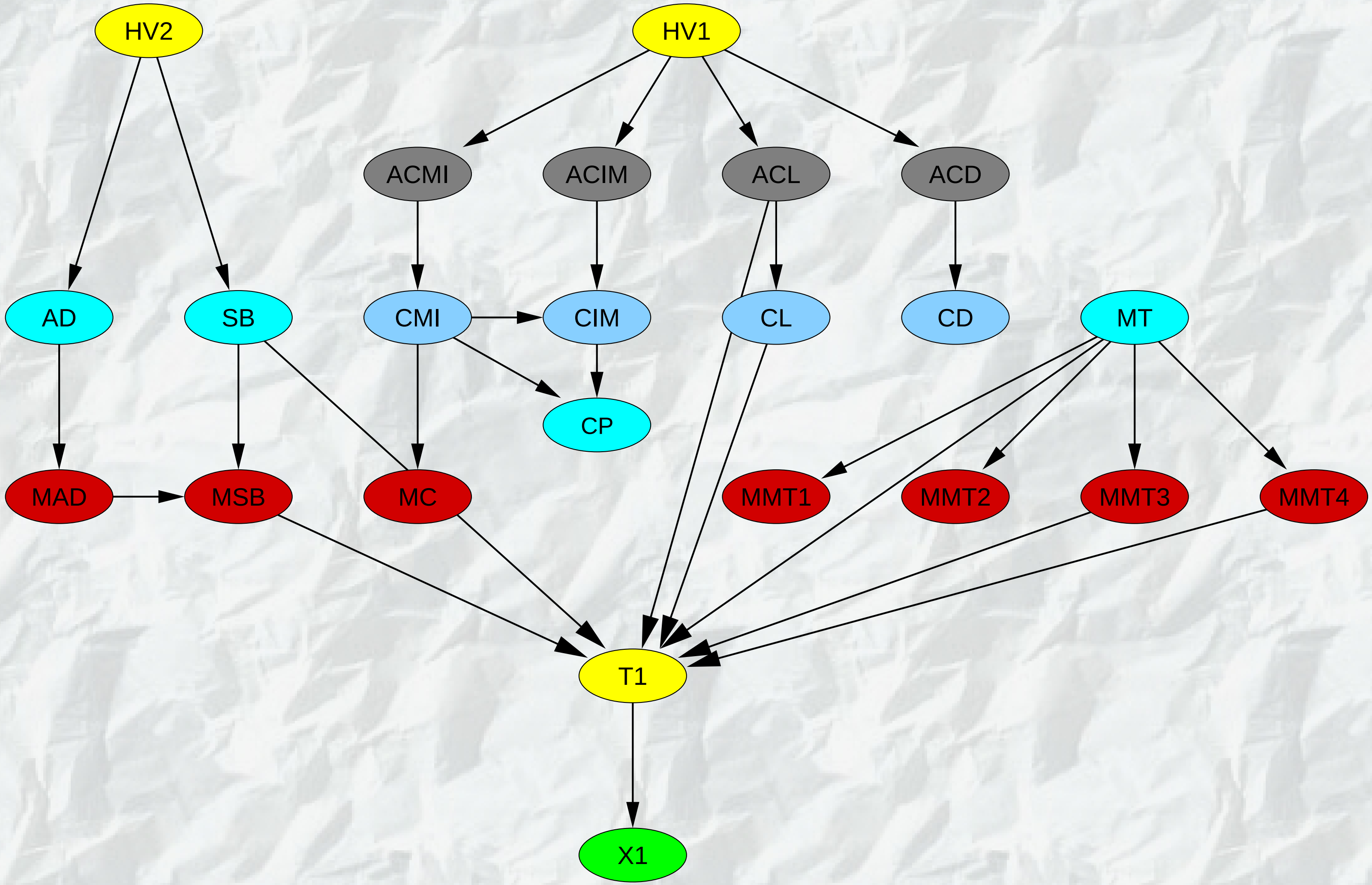
Evidence model of a task

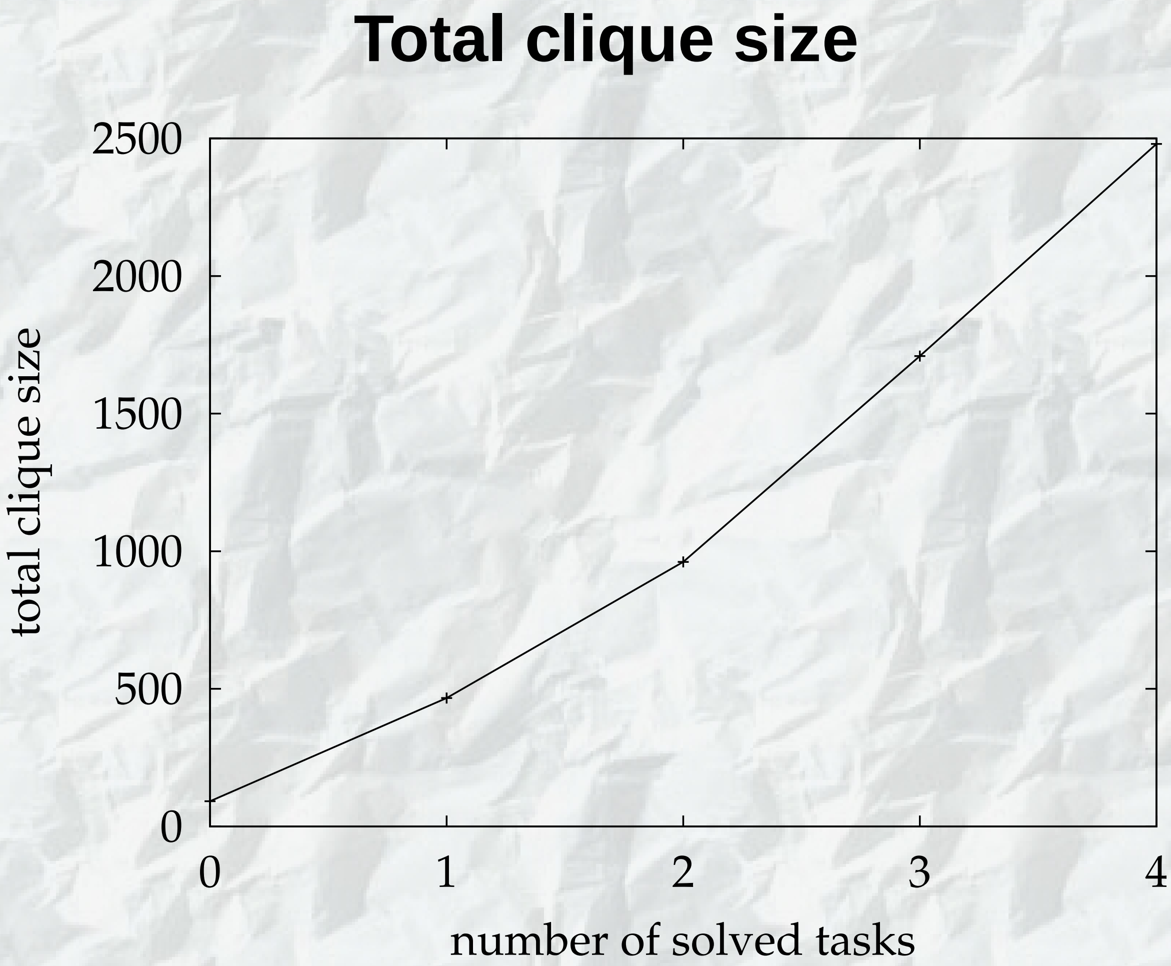
$$\left(\frac{3}{4}\cdot\frac{5}{6}\right)-\frac{1}{8}=\frac{15}{24}-\frac{1}{8}=\frac{5}{8}-\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

$$T_1 \quad \Leftrightarrow \quad MT \;\&\; CL \;\&\; ACL \;\&\; SB \;\&\; \neg MMT3 \;\&\; \neg MMT4 \;\&\; \neg MSB$$



