INVARIANTS TO CONVOLUTION IN ARBITRARY DIMENSIONS

Jan Flusser, Jiří Boldyš and Barbara Zitová

Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 4, 182 08 Prague 8, Czech Republic
E-mail: {flusser,boldys,zitova}@utia.cas.cz

Abstract

Restoration of multidimensional image data which were acquired by a linear imaging system of unknown point-spread function (PSF) is an important problem whose classical solution is based on blind deconvolution (BD). Since BD is an ill-posed and often impossible task, we propose an alternative approach. We introduce a new class of image descriptors which are invariant to convolution of the original image with arbitrary centrosymmetric PSF. The invariants are based on image moments and can be defined in the spectral domain as well as in the spatial domain. The paper presents theoretical results as well as numerical examples and practical applications.

Keywords: Multidimensional imaging, linear system, moments, convolution invariants.

I. Introduction

Restoration of multidimensional image data which were acquired by a real imaging system is the key problem in many application areas such as remote sensing, astronomy and medicine, among others. Most cameras, scanners and other sensors can be modeled as a *linear space-invariant* system, where the relationship between the input $f(\mathbf{x})$ and the acquired image $g(\mathbf{x})$ is described as

$$g(\mathbf{x}) = (f * h)(\mathbf{x}). \tag{1}$$

In the above model, $f(\mathbf{x})$ can be explained as an ideal image of the observed scene, $h(\mathbf{x})$ is the point-spread function (PSF) of the system and $*$ denotes multidimensional convolution.

In many application areas, the PSF is unknown or partially unknown. Nevertheless, it is desirable to find a description of the original object that does not depend on the imaging system. This task has been traditionally solved via *blind deconvolution* (BD) that removes or suppresses the blurring introduced by the PSF of the system. Regardless of the particular method used, BD is an ill-posed problem whose computing complexity can be extremely high and which often does not yield satisfactory results [1], [2], [3], [4], [5].

In this paper, we propose an alternative approach. We introduce a new class of image descriptors (features) which are not affected by the PSF. The only assumptions are the symmetry of the PSF (i.e. $h(\mathbf{x}) = h(-\mathbf{x})$) and the energy-preserving property of the imaging system , i.e.

$$\int_{R^N} h(\mathbf{x})\mathbf{dx} = 1.$$

In this way we avoid the difficult inversion of eq. (1). We do not obtain the complete restoration of the image f, but we are still able to describe its content in a way that is sufficient in most cases. In other words, we present a set of functionals whose domain is a space of multidimensional functions and that fulfill the invariance constraint, i.e. $I(f) = I(f * h)$ for any admissible f and h. Such functionals are called the *blur invariants.*

The rest of the paper is organized as follows. In Section II, basic definitions and propositions are given to build up the necessary mathematical background. Invariants to convolution defined in the Fourier and the spatial domains are introduced in Sections III and IV, respectively. Close relationship between both classes of invariants is shown in Section V. In Section VI, it is shown how to make the blur invariants independent of image contrast, scale and rotation. Robustness to random noise is investigated experimentally in Section VII. Section VIII demonstrates a practical application of the blur invariants to the satellite image-to-image registration. The concluding Section contains a discussion about possible practical applications.

II. Notation and Mathematical Preliminaries

In this Section, we introduce basic terms and relations which will be used later in the paper.

Notation: For $N \geq 1$, $x_i \in \mathcal{R}$, $p_i \in \mathcal{N}_0$, $k_i \in \mathcal{N}_0$ ($\mathcal{R}$ and $\mathcal{N}_0$ denote the sets of real numbers and non-negative integers, respectively) we introduce the N-dimensional vector of coordinates

$$\mathbf{x} \equiv (x_1, \cdots, x_N),$$

the N-dimensional vector of parameters

$$\mathbf{p} \equiv (p_1, \cdots, p_N)$$

and the following notation:

$$\mathbf{dx} \equiv dx_1 \cdots dx_N,$$

$$|\mathbf{p}| \equiv \sum_{i=1}^{N} p_i,$$

$$\mathbf{p}! \equiv \prod_{i=1}^{N} (p_i!),$$

$$\mathbf{x}^{\mathbf{p}} \equiv \prod_{i=1}^{N} x_i^{p_i},$$

$$\binom{\mathbf{p}}{\mathbf{k}} \equiv \prod_{i=1}^{N} \binom{p_i}{k_i}.$$

We also recall the N-dimensional binomial formula

$$(\mathbf{x} + \mathbf{y})^{\mathbf{p}} = \sum_{\mathbf{0} \leq \mathbf{k} \leq \mathbf{p}} \binom{\mathbf{p}}{\mathbf{k}} \mathbf{x}^{\mathbf{p}-\mathbf{k}} \mathbf{y}^{\mathbf{k}}.$$

Definition 1: By N-dimensional *image function* (or *image*) we understand any real function $f(\mathbf{x}) \in L_1(\mathcal{R}^N)$ having a bounded support and nonzero integral.

Definition 2: *Ordinary geometric moment* $m_{\mathbf{p}}^{(f)}$ of order $|\mathbf{p}|$ of the image $f(\mathbf{x})$ is defined by the integral

$$m_{\mathbf{p}}^{(f)} = \int_{R^N} \mathbf{x}^{\mathbf{p}} f(\mathbf{x}) \mathbf{dx}. \tag{2}$$

Definition 3: *Central moment* $\mu_{\mathbf{p}}^{(f)}$ of order $|\mathbf{p}|$ of the image $f(\mathbf{x})$ is defined as

$$\mu_{\mathbf{p}}^{(f)} = \int_{R^N} (\mathbf{x} - \mathbf{x_t})^{\mathbf{p}} f(\mathbf{x}) \mathbf{dx}, \tag{3}$$

where

$$\mathbf{x_t} = \frac{1}{m_{0\ldots0}} (m_{10\ldots0}, m_{01\ldots0}, \cdots, m_{0\ldots01})$$

denotes the centroid of $f(\mathbf{x})$.

Definition 4: Fourier transform (or spectrum) $F(\mathbf{u})$ of the image $f(\mathbf{x})$ is defined as

$$F(\mathbf{u}) = \int_{R^N} f(\mathbf{x}) \cdot e^{-2\pi i \mathbf{u} \cdot \mathbf{x}} \mathbf{dx},$$

where i is the imaginary unit.

Note that the Fourier transform as well as the moments of all orders exist for any image function.

Lemma 1: Let $f(\mathbf{x})$ and $h(\mathbf{x})$ be two image functions and let $g(\mathbf{x}) = (f * h)(\mathbf{x})$. Then $g(\mathbf{x})$ is also an image function and we have, for its moments,

$$m_{\mathbf{p}}^{(g)} = \sum_{\mathbf{0} \le \mathbf{k} \le \mathbf{p}} \binom{\mathbf{p}}{\mathbf{k}} m_{\mathbf{k}}^{(h)} m_{\mathbf{p}-\mathbf{k}}^{(f)}$$

and

$$\mu_{\mathbf{p}}^{(g)} = \sum_{\mathbf{0} \le \mathbf{k} \le \mathbf{p}} \binom{\mathbf{p}}{\mathbf{k}} \mu_{\mathbf{k}}^{(h)} \mu_{\mathbf{p}-\mathbf{k}}^{(f)}$$

for any $\mathbf{p}$.

