Duals and thinnings of some relatives of the contact process

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Abstract: This paper considers contact processes with additional voter model dynamics. For such models, results of Lloyd and Sudbury can be applied to find a self-duality, as well as dualities and thinning relations with systems of random walks with annihilation, branching, coalescence, and deaths. We show that similar relations, which are known from the literature for certain interacting SDE's, can be derived as local mean field limits of the relations of Lloyd and Sudbury.

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1 Introduction

In this paper, we show how certain (mostly well-known) dualities, thinnings, and Poissonization relations for systems of interacting diffusions can be derived by considering these diffusions as 'local mean field limits' of certain interacting particle systems, to which Lloyd-Sudbury theory is applicable. The central results are Propositions 4.2 and 4.5, and Lemma 4.3. This paper is a shortened version of the preprint [7].

2 Lloyd-Sudbury theory

2.1 Introduction

Lloyd and Sudbury [3, 4, 6] have studied dualities for general spin systems which have only two-spin interactions and for which the uniform zero configuration is a trap. Based on algebraic considerations coming from quantum theory, they associate dualities with certain linear operators. The requirement that these operators are the product of operators acting only locally on each site then leads them to consider only duality functions of the form

$$\psi_\eta(x,y) = \prod_{i\in\Lambda} \eta^{x(i)y(i)}, \tag{1}$$

where Λ is a lattice, $x, y \in \{0,1\}^\Lambda$ are spin configurations, and $\eta \in \mathbb{R}\backslash\{1\}$ is a parameter. They show that there are lots of dualities between the models they consider. Moreover, they show that if two models are dual to the same model

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(albeit with a different duality parameter), then one of these models is a thinning of the other.

2.2 Lloyd-Sudbury dualities

Two Markov processes $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ with state spaces S_X and S_Y are usually called *dual* to each other with *duality function* $\psi(\cdot,\cdot)$ if

$$E[\psi(X_t, Y_0)] = E[\psi(X_0, Y_t)] \qquad (t \geq 0), \tag{2}$$

whenever X and Y are independent, for arbitrary initial laws $P[X_0 \in \cdot\,]$ and $P[Y_0 \in \cdot\,]$ on S_X and S_Y, respectively.(In [3, 4], however, the word duality is used in a much more restricted meaning.)

We will in particular be interested in the case that $S_X = S_Y = \{0,1\}^\Lambda$, where Λ is a finite set, and ψ is of the form (1). Most of what we say can be generalized to countable Λ by approximation. We consider Markov processes in $\{0,1\}^\Lambda$ which have only two-spin interactions and for which the uniform 0 configuration is a trap. More precisely, we consider Markov processes $X = (X_t)_{t\geq 0}$ in $\{0,1\}^\Lambda$ with formal generator of the form

$$\begin{aligned} G_{\mathrm{LS}} f(x) := \sum_{i\neq j} q(i,j)\Big\{ & \tfrac{1}{2} a x(i)x(j)\{f(x-\delta_i-\delta_j) - f(x)\} \\ & + b x(i)(1-x(j))\{f(x+\delta_j) - f(x)\} \\ & + c x(i)x(j)\{f(x-\delta_j) - f(x)\} \\ & + d x(i)(1-x(j))\{f(x-\delta_i) - f(x)\} \\ & + e x(i)(1-x(j))\{f(x-\delta_i+\delta_j) - f(x)\}\Big\}, \end{aligned} \tag{3}$$

where a, b, c, d, e are nonnegative constants and $q : \Lambda \times \Lambda \to \mathbb{R}$ is a nonnegative function such that

$$q(i,j) = q(j,i) \quad \text{and} \quad \sum_{j:\, j\neq i} q(i,j) = 1. \tag{4}$$

Here $\delta_i(j) := 1$ if $i = j$ and $\delta_i(j) := 0$ otherwise, and we adopt the convention that sums or suprema over i, j always run over Λ, unless stated otherwise. The letters a, b, c, d, e denote annihilation, branching, coalescence, death, and exclusion, respectively. We may interpret q as the jump rates of a continuous time Markov process on Λ, called the *underlying motion* of the interacting particle system X. We call the Markov process X with formal generator (3) the Lloyd-Sudbury model with underlying motion kernel q and parameters a, b, c, d, e, or shortly, the (q, a, b, c, d, e)-*LSM.*

We cite the next proposition from [6, formula (9)], which is a simplification of [3, formula (21)]. If two processes are dual with the duality function in (1), then we say that they are dual with *duality parameter* η.