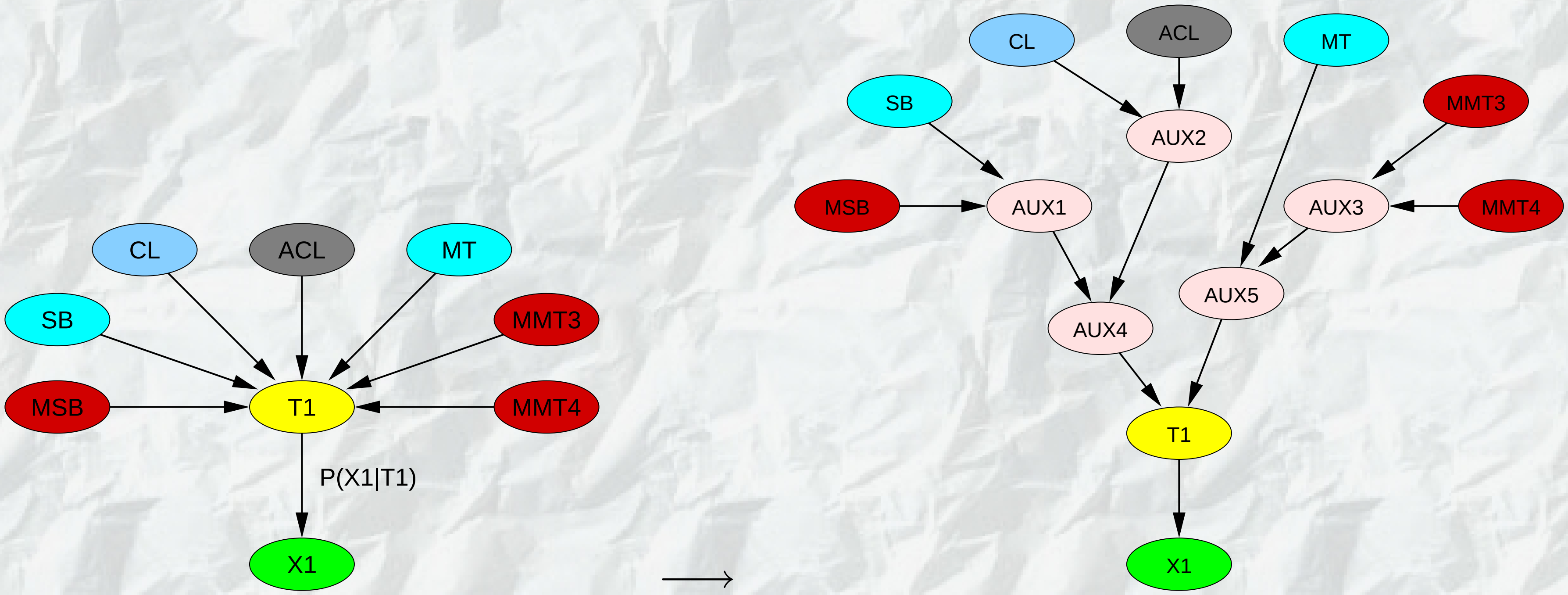
Original model + one evidence model

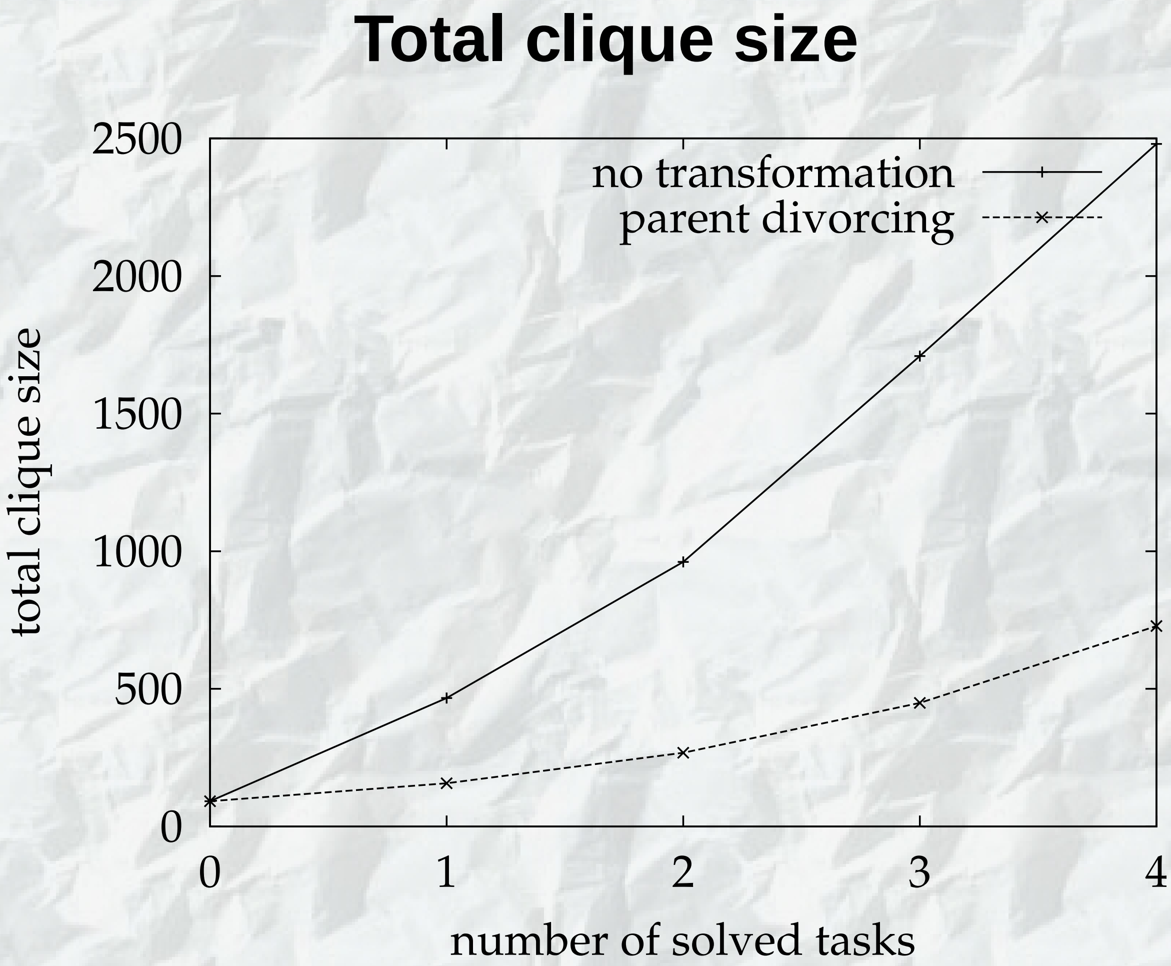