Proof: Since $g(\mathbf{x})$ has a bounded support and

$$\int_{R^N} g(\mathbf{x}) \mathbf{dx} = \int_{R^N} f(\mathbf{x}) \mathbf{dx} \cdot \int_{R^N} h(\mathbf{x}) \mathbf{dx},$$

$g(\mathbf{x})$ is an image function. Now we prove the first equality only, the proof of the second one is similar.

$$m_{\mathbf{p}}^{(g)} = \int_{R^N} \mathbf{x}^{\mathbf{p}} g(\mathbf{x}) \mathbf{dx} = \int_{R^N} \mathbf{x}^{\mathbf{p}} (f * h)(\mathbf{x}) \mathbf{dx} = \int_{R^N} \mathbf{x}^{\mathbf{p}} (\int_{R^N} h(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) \mathbf{dy}) \mathbf{dx} =$$

$$= \int_{R^N} h(\mathbf{y}) (\int_{R^N} \mathbf{x}^{\mathbf{p}} f(\mathbf{x} - \mathbf{y}) \mathbf{dx}) \mathbf{dy} = \int_{R^N} h(\mathbf{y}) (\int_{R^N} (\mathbf{x} + \mathbf{y})^{\mathbf{p}} f(\mathbf{x}) \mathbf{dx}) \mathbf{dy} =$$

$$= \int_{R^N} h(\mathbf{y}) (\int_{R^N} \sum_{\mathbf{0} \le \mathbf{k} \le \mathbf{p}} \binom{\mathbf{p}}{\mathbf{k}} \mathbf{x}^{\mathbf{p}-\mathbf{k}} \mathbf{y}^{\mathbf{k}} f(\mathbf{x}) \mathbf{dx}) \mathbf{dy} = \sum_{\mathbf{0} \le \mathbf{k} \le \mathbf{p}} \binom{\mathbf{p}}{\mathbf{k}} m_{\mathbf{k}}^{(h)} m_{\mathbf{p}-\mathbf{k}}^{(f)} \qquad \square$$

Lemma 2: Let $h(\mathbf{x})$ be a centrally symmetric image function, i.e. $h(\mathbf{x}) = h(-\mathbf{x})$. Then

- $\mu_{\mathbf{p}}^{(h)} = m_{\mathbf{p}}^{(h)}$ for every $\mathbf{p}$;
- If $|\mathbf{p}|$ is odd, then $\mu_{\mathbf{p}}^{(h)} = 0$.

Lemma 3: The relationship between the Fourier transform of an image and the geometric moments is expressed by the following equation:

$$F(\mathbf{u}) = \sum_{\mathbf{0}\le\mathbf{k}} \frac{(-2\pi i)^{|\mathbf{k}|}}{\mathbf{k}!} m_{\mathbf{k}}^{(f)} \cdot \mathbf{u}^{\mathbf{k}}.$$

The assertions of Lemmas 2 and 3 can be easily proven just using the definitions of moments and of the Fourier transform.

III. Invariants in the Spectral Domain

In this Section, the blur invariants in the Fourier spectral domain are investigated.

Theorem 1: Tangent of the Fourier transform phase is a blur invariant.

Proof: Due to the well-known convolution theorem, the corresponding relation to eq. (1) in the spectral domain has the form

$$G(\mathbf{u}) = F(\mathbf{u}) \cdot H(\mathbf{u}), \tag{4}$$

where $G(\mathbf{u})$, $F(\mathbf{u})$ and $H(\mathbf{u})$ are the Fourier transforms of the functions $g(\mathbf{x})$, $f(\mathbf{x})$ and $h(\mathbf{x})$, respectively. Considering the amplitude and phase separately, we get

$$|G(\mathbf{u})| = |F(\mathbf{u})| \cdot |H(\mathbf{u})| \tag{5}$$

and

$$\mathrm{ph}G(\mathbf{u}) = \mathrm{ph}F(\mathbf{u}) + \mathrm{ph}H(\mathbf{u}). \tag{6}$$

(Note that the last equation is correct only for those points where $G(\mathbf{u}) \neq 0$; $\mathrm{ph}G(\mathbf{u})$ is not defined otherwise. The sign "+" means here addition modulo 2π.)

Due to the central symmetry of $h(\mathbf{x})$, its Fourier transform $H(\mathbf{u})$ is real (that means the phase of $H(\mathbf{u})$ is only a two-valued function):

$$\mathrm{ph}H(\mathbf{u}) \in \{0; \pi\}.$$

It follows immediately from the periodicity of the tangent that

$$\tan(\mathrm{ph}G(\mathbf{u})) = \tan(\mathrm{ph}F(\mathbf{u}) + \mathrm{ph}H(\mathbf{u})) = \tan(\mathrm{ph}F(\mathbf{u})). \tag{7}$$

Thus, $\tan(\mathrm{ph}G(\mathbf{u}))$ is invariant under convolution of the original image with any centrally symmetric PSF. □

IV. Invariants in the Space Domain

In this Section, blur invariants based on image moments are introduced.

Theorem 2: Let $f(\mathbf{x})$ be an image function. Let us define the following function $C^{(f)} : \mathcal{N}_0^N \to \mathcal{R}$.
If $|\mathbf{p}|$ is even then

$$C(\mathbf{p})^{(f)} = 0.$$

If $|\mathbf{p}|$ is odd then

$$C(\mathbf{p})^{(f)} = \mu_{\mathbf{p}}^{(f)} - \frac{1}{\mu_{\mathbf{0}}^{(f)}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p} \\ 0<|\mathbf{n}|<|\mathbf{p}|}} \binom{\mathbf{p}}{\mathbf{n}} C(\mathbf{p}-\mathbf{n})^{(f)} \cdot \mu_{\mathbf{n}}^{(f)}. \tag{8}$$

Then $C(\mathbf{p})$ is a blur invariant for any $\mathbf{p}$. The number $r = |\mathbf{p}|$ is called the order of the invariant.

Proof: The statement of the Theorem is trivial for any even r. Let us prove the statement for odd r by induction.

- $r = 1$

$$\begin{aligned} C(1,0,\cdots,0)^{(g)} &= \mu_{100\cdots0}^{(g)} = 0, \\ C(1,0,\cdots,0)^{(f)} &= \mu_{100\cdots0}^{(f)} = 0, \\ &\cdots \end{aligned}$$

  regardless of f and g, because the central moments of order one are zero by definition.

