Inference in Bayesian networks

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Simple diagnostic example - 1

A medical doctor inspects a patient.
Possible diagnosis are: tuberculosis, lung cancer, or bronchitis.

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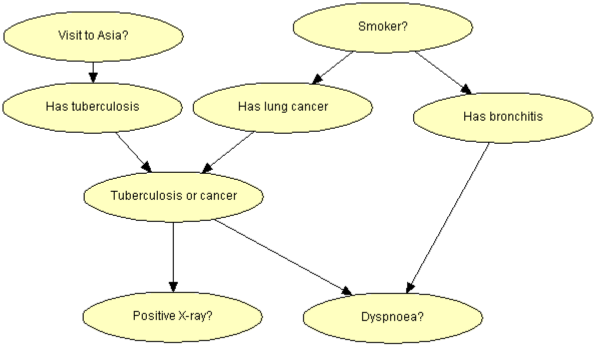
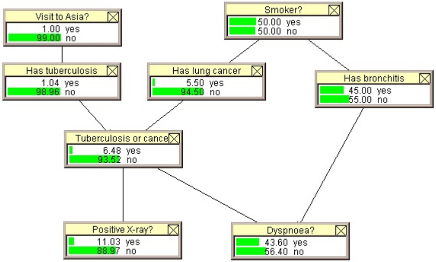


Simple diagnostic example - 2

We do not know anything about the patient.



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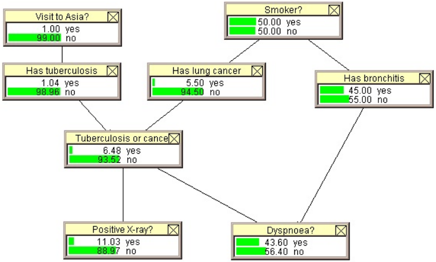
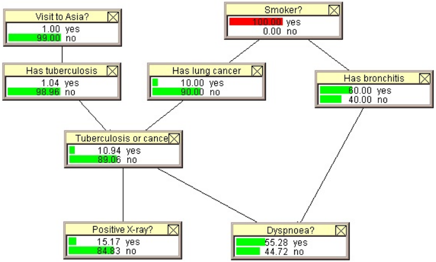


The patient is a smoker.



Simple diagnostic example - 3

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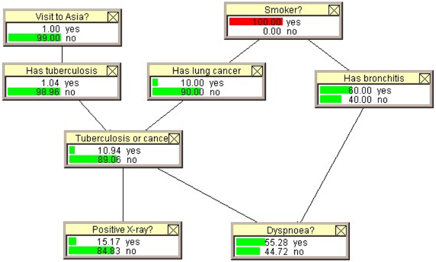

Simple diagnostic example - 3

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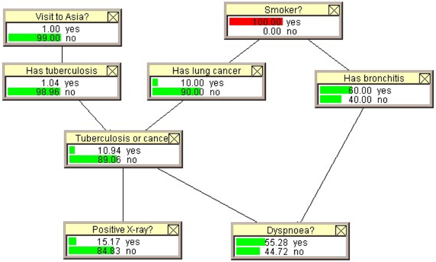
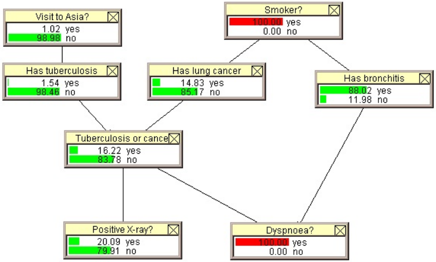
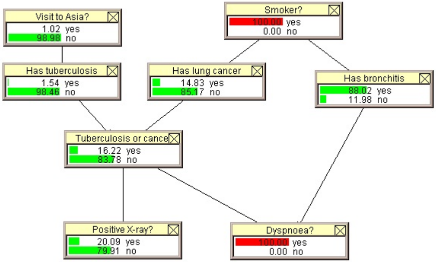


... and he suffers from dyspnoea.