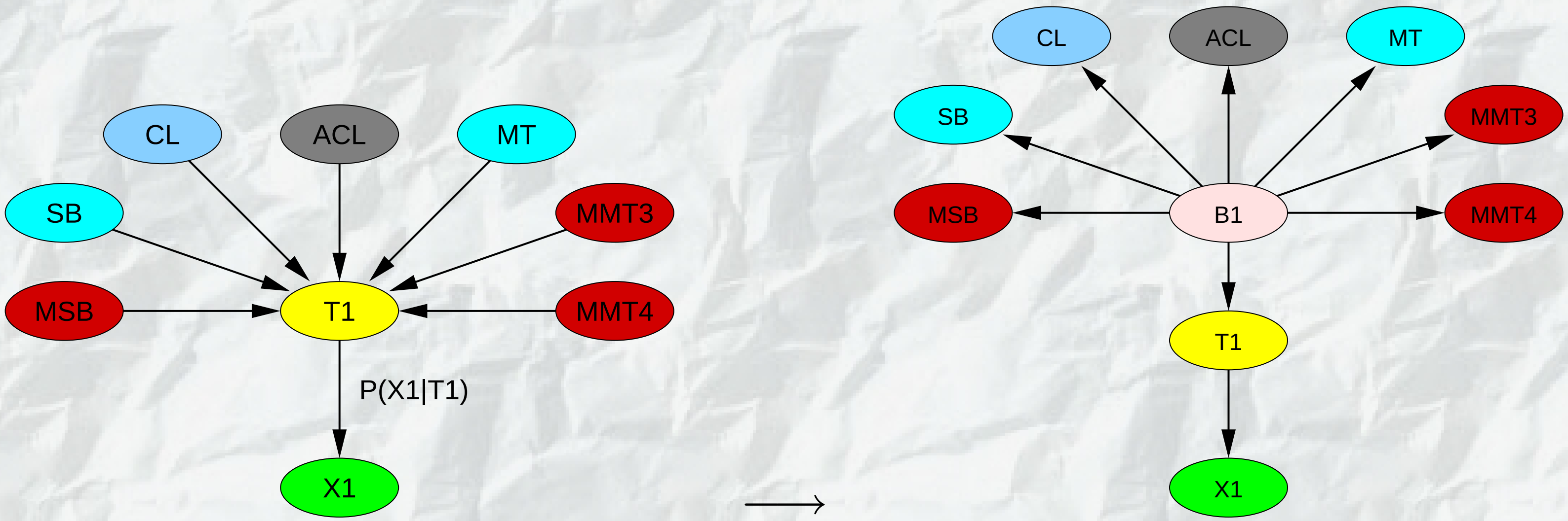
Total clique size
total clique size
2500
2000
1500
1000
500
0
0
1
2
3
4
number of solved tasks