- $r = 3$
  The invariants of the 3rd order are $C(3,0,\cdots,0), C(2,1,0,\cdots,0), C(1,1,1,0,\cdots,0)$ and others with permuted indices. Evaluating from the recursive definition (8) we get explicitly:

$$\begin{aligned} &C(3,0,\cdots,0) = \mu_{30\cdots0}, \\ &C(2,1,0,\cdots,0) = \mu_{210\cdots0}, \\ &C(1,1,1,0,\cdots,0) = \mu_{1110\cdots0}, \\ &\cdots \end{aligned}$$

  Let us show the evaluation for $C(2,1,0,\cdots,0)$; the proofs for the other invariants are similar. Applying Lemma 1 we get

$$\begin{aligned} &C(2,1,0,\cdots,0)^{(g)} = \\ &= \mu_{210\cdots0}^{(g)} = \sum_{k_1=0}^{2} \sum_{k_2=0}^{1} \binom{2}{k_1}\binom{1}{k_2} \mu_{k_1k_20\cdots0}^{(h)} \mu_{2-k_1,1-k_2,0,\cdots,0}^{(f)} = \mu_{210\cdots0}^{(f)} \mu_{00\cdots0}^{(h)} = \mu_{210\cdots0}^{(f)} = \\ &= C(2,1,0,\cdots,0)^{(f)}. \end{aligned}$$

- Let us assume the Theorem valid for all invariants of orders $1, 3, \cdots, r-2$. Using Lemma 1 we get

$$\begin{aligned} C(\mathbf{p})^{(g)} &= \mu_{\mathbf{p}}^{(g)} - \frac{1}{\mu_{\mathbf{0}}^{(g)}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p} \\ 0<|\mathbf{n}|<|\mathbf{p}|}} \binom{\mathbf{p}}{\mathbf{n}} C(\mathbf{p}-\mathbf{n})^{(g)} \cdot \mu_{\mathbf{n}}^{(g)} = \\ &= \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{p}} \binom{\mathbf{p}}{\mathbf{k}} \mu_{\mathbf{k}}^{(h)} \mu_{\mathbf{p}-\mathbf{k}}^{(f)} - \frac{1}{\mu_{\mathbf{0}}^{(f)}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p} \\ 0<|\mathbf{n}|<|\mathbf{p}|}} \binom{\mathbf{p}}{\mathbf{n}} C(\mathbf{p}-\mathbf{n})^{(f)} \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mu_{\mathbf{k}}^{(h)} \mu_{\mathbf{n}-\mathbf{k}}^{(f)}. \end{aligned}$$

  Grouping the terms with the vector index $\mathbf{k} = \mathbf{0}$ together to produce the term $C(\mathbf{p})^{(f)}$ we get

$$C(\mathbf{p})^{(g)} = C(\mathbf{p})^{(f)} + \sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{p} \\ 0<|\mathbf{k}|}} \binom{\mathbf{p}}{\mathbf{k}} \mu_{\mathbf{k}}^{(h)} \mu_{\mathbf{p}-\mathbf{k}}^{(f)} - \frac{1}{\mu_{\mathbf{0}}^{(f)}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p} \\ 0<|\mathbf{n}|<|\mathbf{p}|}} \sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{n} \\ 0<|\mathbf{k}|}} \binom{\mathbf{p}}{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} C(\mathbf{p}-\mathbf{n})^{(f)} \mu_{\mathbf{k}}^{(h)} \mu_{\mathbf{n}-\mathbf{k}}^{(f)}$$

Using the identity of binomial coefficients

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c},$$

we get by rearranging the order of the summation

$$\begin{aligned}
&C(\mathbf{p})^{(g)} = \\
= \; & C(\mathbf{p})^{(f)} + \sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{p}\\ 0<|\mathbf{k}|}} \binom{\mathbf{p}}{\mathbf{k}} \mu_{\mathbf{k}}^{(h)}\mu_{\mathbf{p}-\mathbf{k}}^{(f)} - \frac{1}{\mu_{\mathbf{0}}^{(f)}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p}\\ 0<|\mathbf{n}|<|\mathbf{p}|}} \sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{n}\\ 0<|\mathbf{k}|}} \binom{\mathbf{p}}{\mathbf{k}}\binom{\mathbf{p}-\mathbf{k}}{\mathbf{n}-\mathbf{k}} C(\mathbf{p}-\mathbf{n})^{(f)}\mu_{\mathbf{k}}^{(h)}\mu_{\mathbf{n}-\mathbf{k}}^{(f)} = \\
= \; & C(\mathbf{p})^{(f)} + \sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{p}\\ 0<|\mathbf{k}|}} \binom{\mathbf{p}}{\mathbf{k}} \mu_{\mathbf{k}}^{(h)}\mu_{\mathbf{p}-\mathbf{k}}^{(f)} - \frac{1}{\mu_{\mathbf{0}}^{(f)}} \sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{p}\\ 0<|\mathbf{k}|}} \sum_{\substack{\mathbf{k}\le\mathbf{n}\le\mathbf{p}\\ |\mathbf{n}|<|\mathbf{p}|}} \binom{\mathbf{p}}{\mathbf{k}}\binom{\mathbf{p}-\mathbf{k}}{\mathbf{n}-\mathbf{k}} C(\mathbf{p}-\mathbf{n})^{(f)}\mu_{\mathbf{k}}^{(h)}\mu_{\mathbf{n}-\mathbf{k}}^{(f)} = \\
= \; & C(\mathbf{p})^{(f)} + \sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{p}\\ 0<|\mathbf{k}|}} \binom{\mathbf{p}}{\mathbf{k}} \mu_{\mathbf{k}}^{(h)} \left[\mu_{\mathbf{p}-\mathbf{k}}^{(f)} - \frac{1}{\mu_{\mathbf{0}}^{(f)}} \sum_{\substack{\mathbf{k}\le\mathbf{n}\le\mathbf{p}\\ |\mathbf{n}|<|\mathbf{p}|}} \binom{\mathbf{p}-\mathbf{k}}{\mathbf{n}-\mathbf{k}} C(\mathbf{p}-\mathbf{n})^{(f)}\mu_{\mathbf{n}-\mathbf{k}}^{(f)}\right] = \\
= \; & C(\mathbf{p})^{(f)} + \sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{p}\\ 0<|\mathbf{k}|}} \binom{\mathbf{p}}{\mathbf{k}} \mu_{\mathbf{k}}^{(h)} \left[\mu_{\mathbf{p}-\mathbf{k}}^{(f)} - \frac{1}{\mu_{\mathbf{0}}^{(f)}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p}-\mathbf{k}\\ |\mathbf{n}|<|\mathbf{p}-\mathbf{k}|}} \binom{\mathbf{p}-\mathbf{k}}{\mathbf{n}} C(\mathbf{p}-\mathbf{n}-\mathbf{k})^{(f)}\mu_{\mathbf{n}}^{(f)}\right],
\end{aligned}$$

which we can rewrite as

$$C(\mathbf{p})^{(g)} = C(\mathbf{p})^{(f)} + \sum_{\substack{\mathbf{0}\le\mathbf{k}\le\mathbf{p}\\ 0<|\mathbf{k}|}} \binom{\mathbf{p}}{\mathbf{k}} \mu_{\mathbf{k}}^{(h)} \cdot D_{\mathbf{k}} \tag{9}$$

where

$$D_{\mathbf{k}} = \mu_{\mathbf{p}-\mathbf{k}}^{(f)} - \frac{1}{\mu_{\mathbf{0}}^{(f)}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p}-\mathbf{k}\\ |\mathbf{n}|<|\mathbf{p}-\mathbf{k}|}} \binom{\mathbf{p}-\mathbf{k}}{\mathbf{n}} C(\mathbf{p}-\mathbf{n}-\mathbf{k})^{(f)}\mu_{\mathbf{n}}^{(f)}.$$