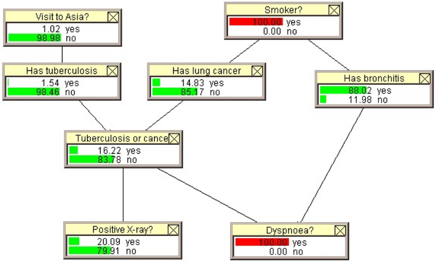
Simple diagnostic example - 4

The patient is a smoker
and he suffers from dyspnoea.

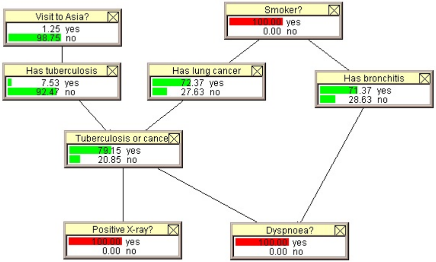


Simple diagnostic example - 4

The patient is a smoker
and he suffers from dyspnoea.

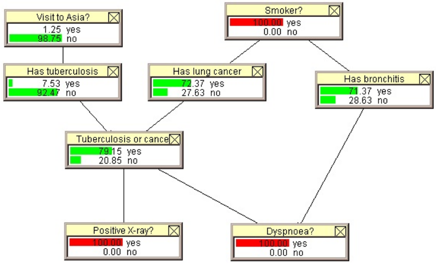


... and his X-ray is positive



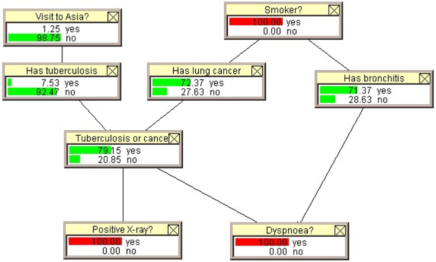
Simple diagnostic example - 5

The patient is a smoker,
he suffers from dyspnoea
and his X-ray is positive

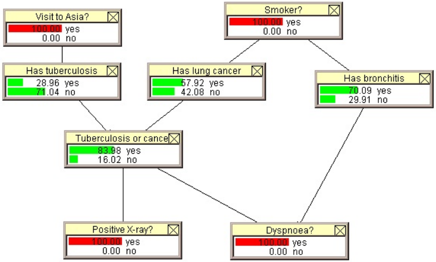


Simple diagnostic example - 5

The patient is a smoker,
he suffers from dyspnoea
and his X-ray is positive

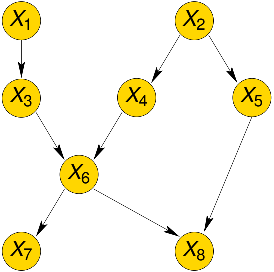


... and he visited Asia recently.



Example written more formally

X_1	"Visit to Asia?"
X_2	"Smoker?"
X_3	"Has tuberculosis?"
X_4	"Has lung cancer?"
X_5	"Has bronchitis?"
X_6	"Tuberculosis or cancer"
X_7	"Positive X-ray?"
X_8	"Dyspnoea"



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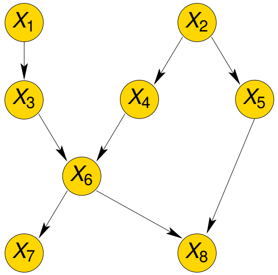

The joint probability distribution defined by the Bayesian network:

$$P(X_1, X_2, \ldots, X_8) \quad = \quad \prod_{i=1}^{8} P(X_i | \{X_j\}_{j \in Pa(i)})$$

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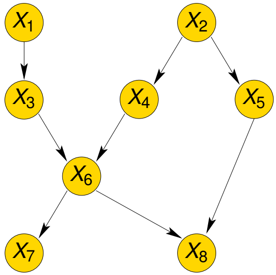

The joint probability distribution defined by the Bayesian network:

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\begin{aligned}
P(X_1, X_2, \ldots, X_8) &= \prod_{i=1}^{8} P(X_i | \{X_j\}_{j \in Pa(i)}) \\
&= P(X_8|X_6, X_5) \cdot P(X_7|X_6) \cdot P(X_6|X_3, X_4) \\
&\quad \cdot P(X_5|X_2) \cdot P(X_4|X_2) \cdot P(X_3|X_1) \cdot P(X_2) \cdot P(X_1)
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Conditional probability