Hierarchical evidence model (=parent divorcing)




Total clique size
total clique size
number of solved tasks
no transformation
parent divorcing
2500
2000
1500
1000
500
0
0
1
2
3
4

Factorized evidence model



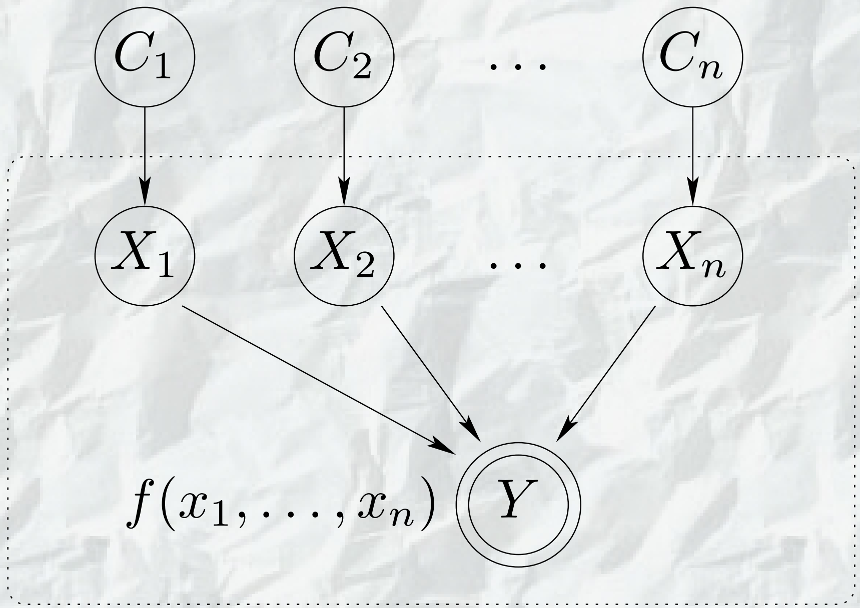
$$\psi(T1, MSB, SB, CL, ACL, MT, MMT3, MMT4) \; = \sum_{B1} \left(\begin{array}{l} \varphi(T1, B1) \cdot \varphi(B1, MSB) \cdot \varphi(B1, SB) \cdot \varphi(B1, CL) \cdot \\ \varphi(B1, ACL) \cdot \varphi(B1, MT) \cdot \varphi(B1, MMT3) \cdot \varphi(B1, MMT4) \end{array} \right)$$

Functional dependence

Y is functionally dependent on $X_1, \ldots, X_n$ if

$$\psi(y, x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } y = f(x_1, \ldots, x_n) \\ 0 & \text{otherwise.} \end{cases}$$

Example - a model with independence of causal influence



Factorization of MAX

$$y = f(x_1, x_2) \; = \; \max\,\{x_1, x_2\}$$

$$P(Y|X_1, X_2) = \textstyle\sum_R h(Y,R) \cdot g_1(X_1,R) \cdot g_2(X_1,R)$$

	0			+1			+2		
	0	+1	+2	0	+1	+2	0	+1	+2
0	1								
+1		1		1	1				
+2			1			1	1	1	1