If $|\mathbf{k}|$ is odd then Lemma 2 implies $\mu_{\mathbf{k}}^{(h)} = 0$. If $|\mathbf{k}|$ is even then it follows from the definition (8)

$$C(\mathbf{p}-\mathbf{k}) = \mu_{\mathbf{p}-\mathbf{k}} - \frac{1}{\mu_{\mathbf{0}}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p}-\mathbf{k}\\ 0<|\mathbf{n}|<|\mathbf{p}-\mathbf{k}|}} \binom{\mathbf{p}-\mathbf{k}}{\mathbf{n}} C(\mathbf{p}-\mathbf{k}-\mathbf{n}) \cdot \mu_{\mathbf{n}}.$$

Consequently,

$$D_{\mathbf{k}} = C(\mathbf{p}-\mathbf{k})^{(f)} - \frac{1}{\mu_{\mathbf{0}}^{(f)}} C(\mathbf{p}-\mathbf{k})^{(f)}\mu_{\mathbf{0}}^{(f)} = 0.$$

Thus, (9) implies $C(\mathbf{p})^{(g)} = C(\mathbf{p})^{(f)}$ for every $\mathbf{p}$. $\square$

Applying the recursive formula (8), we can construct the invariants of any order and express them in explicit form. By permutations of indices in formulas listed below, it is possible to obtain the set of all invariants of the 3rd and 5th orders:

- 3rd order:

$$\begin{aligned}
C(3,0,\cdots,0) &= \mu_{30\cdots0},\\
C(2,1,0,\cdots,0) &= \mu_{210\cdots0},\\
C(1,1,1,0,\cdots,0) &= \mu_{1110\cdots0}.
\end{aligned}$$

- 5th order:

$$\begin{aligned}
C(5,0,\cdots,0) &= \mu_{50\cdots0} - \frac{10\mu_{30\cdots0}\mu_{20\cdots0}}{\mu_{0\cdots0}}, \\
C(4,1,0,\cdots,0) &= \mu_{410\cdots0} - \frac{2}{\mu_{0\cdots0}}(3\mu_{210\cdots0}\mu_{20\cdots0} + 2\mu_{30\cdots0}\mu_{110\cdots0}), \\
C(3,2,0,\cdots,0) &= \mu_{320\cdots0} - \frac{1}{\mu_{0\cdots0}}(3\mu_{120\cdots0}\mu_{20\cdots0} + \mu_{30\cdots0}\mu_{020\cdots0} + 6\mu_{210\cdots0}\mu_{110\cdots0}), \\
C(3,1,1,0,\cdots,0) &= \mu_{3110\cdots0} - \frac{1}{\mu_{0\cdots0}}(\mu_{30\cdots0}\mu_{0110\cdots0} + 3\mu_{210\cdots0}\mu_{1010\cdots0} \\
&\quad +3\mu_{2010\cdots0}\mu_{110\cdots0} + 3\mu_{1110\cdots0}\mu_{20\cdots0}), \\
C(2,2,1,0,\cdots,0) &= \mu_{2210\cdots0} - \frac{1}{\mu_{0\cdots0}}(\mu_{0210\cdots0}\mu_{20\cdots0} + 4\mu_{1110\cdots0}\mu_{110\cdots0} + \mu_{2010\cdots0}\mu_{020\cdots0} \\
&\quad +2\mu_{120\cdots0}\mu_{1010\cdots0} + 2\mu_{210\cdots0}\mu_{0110\cdots0}), \\
C(1,1,1,1,1,0,\cdots,0) &= \mu_{111110\cdots0} - \frac{1}{\mu_{0\cdots0}}(\mu_{001110\cdots0}\mu_{110\cdots0} + \mu_{010110\cdots0}\mu_{1010\cdots0} \\
&\quad +\mu_{011010\cdots0}\mu_{10010\cdots0} + \mu_{01110\cdots0}\mu_{100010\cdots0} + \mu_{100110\cdots0}\mu_{0110\cdots0} \\
&\quad +\mu_{101010\cdots0}\mu_{01010\cdots0} + \mu_{10110\cdots0}\mu_{010010\cdots0} + \mu_{110010\cdots0}\mu_{00110\cdots0} \\
&\quad +\mu_{11010\cdots0}\mu_{001010\cdots0} + \mu_{1110\cdots0}\mu_{000110\cdots0}).
\end{aligned}$$

If we use ordinary geometric moments instead of the central ones in definition (8), we get another set of blur invariants (let us denote them $M(\mathbf{p})$). Unlike $C(\mathbf{p})$'s, $M(\mathbf{p})$'s depend on the shift of the coordinate origin.

V. Relationship between Fourier Domain Invariants and Spatial Domain Invariants

In this Section, a close relationship between the Fourier transform phase and the moment-based blur invariants is presented.

Theorem 5: Tangent of the Fourier transform phase of any image $f(\mathbf{x})$ can be expanded into power series (except at the points in which $F(\mathbf{u}) = 0$ or $\mathrm{ph}F(\mathbf{u}) = \pm\pi/2$)

$$\tan(\mathrm{ph}F(\mathbf{u})) = \sum_{\mathbf{0}\le\mathbf{k}} c_{\mathbf{k}}\mathbf{u}^{\mathbf{k}}, \tag{10}$$

where

$$c_{\mathbf{k}} = \frac{(-1)^{(|\mathbf{k}|-1)/2}\cdot(-2\pi)^{|\mathbf{k}|}}{\mathbf{k}!\cdot m_{\mathbf{0}}} M(\mathbf{k}). \tag{11}$$

Proof: Lemma 3 implies that

$$\mathrm{Re}F(\mathbf{u}) = \sum_{\substack{\mathbf{0}\le\mathbf{n}\\ |\mathbf{n}|\text{even}}} \frac{(-2\pi i)^{|\mathbf{n}|}}{\mathbf{n}!} m_{\mathbf{n}}\cdot\mathbf{u}^{\mathbf{n}} = \sum_{\substack{\mathbf{0}\le\mathbf{n}\\ |\mathbf{n}|\text{even}}} \frac{(-1)^{|\mathbf{n}|/2}(-2\pi)^{|\mathbf{n}|}}{\mathbf{n}!} m_{\mathbf{n}}\cdot\mathbf{u}^{\mathbf{n}}$$

and

$$\mathrm{Im}F(\mathbf{u}) = \frac{1}{i}\sum_{\substack{\mathbf{0}\le\mathbf{n}\\ |\mathbf{n}|\text{odd}}} \frac{(-2\pi i)^{|\mathbf{n}|}}{\mathbf{n}!} m_{\mathbf{n}}\cdot\mathbf{u}^{\mathbf{n}} = \sum_{\substack{\mathbf{0}\le\mathbf{n}\\ |\mathbf{n}|\text{odd}}} \frac{(-1)^{(|\mathbf{n}|-1)/2}(-2\pi)^{|\mathbf{n}|}}{\mathbf{n}!} m_{\mathbf{n}}\cdot\mathbf{u}^{\mathbf{n}}.$$