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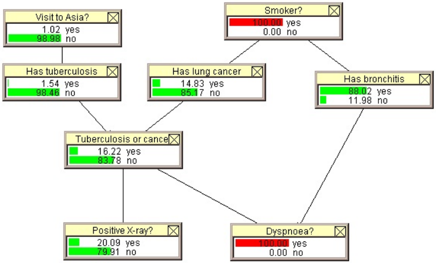

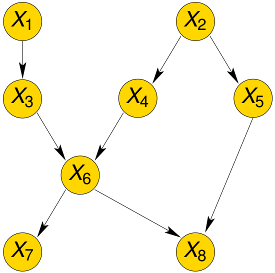

The conditional probability distribution corresponding to the query:

$$P(X_3, | X_2 = 1, X_8 = 1) \quad = \quad \frac{P(X_2 = 1, X_3, X_8 = 1)}{P(X_2 = 1, X_8 = 1)}$$

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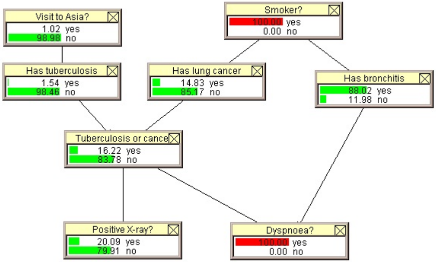

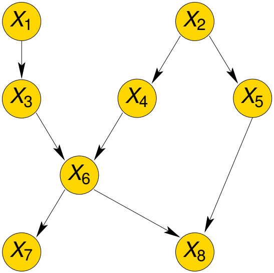

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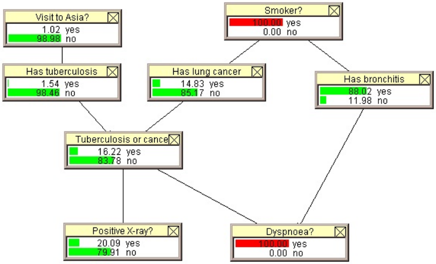

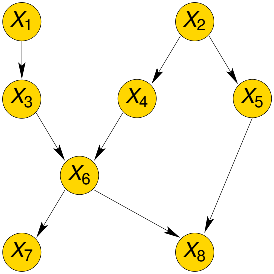

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Distributive law for probability distributions

$$P(X,Y)\cdot P(Y,Z) = \begin{array}{|c|cc|}\hline & X=0 & X=1 \\ \hline Y=0 & a & c \\ Y=1 & b & d \\ \hline\end{array} \cdot \begin{array}{|c|cc|}\hline & Z=0 & Z=1 \\ \hline Y=0 & e & g \\ Y=1 & f & h \\ \hline\end{array}$$

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$$= P(X,Y)\cdot\left(\sum_Z P(Y,Z)\right)$$

Direct computation of a marginal probability

$$P(X_2 = 1, X_3, X_8 = 1)$$
$$= \sum_{X_1, X_4, X_5, X_6, X_7} P(X_1, X_2 = 1, X_3, \ldots, X_7, X_8 = 1)$$

Direct computation of a marginal probability

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\begin{aligned}
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& = \sum_{X_1, X_4, X_5, X_6, X_7} P(X_1, X_2 = 1, X_3, \ldots, X_7, X_8 = 1) \\
& = \sum_{X_1, X_4, X_5, X_6, X_7} \left(\begin{array}{l} P(X_8 = 1 | X_6, X_5) \cdot P(X_7 | X_6) \cdot P(X_6 | X_3, X_4) \\ \cdot P(X_5 | X_2 = 1) \cdot P(X_4 | X_2 = 1) \cdot P(X_3 | X_1) \\ \cdot P(X_2 = 1) \cdot P(X_1) \end{array} \right)
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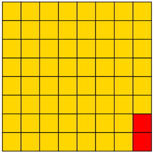
\end{aligned}
$$

- $\psi(X_1, X_2 = 1, X_3, \ldots, X_7, X_8 = 1)$
  $\rightarrow \psi(X_2 = 1, X_3, X_8 = 1)$



Direct computation of a marginal probability

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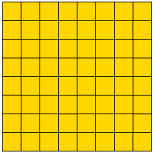

The largest table has size $2^6 = 64$.