$=$

$\sum_{r_1,r_2,r_3}($

	r_1	r_2	r_3
0	1		
+1	-1	1	
+2		-1	1

$\times$

	r_1	r_2	r_3
0	1	1	1
+1		1	1
+2			1

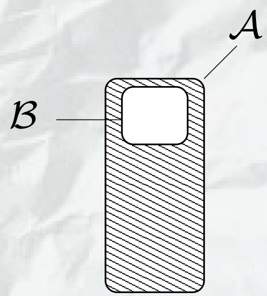
$\times$

	r_1	r_2	r_3
0	1	1	1
+1		1	1
+2			1

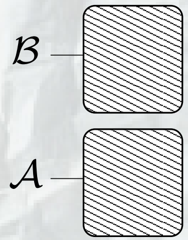
$)$

Proper difference and disjunctive union

- If $\mathcal{A} \supseteq \mathcal{B}$ then proper difference of $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A} \ominus \mathcal{B} = \{\mathbf{x} \in \mathcal{A} \wedge \mathbf{x} \notin \mathcal{B}\}$.



- If $\mathcal{A} \cap \mathcal{B} = \emptyset$ then disjunctive union of $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A} \oplus \mathcal{B} = \{\mathbf{x} \in \mathcal{A} \vee \mathbf{x} \in \mathcal{B}\}$.

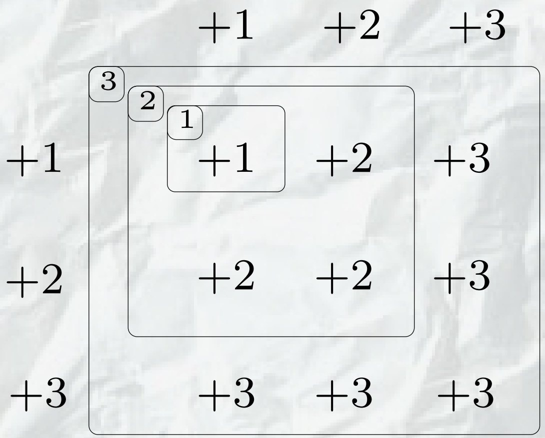


Minimal base of rectangles (MBR)

For a given partition $\mathcal{Y}_1, \ldots, \mathcal{Y}_q$ of $\mathcal{X} = \times_{i=1}^{n} \mathcal{X}_i$
find a set $\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\}$ of minimal cardinality such that:

- for $j = 1, \ldots, k$ set $\mathcal{R}_j$ is a rectangle,
  i.e. $\mathcal{R}_j = \times_{i=1}^{n} \mathcal{D}_i$, $\emptyset \neq \mathcal{D}_i \subseteq \mathcal{X}_i$,
- each element $\mathcal{Y}_\ell, \ell = 1, 2, \ldots q$ of the partition can be generated from base $\{\mathcal{R}_1, \ldots, \mathcal{R}_k\}$ using operations $\ominus$ and $\oplus$.

Factorization of MAX: problem reformulation



Partition by values of Y:

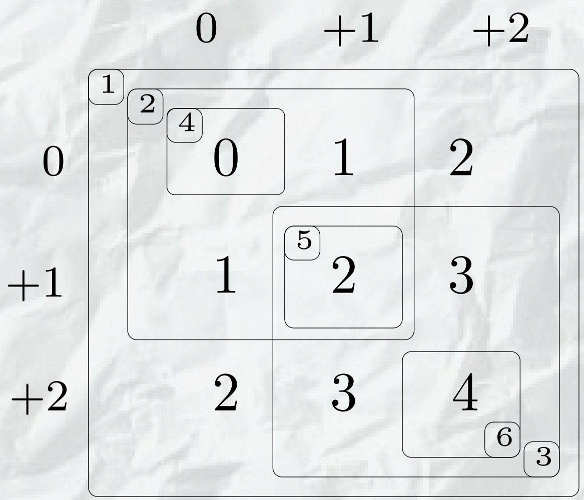
$$\begin{aligned}
\mathcal{Y}_1 &= \{\ (+1,+1)\ \} \\
\mathcal{Y}_2 &= \{\ (+1,+2),(+2,+1),(+2,+2)\ \} \\
\mathcal{Y}_3 &= \{\ (+1,+3),(+2,+3),(+3,+1), \\
&\qquad (+3,+2)(+3,+3)\ \}
\end{aligned}$$

Rectangular subspaces:

$$\begin{aligned}
\mathcal{R}_1 &= \{\ (+1,+1)\ \} \\
\mathcal{R}_2 &= \{\ (+1,+1),(+1,+2),(+2,+1),(+2,+2)\ \} \\
\mathcal{R}_3 &= \mathcal{X}_1 \times \mathcal{X}_2
\end{aligned}$$

$\mathcal{Y}_1 = \mathcal{R}_1$, $\mathcal{Y}_2 = \mathcal{R}_2 \ominus \mathcal{R}_1$, and $\mathcal{Y}_3 = \mathcal{R}_3 \ominus \mathcal{R}_1$.