Thus, $\tan(\mathrm{ph}F(\mathbf{u}))$ is a ratio of two absolutely convergent power series and therefore it can be also expressed as a power series

$$\tan(\mathrm{ph}F(\mathbf{u})) = \frac{\mathrm{Im}F(\mathbf{u})}{\mathrm{Re}F(\mathbf{u})} = \sum_{\mathbf{0}\le\mathbf{k}} c_{\mathbf{k}}\mathbf{u}^{\mathbf{k}}$$

where the coefficients $c_{\mathbf{k}}$ satisfy

$$\sum_{\substack{\mathbf{0}\le\mathbf{n}\\ |\mathbf{n}|\mathrm{odd}}} \frac{(-1)^{(|\mathbf{n}|-1)/2}(-2\pi)^{|\mathbf{n}|}}{\mathbf{n}!} m_{\mathbf{n}} \cdot \mathbf{u}^{\mathbf{n}} = \sum_{\substack{\mathbf{0}\le\mathbf{n}\\ |\mathbf{n}|\mathrm{even}}} \frac{(-1)^{|\mathbf{n}|/2}(-2\pi)^{|\mathbf{n}|}}{\mathbf{n}!} m_{\mathbf{n}} \cdot \mathbf{u}^{\mathbf{n}} \cdot \sum_{\mathbf{0}\le\mathbf{k}} c_{\mathbf{k}}\mathbf{u}^{\mathbf{k}}. \tag{12}$$

As follows immediately from (12), if $|\mathbf{k}|$ is even then $c_{\mathbf{k}} = 0$. Let's prove by induction that $c_{\mathbf{k}}$ has the form (11) if $|\mathbf{k}|$ is odd.

- $|\mathbf{k}| = 1$
  As follows from (12),
  $$c_{10\cdots0} = \frac{-2\pi m_{10\cdots0}}{m_{0\cdots0}}.$$
  On the other hand, evaluation of the right-hand side of eq. (11) for $\mathbf{k} = (1, 0, \cdots, 0)$ yields
  $$\frac{(-2\pi)(-1)^0}{1!\cdot 0!\cdots 0!\cdot m_{\mathbf{0}}} M(1, 0, \cdots, 0) = \frac{-2\pi m_{10\cdots0}}{m_{0\cdots0}} = c_{10\cdots0}.$$

- Let's suppose the assertion has been proven for all $\mathbf{k}$, $|\mathbf{k}| \le r$, where r is an odd integer and let $|\mathbf{p}| = r + 2$. It follows from(12) that
  $$\frac{(-1)^{(|\mathbf{p}|-1)/2}(-2\pi)^{|\mathbf{p}|}}{\mathbf{p}!} m_{\mathbf{p}} = m_{\mathbf{0}}c_{\mathbf{p}} + \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p}\\ 0<|\mathbf{n}|}} \frac{(-1)^{|\mathbf{n}|/2}(-2\pi)^{|\mathbf{n}|}}{\mathbf{n}!} m_{\mathbf{n}}c_{\mathbf{p}-\mathbf{n}}.$$
  Introducing (11) into the right-hand side we get
  $$\frac{(-1)^{(|\mathbf{p}|-1)/2}(-2\pi)^{|\mathbf{p}|}}{\mathbf{p}!} m_{\mathbf{p}} = m_{\mathbf{0}}c_{\mathbf{p}} + \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p}\\ 0<|\mathbf{n}|<|\mathbf{p}|}} \frac{(-1)^{(|\mathbf{p}|-1)/2}(-2\pi)^{|\mathbf{p}|}}{\mathbf{n}!\cdot(\mathbf{p}-\mathbf{n})!\cdot m_{\mathbf{0}}} M(\mathbf{p}-\mathbf{n}) \cdot m_{\mathbf{n}}$$
  and, consequently,
  $$c_{\mathbf{p}} = \frac{(-1)^{(|\mathbf{p}|-1)/2}(-2\pi)^{|\mathbf{p}|}}{\mathbf{p}!\cdot m_{\mathbf{0}}} \cdot \left[m_{\mathbf{p}} - \frac{1}{m_{\mathbf{0}}} \sum_{\substack{\mathbf{0}\le\mathbf{n}\le\mathbf{p}\\ 0<|\mathbf{n}|<|\mathbf{p}|}} \binom{\mathbf{p}}{\mathbf{n}} M(\mathbf{p}-\mathbf{n}) \cdot m_{\mathbf{n}} \right].$$
  Using the M-analogy of eq. (8) we finally get
  $$c_{\mathbf{p}} = \frac{(-1)^{(|\mathbf{p}|-1)/2}(-2\pi)^{|\mathbf{p}|}}{\mathbf{p}!\cdot m_{\mathbf{0}}} M(\mathbf{p}). \qquad \square$$

VI. Additional Invariance

In this Section, we propose how to make the blur invariants $C(\mathbf{p})$ invariant also to contrast changes, scaling and rotation, that is desirable in many applications.

The global change of contrast can be modeled as a multiplication of the image and a positive constant, i.e.

$$f'(\mathbf{x}) = \alpha \cdot f(\mathbf{x}).$$

Let us define the normalized blur invariants $C_n(\mathbf{p})$ as

$$C_n(\mathbf{p}) = \frac{C(\mathbf{p})}{\mu_{\mathbf{0}}}.$$

$C_n(\mathbf{p})$ is still invariant to blurring and, moreover, it is invariant to the change of contrast too. It should be noted that $C_n(\mathbf{p})$ remains invariant to convolution even if $\mu_{\mathbf{0}}^{(h)} \neq 1$.

The isotropic scaling can be described as the coordinate transform

$$\mathbf{x}' = s \cdot \mathbf{x},$$

where the scaling factor s is a positive constant. Scaling invariance can be reached using the normalized central moments

$$\nu_{\mathbf{p}} = \frac{\mu_{\mathbf{p}}}{\mu_{\mathbf{0}}^{\frac{|\mathbf{p}|}{N}+1}}$$

in eq. (8). The invariance to convolution is still preserved.

Finding rotation invariants is much more complicated task. We present here a solution in two dimensions only.

The rotation around the origin by angle θ can be described as the coordinate transform

$$\begin{aligned} x_1' &= x_1 \cos\theta + x_2 \sin\theta, \\ x_2' &= -x_1 \sin\theta + x_2 \cos\theta. \end{aligned}$$

Let us define the *complex moment* $\gamma_{pq}^{(f)}$ as

$$\gamma_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 + ix_2)^p (x_1 - ix_2)^q f(x_1, x_2)\, dx_1 dx_2. \tag{13}$$

Its transformation under rotation is

$$\gamma_{pq}' = e^{-i(p-q)\theta} \cdot \gamma_{pq}. \tag{14}$$

Now we modify the definition of the blur invariants (8) by substituting the complex moments for the regular ones. We denote the new functionals as $K(p,q)$. It was proven in [9] that the $K(p,q)$'s are invariant to convolution and that they behave under rotation in the same way as the complex moments themselves. Thus, any product

$$\prod_{j=1}^{n} K(p_j, q_j)^{k_j} \tag{15}$$

where $n \geq 1$ and $k_j, p_j, q_j; j = 1, \cdots, n$, are non-negative integers such that

$$\sum_{j=1}^{n} k_j (p_j - q_j) = 0$$

is invariant both to convolution and rotation.