Efficient computation of a marginal probability

$$P(X_2 = 1, X_3, X_8 = 1) \quad = P(X_2 = 1) \cdot \sum_{X_1} (P(X_3|X_1) \cdot P(X_1))$$

$$\cdot \sum_{X_6} \left(\begin{array}{l} \sum_{X_7} P(X_7|X_6) \\ \cdot \sum_{X_4} (P(X_6|X_3, X_4) \cdot P(X_4|X_2 = 1)) \\ \cdot \sum_{X_5} (P(X_8 = 1|X_6, X_5) \cdot P(X_5|X_2 = 1)) \end{array} \right)$$

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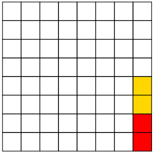
- $\sum_{X_5} \psi(X_5, X_6) \to \psi(X_6)$



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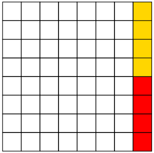
- $\sum_{X_5} \psi(X_5, X_6) \rightarrow \psi(X_6)$
- $\sum_{X_4} \psi(X_3, X_4, X_6) \rightarrow \psi(X_3, X_6)$



Efficient computation of a marginal probability

$$P(X_2 = 1, X_3, X_8 = 1) \quad = P(X_2 = 1) \cdot \sum_{X_1} \left(P(X_3|X_1) \cdot P(X_1) \right)$$

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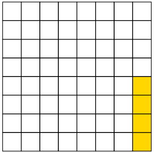
- $\sum_{X_5} \psi(X_5, X_6) \to \psi(X_6)$
- $\sum_{X_4} \psi(X_3, X_4, X_6) \to \psi(X_3, X_6)$
- $\sum_{X_7} \psi(X_6, X_7) \to 1$



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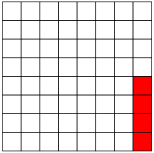
- $\sum_{X_4} \psi(X_3, X_4, X_6) \rightarrow \psi(X_3, X_6)$
- $\sum_{X_7} \psi(X_6, X_7) \rightarrow 1$
- $\psi(X_6) \cdot \psi(X_3, X_6) \rightarrow \psi'(X_3, X_6)$



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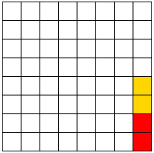

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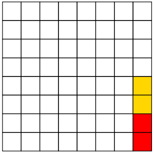

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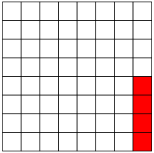

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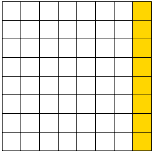

The largest table has size $2^3 = 8$.

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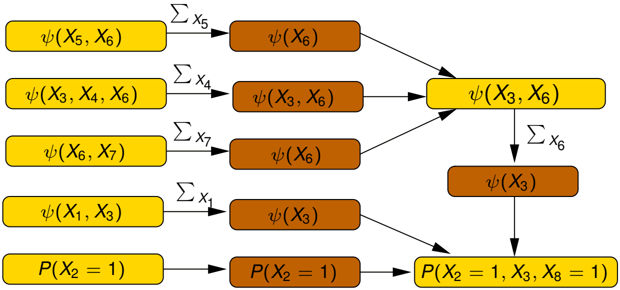

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- Generally, computing a marginal probability in a Bayesian network is NP-hard.

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- Junction Tree Algorithm

Domain graph

Recall, the joint probability distribution of the Bayesian network

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | \{X_j\}_{j \in Pa(i)})$$

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The domain graph is an undirected graph with variables $X_1, \ldots, X_n$ as nodes and with an edge between a pair of variables X_a, X_b if there exists $P(X_i | \{X_j\}_{j \in Pa(i)})$ such that $X_a, X_b \in D_i$.