Minimal base of rectangles for ADD



$$\mathcal{Y}_1 = \mathcal{R}_4$$

$$\mathcal{Y}_2 = \mathcal{R}_2 \ominus \mathcal{R}_4 \ominus \mathcal{R}_5$$

$$\mathcal{Y}_3 = (\mathcal{R}_1 \ominus (\mathcal{R}_2 \ominus \mathcal{R}_5)) \ominus (\mathcal{R}_3 \ominus \mathcal{R}_5)$$

$$\mathcal{Y}_4 = \mathcal{R}_3 \ominus \mathcal{R}_5 \ominus \mathcal{R}_6$$

$$\mathcal{Y}_5 = \mathcal{R}_6$$

Correspondence of MBR and factorization

$$\begin{aligned}
\mathcal{Y}_1 &= \mathcal{R}_4 \\
\mathcal{Y}_2 &= \mathcal{R}_2 \ominus \mathcal{R}_4 \ominus \mathcal{R}_5 \\
\mathcal{Y}_3 &= (\mathcal{R}_1 \ominus (\mathcal{R}_2 \ominus \mathcal{R}_5)) \ominus (\mathcal{R}_3 \ominus \mathcal{R}_5) \\
\mathcal{Y}_4 &= \mathcal{R}_3 \ominus \mathcal{R}_5 \ominus \mathcal{R}_6 \\
\mathcal{Y}_5 &= \mathcal{R}_6 \quad ,
\end{aligned}$$

Hidden variable B has one state for each rectangle

$h(y, b)$

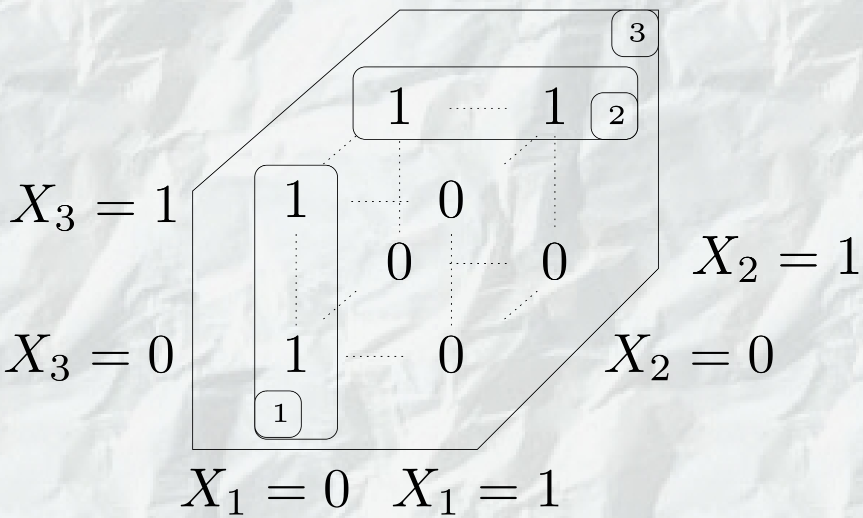
	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$
y_1	0	0	0	+1	0	0
y_2	0	+1	0	−1	−1	0
y_3	+1	−1	−1	0	+2	0
y_4	0	0	+1	0	−1	−1
y_5	0	0	0	0	0	+1