The blur and rotation invariants in 2D are discussed comprehensively in our recent paper [9]. Unfortunately, the approach based on the complex moments cannot be easily generalized for $N > 2$.

VII. Testing the Numerical Properties

The above presented theory has dealt with a continuous representation of the images. In discrete domain, the invariance properties might be disturbed due to discretization and quantization effects and also due to the round-off errors of the calculations. The aim of the accomplished experiment is to investigate the behavior of the blur invariants under discrete convolution and also to evaluate their robustness to additive random noise.

We have done numerous experiments in 1D, 2D and 3D using various images and employing the blur invariants up to the 7th order. We present here selected results achieved in 2D, the behavior of the invariants in all other cases is analogous.

(a) (b) (c)

Figure 1: Examples of the test images: (a) original image, (b) blurred image, (c) blurred image with an additive noise.

In this experiment, we took a part of Lena image sized 101×101 pixels with zero border 30 pixels wide (see Fig.1(a)). Normalized uniform square masks of different sizes (3×3, 5×5, 7×7, 9×9, 11×11, 13×13 and 15×15) were employed as the blurring filters and the original image was convolved with all of them. Each blurred image was corrupted by zero-mean Gaussian noise to get various signal to noise ratios (SNR) from 62 dB to 2 dB. On each level of SNR, twenty realizations of noise were generated and the mean values of the particular invariants were used for robustness evaluation. Figs.1(b) and 1(c) show two examples of the degraded images.

The invariants $C(2,1)$ and $C(7,0)$ as well as their relative errors were computed on all 168 test images. Fig.2 shows how the relative errors of the invariants depend on the image blur and the SNR. The influence of the image blur is negligible as can be expected from theoretical considerations. The maximum relative error of $C(2,1)$ caused by the noise is 1.6%, the same of $C(7,0)$ is a bit higher – 4%. Such errors are still acceptable. Images with the highest errors of invariants have SNR 2 dB but in most practical applications we usually assume SNR higher than 10 dB. Under such circumstances, the invariants show sufficient robustness. The noise impact is more significant in the case of the invariants of higher orders. This is in accordance with the well-known fact that the higher-order moments are more vulnerable to noise.

VIII. Application to Satellite Image Registration

In this Section, we present an application of 2D blur-rotation invariants (15) to the registration of satellite images.

Image registration in general is the process of overlaying two or more images of the same scene acquired from different viewpoints, by different sensors and/or at different times so that the pixels of the same coordinates in the images correspond to the same part of the scene. Image

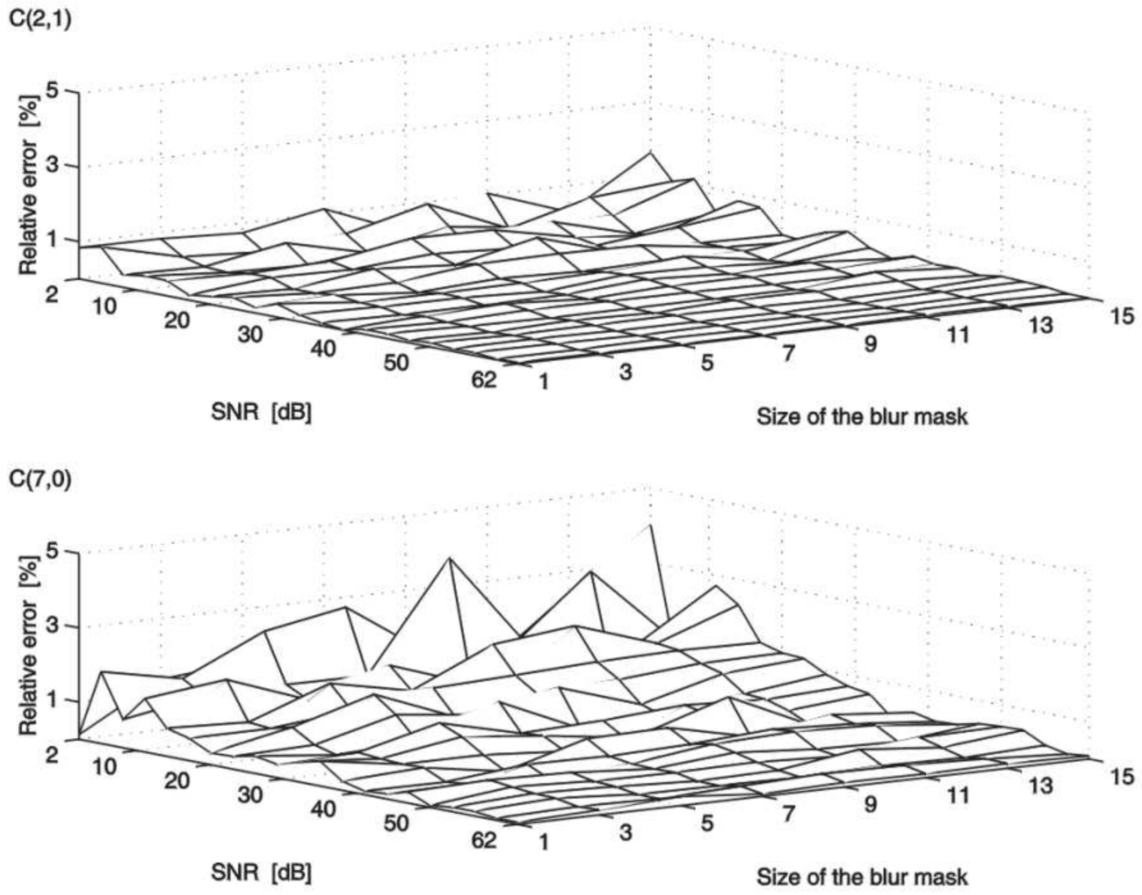

Figure 2: The relative errors of the invariants $C(2,1)$ and $C(7,0)$ for blurred and noisy images. The size of the blurring filters is from 1×1 (no blur) to 15×15, the SNR is from 2 dB to 62 dB.

registration is required as a pre-processing stage in analysis of remotely sensed data, medical image analysis, image fusion, in automatic change detection and scene monitoring, among others.

Regardless of the image data involved and of the particular application, image registration usually consists of the four major steps.

First, control points (CP) are detected both in reference and sensed images. Edge intersections, objects centroids or significant contour points can be considered for this purpose. The correspondence between the CP sets in the reference and sensed images is then established. Matching methods are based on the image content (cross-correlation, mutual information) or on symbolic description of the CP sets (parameter clustering, graph matching, relaxation). Matching is usually the most difficult part of the registration. After the CP sets have been matched, the type and parameters of spatial transform between the reference and sensed images are estimated. The mapping function can be global or local, depending on the type of the image distortions. Finally, the sensed image is transformed and overlayed over the reference one.