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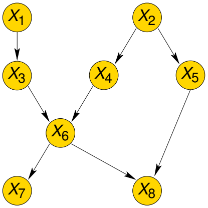

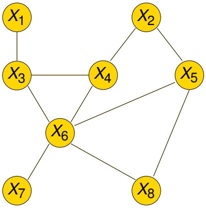

Triangulated graph

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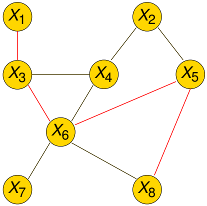

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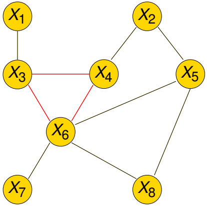

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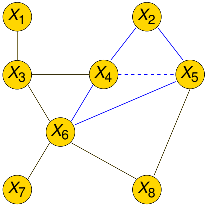

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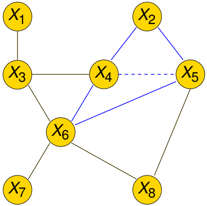

An undirected graph is triangulated if it does not contain a cycle of length four or more without a chord.

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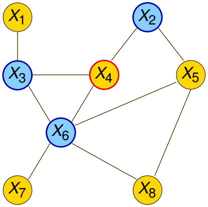

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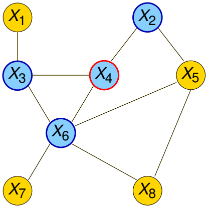

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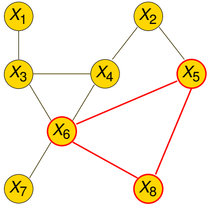

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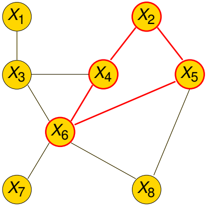

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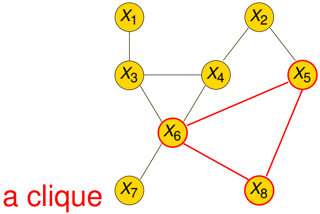

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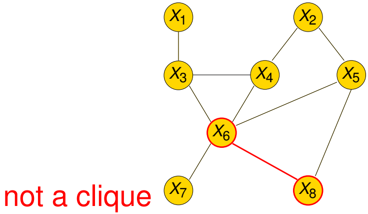

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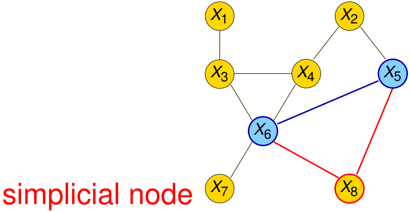

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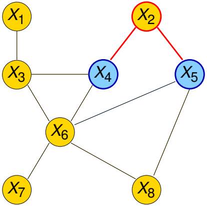

Elimination sequence

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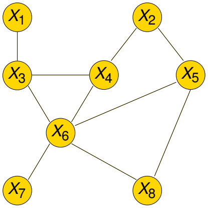

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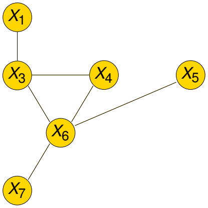
nodes with evidence eliminated



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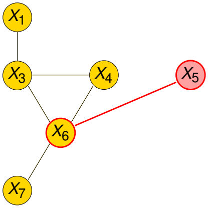
eliminating X_5



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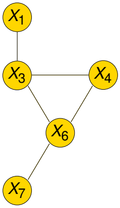

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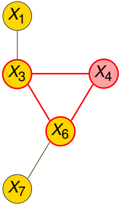
$$X_5, X_4, X_7, X_6, X_1$$

eliminating X_4



Elimination sequence

$$P(X_2 = 1, X_3, X_8 = 1) \quad = P(X_2 = 1) \cdot \sum_{X_1} (P(X_3|X_1) \cdot P(X_1))$$

$$\cdot \sum_{X_6} \left(\begin{array}{l} \sum_{X_7} P(X_7|X_6) \\ \cdot \sum_{X_4} (P(X_6|X_3, X_4) \cdot P(X_4|X_2 = 1)) \\ \cdot \sum_{X_5} (P(X_8 = 1|X_6, X_5) \cdot P(X_5|X_2 = 1)) \end{array} \right)$$