$g_1(x_1, b), \quad g_2(x_2, b)$

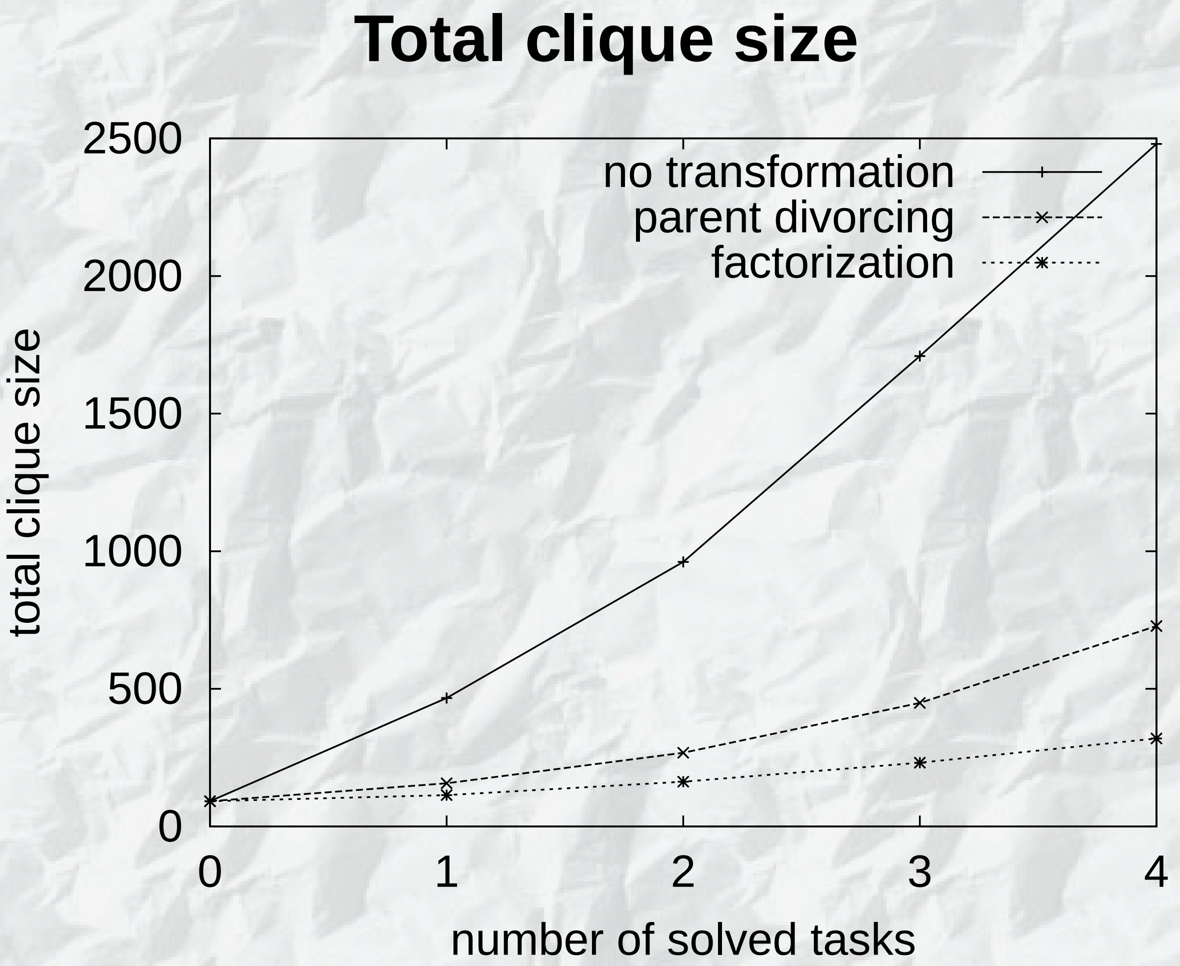
	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$
x_1	1	1	0	1	0	0
x_2	1	1	1	0	1	0
x_3	1	0	1	0	0	1

A Boolean function

$$\begin{aligned} Y &= (X_1 \vee X_2) \Rightarrow (X_2 \wedge X_3) \\ &= (\neg X_1 \wedge \neg X_2) \vee (X_2 \wedge X_3) \end{aligned}$$



$$\begin{aligned} \mathcal{Y}_0 &= \mathcal{R}_3 \ominus (\mathcal{R}_2 \oplus \mathcal{R}_1) \\ \mathcal{Y}_1 &= \mathcal{R}_2 \oplus \mathcal{R}_1 \end{aligned}$$



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