The invariants enter the process of image registration in the second step, CP matching. They are calculated over a circular neighborhood of each CP candidate detected earlier. After that, the correspondence is established by minimum distance rule with thresholding in the Euclidean space of the invariants. Herein described application uses the blur-rotation invariants for registration of satellite images, that are rotated and shifted one another and differently blurred. In practice, the blurring function is often an unknown composite function describing the degradation effects of the sensor and the atmosphere.

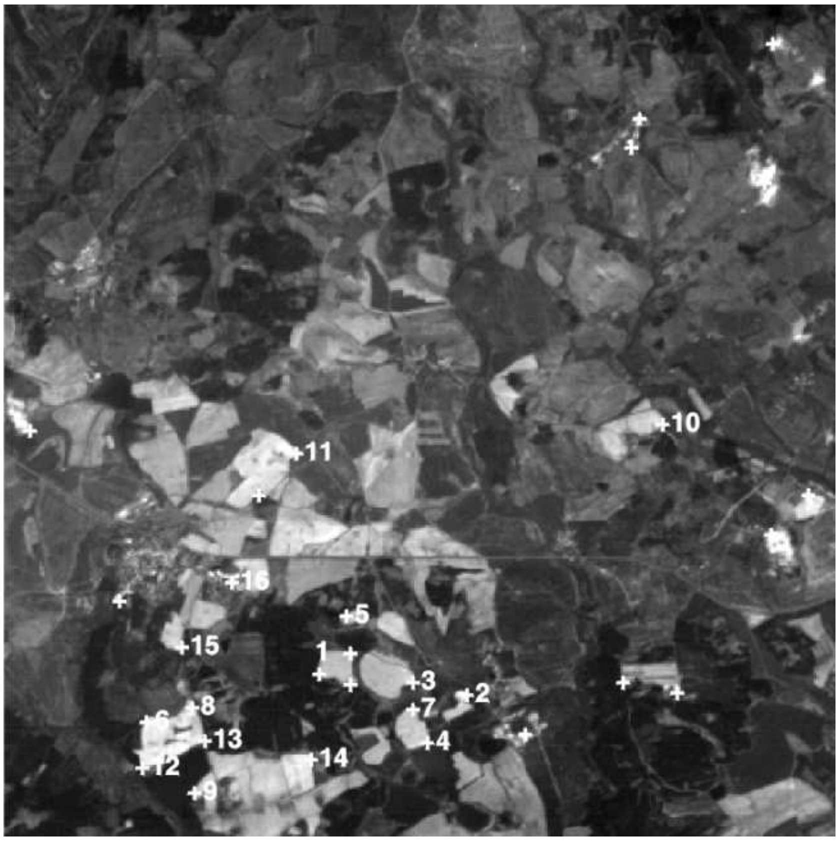

Figure 3: Reference image – SPOT subscene of the size 400x400 pixels – with the detected control point candidates. Numbered CPCs have their counterparts in the sensed image.

The experiment was performed on real satellite data with simulated blurring and rotation. The reference image of the size 400×400 pixels was extracted from the SPOT subscene, band 2, Czech Republic (see Fig. 3). The sensed image of the size 325×325 pixels was extracted from the different SPOT subscene, band 2, from the same flight covering approximately the same ground. It was then rotated by 15 degrees and the nonideal acquisition was simulated by blurring with the 7x7 averaging mask (see Fig. 4).

To find CP candidates (CPCs) in the both frames, a method developed particularly for detection of corner-like dominant points in blurred images [11] was employed. Thirty CPCs selected in the reference and sensed images are depicted in Figs. 3 and 4, respectively.

The sensed image is transformed using the rigid-body mapping function whose coefficients were calculated via least-square technique by means of the matched CPs. Inter-pixel gray values are estimated via bilinear interpolation. The co-registered images are shown in Fig. 5. Intensity values in the overlapped area are calculated as the mean of the corresponding intensity values of the reference and sensed images.

IX. Conclusion

The paper is devoted to the image features which are invariant to convolution with a centrally symmetric filter. The invariants in the spectral domain as well as in the spatial domain are derived and the relationship between them is investigated.

The assumption of centrosymmetry is not a significant limitation for practical utilisation of

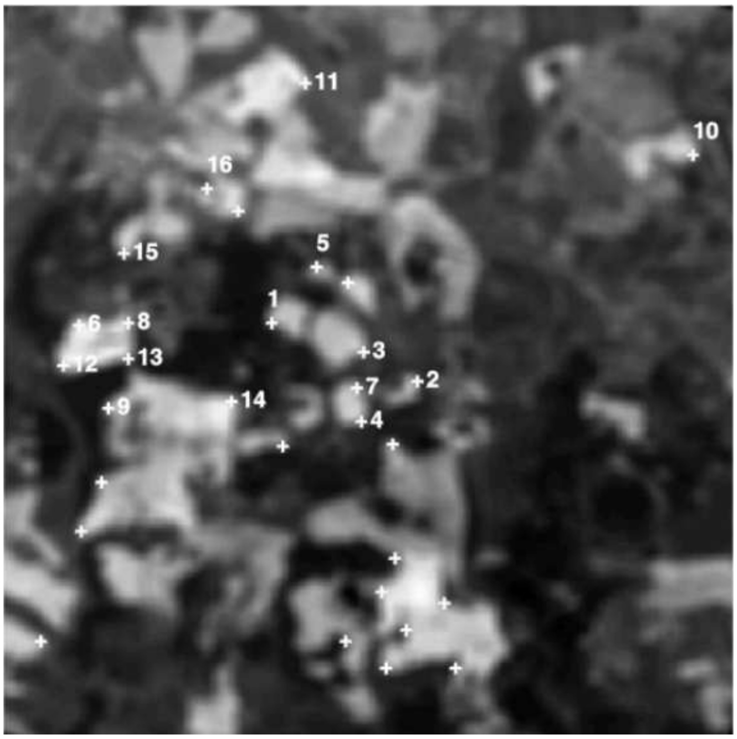

Figure 4: Sensed image with the CPCs – different SPOT subscene of the size 325x325 pixels, taken during the same flight and covering approximately the same ground. The image was rotated by 15 degrees, the nonideal acquisition was simulated by blurring with the 7x7 averaging mask. Numbered CPCs have their counterparts in the sensed image.

the method. Most real sensors and imaging systems, both optical and non-optical ones, have the PSF with certain degree of symmetry. In many cases, they have even higher symmetries, such as axial or radial symmetry. Thus, the central symmetry is general enough to describe almost all practical situations.

Practical applications of the proposed invariants may be found in object recognition in blurred environment, in template matching, feature-based image registration and in related tasks. If $N = 2$, the blur invariants can be emloyed in template matching on satellite images. Satellite images are often blurred according to (1) due to the composite sensor PSF and atmospheric turbulence [6]. Astronomical images are also degraded by a low-pass filtering due to nonideal observational conditions [7]. Another possible application is in the area of video surveillance and person authentification where face recognition from defocused photographs is often required.