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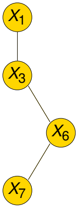

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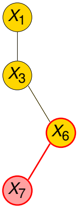
$$X_5, X_4, X_7, X_6, X_1$$

eliminating X_7



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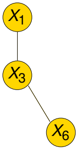

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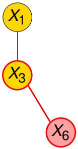

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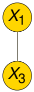

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X_1 eliminated

Perfect elimination sequence

- When eliminating a variable X we need to multiply all probability tables containing this variable.

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- The sequence of the previous example was perfect.

A non-perfect sequence

Compute $P(X_3)$, i.e., a marginal probability of X_3 without any evidence.

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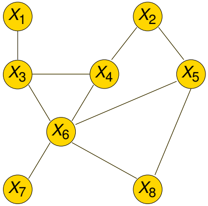
domain graph



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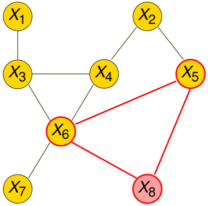
Compute $P(X_3)$, i.e., a marginal probability of X_3 without any evidence.

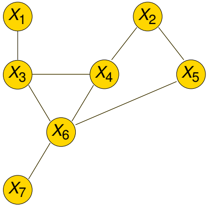
eliminating X_8



A non-perfect sequence

Compute $P(X_3)$, i.e., a marginal probability of X_3 without any evidence.

X_8 eliminated



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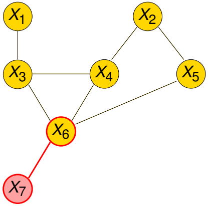
eliminating X_7



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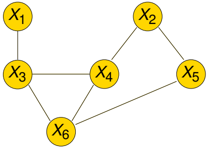
X_7 eliminated



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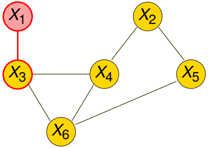
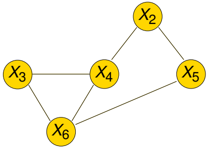


eliminating X_1

A non-perfect sequence

Compute $P(X_3)$, i.e., a marginal probability of X_3 without any evidence.

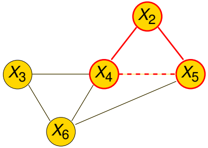
X_1 eliminated
now, there is no simplicial node among nodes to be eliminated



A non-perfect sequence

Compute $P(X_3)$, i.e., a marginal probability of X_3 without any evidence.

eliminating X_2
filling in edge $\{X_4, X_5\}$



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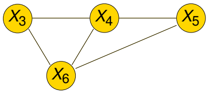
X_2 eliminated



A non-perfect sequence

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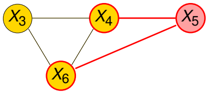
eliminating X_5



A non-perfect sequence

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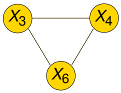
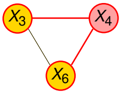
X_5 eliminated



A non-perfect sequence

Compute $P(X_3)$, i.e., a marginal probability of X_3 without any evidence.

eliminating X_4



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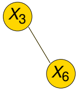

A non-perfect sequence

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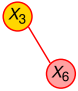
eliminating X_6



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Observe that the added edge is exactly the same as the one we added to make the graph triangulated.

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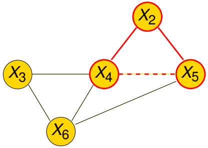

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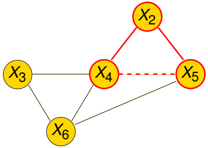

In a triangulated graph we can **always** find a perfect elimination sequence!

Join tree

Join tree of an undirected graph is a undirected graph such that:

- nodes corresponds to cliques of the graph

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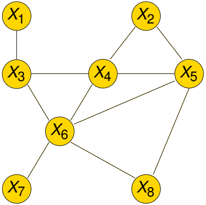

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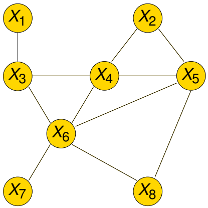

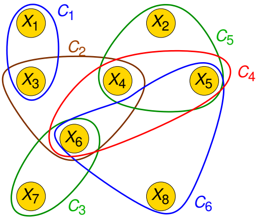

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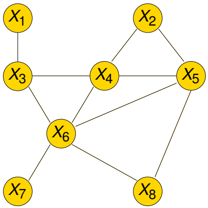

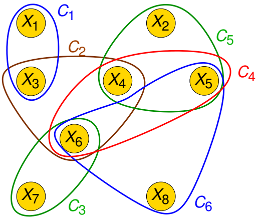

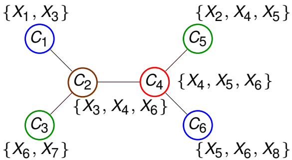

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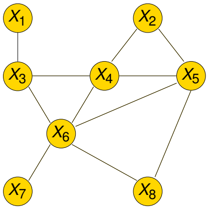

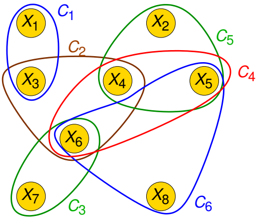

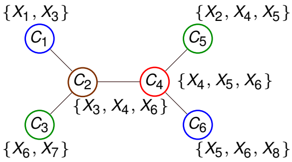

For a triangulated graph we can **always** construct a join tree.

Junction tree

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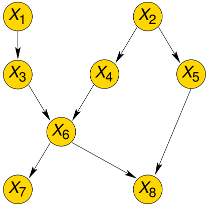

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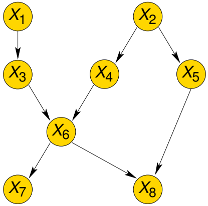

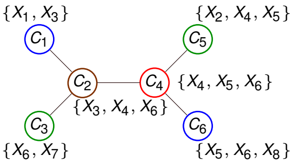

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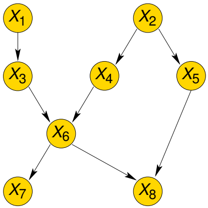

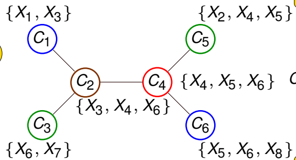

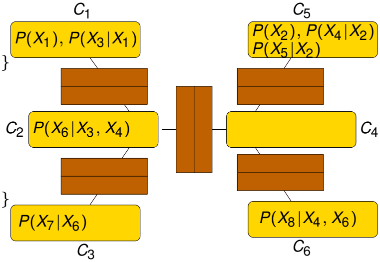

Computation of marginal probabilities in junction tree

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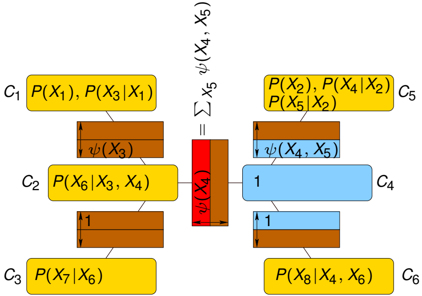

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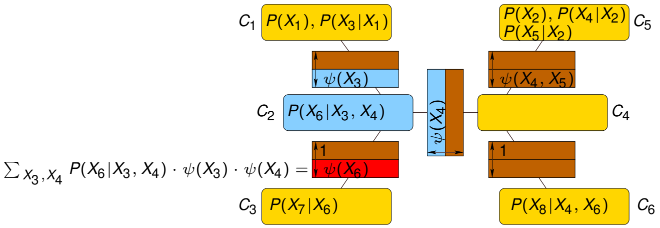