If $N = 3$ or $N = 4$, the proposed method can be applied to registration of volumetric medical images degraded by blurring. Although numerous registration algorithms have been described in the recent years [8], no one has been designed particularly for blurred data.

Application of this method in practice would require to make the invariants at least rotationally independent. Unfortunatelly, this is not straightforward. Recently we have proposed a solution in 2-D that is based on complex moments [9]. General blur-rotation invariants in dimensions higher than two are, however, still under investigation.

From the practical point of view, a very important issue is robustness of the invariants to random noise, discretization errors and other factors. Usually the higher degree of invariance, the less stable they are. A common approach to increasing robustness is to use certain proper functions of the invariants that may be not strictly invariant but are more stable [10]. The

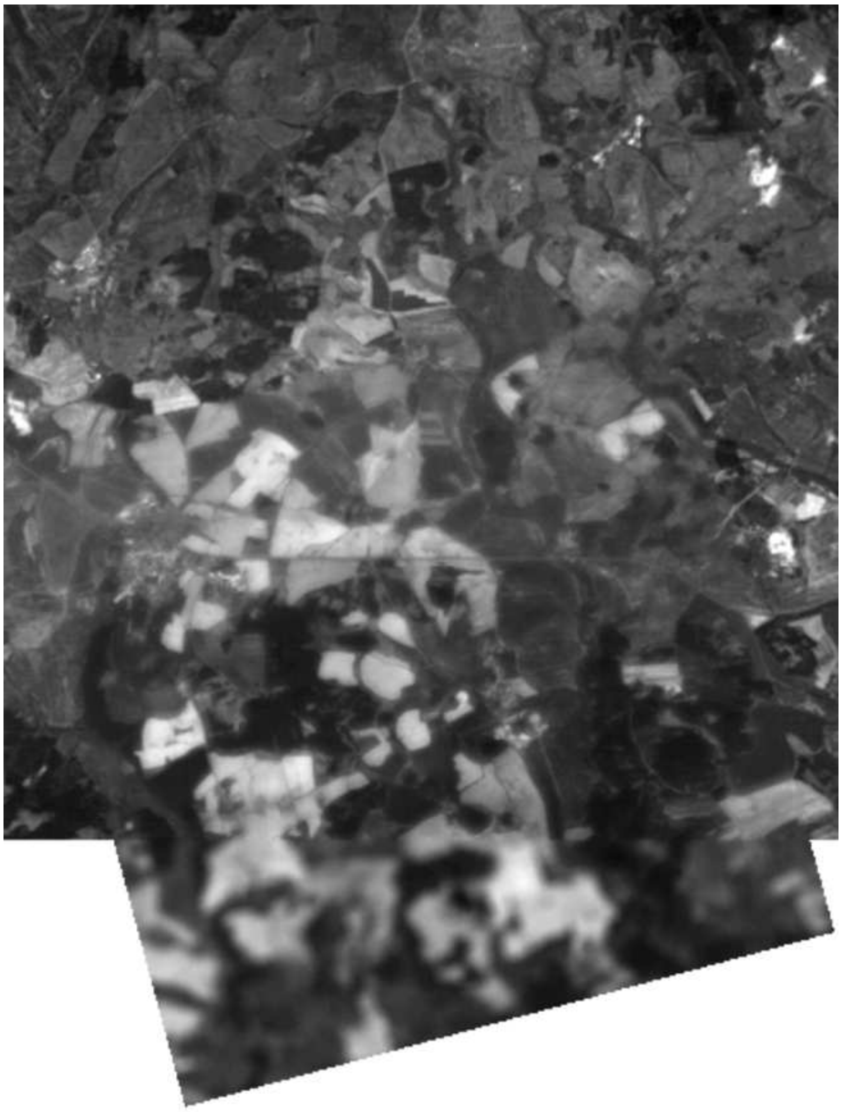

Figure 5: The registered images. Intensity values in the overlapped area are calculated as the mean of the corresponding intensity values of the reference and sensed images.

experiments presented in Section VII show it is not necessary here in the case of additive random noise – the robustness of our blur invariants is sufficient. This is mainly due to the fact that the moments are calculated by integration over the image area and therefore the zero-mean noise is averaged out.

In practice we often use the invariants of a part of the image only. In such a case the gray values along the boundary are influenced by the pixels from the outside and convolution is not well defined within the region of interest. The robustness to this so-called "boundary effect" depends on the size of the region of interest and on the size of the PSF support. The robustness can be very low when both sizes are comparable, which prevents from using the blur invariants in such cases.

Acknowledgment

This work has been supported by the grant No. 4178-3 of the Czech Ministry of Health and by the Research Project No. K 1075601 of the Czech Academy of Sciences.

References

[1] D. Kundur and D. Hatzinakos, "Blind image deconvolution," *IEEE Signal Processing Magazine*, vol. 13, no. 3, pp. 43–64, 1996.

[2] M. I. Sezan and A. M. Tekalp, "Survey of recent developments in digital image restoration," *Optical Engineering*, vol. 29, pp. 393–404, 1990.

[3] R. G. Lane and R. H. T. Bates, "Automatic multidimensional deconvolution," *J. Optical Soc. Amer. A*, vol. 4, pp. 180–188, 1987.

[4] R. L. Lagendijk, J. Biemond, and D. E. Boekee, "Identification and restoration of noisy blurred images using the expectation-maximization algorithm," *IEEE Trans. Acous., Speech, Signal Processing*, vol. 38, pp. 1180–1191, 1990.

[5] Y. Yang, N. P. Galatsanos, and H. Stark, "Projection-based blind deconvolution," *J. Optical Soc. Amer. A*, vol. 11, pp. 2401–2409, 1994.

[6] S. E. Reichenbach, D. E. Koehler, and D. W. Strelow, "Restoration and reconstruction of AVHRR images," *IEEE Trans. Geoscience and Remote Sensing*, vol. 33, pp. 997–1007, 1995.

[7] R. Krejčí, S. Šimberová, and J. Flusser, "A new multichannel blind deconvolution method and its application to solar images," in *Proceedings of the 14th International Conference on Pattern Recognition* (A. Jain, S. Venkatesh, and B. C. Lovell, eds.), (Los Alamitos), pp. 1765–1767, IEEE, August 1998.

[8] J. B. A. Maintz and M. A. Viergever, "A survey of medical image registration," *Medical Image Analysis*, vol. 2, no. 1, pp. 1–36, 1998.

[9] J. Flusser and B. Zitová, ‘Combined invariants to linear filtering and rotation," *Intl. J. Pattern Recognition Art. Intell.*, vol. 13, no. (8), 1999.

[10] L. M. J. Florack, B. M. ter Haar Romeny, J. J. Koenderink and M. A. Viergever, "General intensity transformations and differential invariants," *J. of Mathematical Imaging and Vision*, vol. 4, no. 2, pp. 171–187, 1994.

[11] B. Zitová, J. Kautský, G. Peters and J. Flusser, "Robust detection of significant points in multiframe images," *Pattern Recognition Letters*, 20:199–206, 1999.