Computation of marginal probabilities in junction tree

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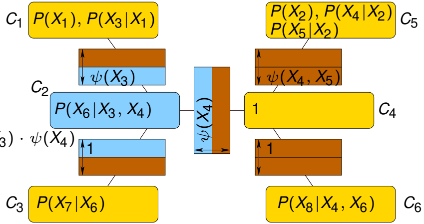
$$P(X_3) = \sum_{X_4, X_6} P(X_6|X_3, X_4) \cdot \psi(X_3) \cdot \psi(X_4)$$



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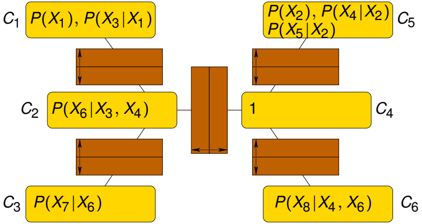
- The process is finished when all nodes have all mailboxes full.

Example of the computation of $P(X_3)$ in junction tree



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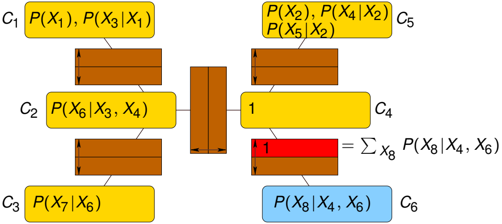

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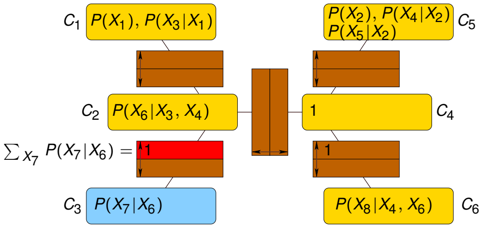

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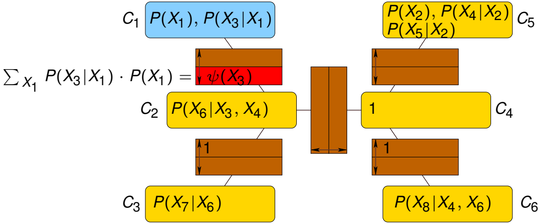

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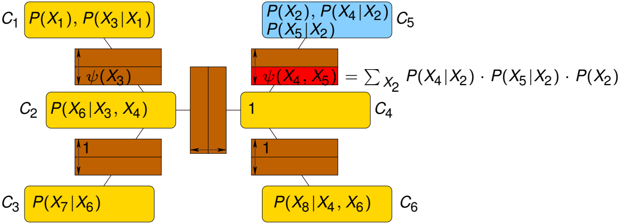

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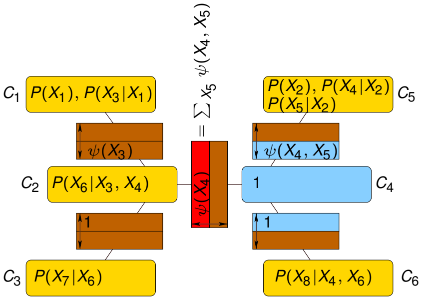

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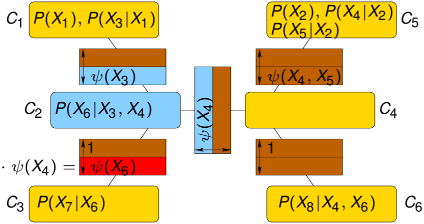

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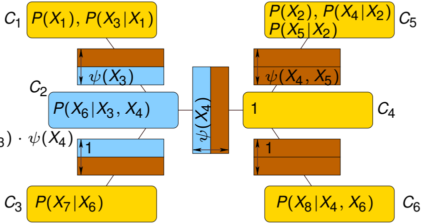

Other inference methods

- Exact methods: Lauritzen-Spiegelhalter method, Shenoy-Shafer method, Lazy propagation (A. Madsen and F. V. Jensen), Variable elimination - e.g., Bucket elimination (R. Dechter et al.).

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- Exact methods exploiting local structure of tables: Algebraic circuits (A. Darwiche et al.).

Other tasks solved in Bayesian networks

- Maximum aposteriori configuration (MAP)

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- Decision making maximizing expected utility when also utility and decision nodes are included (decision diagrams)

References

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