Proposition 2.1. (Dualities between Lloyd-Sudbury models) *The (q,a,b,c,d,e)-LSM and (q,a',b',c',d',e')-LSM are dual with duality parameter $\eta \in \mathbb{R}\backslash\{1\}$, provided that*

$$a' = a + 2\eta\gamma, \quad b' = b + \gamma, \quad c' = c - (1+\eta)\gamma, \quad d' = d + \gamma, \quad e' = e - \gamma, \tag{5}$$

where $\gamma := (a + c - d + \eta b)/(1-\eta)$. In particular, each model with $b > 0$ and $\eta := (d - a - c)/b \neq 1$ is self-dual with parameter η.

2.3 Thinnings

In this section, we recall from [4] that if two (q,a,b,c,d,e)-LSM's have the same dual (in the sense of Proposition 2.1), then one is a thinning of the other. Here, in general, we interpret the space $\mathbb{N}^\Lambda$ as the space of all particle configurations x on Λ, where each site $i \in \Lambda$ can be occupied by $x(i) = 0, 1, \ldots$ particles. A u-*thinning* of a particle configuration $x \in \mathbb{N}^\Lambda$, denoted by $\mathrm{Thin}_u(x)$, is then obtained by independently throwing away particles, where a particle at $i \in \Lambda$ is kept with probability $u(i)$. If x and / or u are random, then we throw away particles independently conditional on x and u.

The next lemma gives suficient conditions for one Lloyd-Sudbury model X^1 to be a v-thinning of another Lloyd-Sudbury model X^2.

Lemma 2.2. (Thinnings of Lloyd-Sudbury models) *Consider (q,a^k,b^k,c^k,d^k,e^k)-LSM's X^k $(k=0,1,2)$. Assume that for $k=1,2$, X^k is dual to X^0 with the duality function from (1) and duality parameter η^{k0}. Then*

$$P[X^1_0 \in \cdot\,] = P[\mathrm{Thin}_v(X^2_0) \in \cdot\,] \quad \textit{implies} \quad P[X^1_t \in \cdot\,] = P[\mathrm{Thin}_v(X^2_t) \in \cdot\,] \tag{6}$$

$(t \geq 0)$, provided that $v := (1-\eta^{20})/(1-\eta^{10}) \in [0,1]$.

Proof This is more or less [6, Theorem 2.1] (although the formulation of thinning is a bit different there). $\square$

3 Contact-voter models

In this section, we apply the results from the previous section to mixtures of the contact process and the voter model. Let q be as in (4) and let $r,s,m \geq 0$ be constants. By definition, the contact-voter process with underlying motion kernel q and parameters r,s,m, shortly the (q,r,s,m)-*CVP*, is the Markov process $X = (X_t)_{t\geq 0}$ in $\{0,1\}^\Lambda$ with formal generator

$$\begin{aligned} G_{\mathrm{CVP}} f(x) := \sum_{i\neq j} q(i,j)\Big\{&(r+s)x(i)(1-x(j))\{f(x+\delta_j) - f(x)\} \\ &+ r(1-x(i))x(j)\{f(x-\delta_j) - f(x)\}\Big\} \\ &+ m\sum_i x(i)\{f(x-\delta_i) - f(x)\}. \end{aligned} \tag{7}$$

We may interpret $X_t(i) \in \{0,1\}$ as the genetic type of an organism living at time $t \geq 0$ at the site $i \in \Lambda$, where 1 is the fitter type. An organism living at site i invades a site j with rate $rq(i,j)$ if it is of type 0, and with rate $(r+s)q(i,j)$ if it is of type 1. In addition, organisms of type 1 mutate with rate m to organisms of type 0. We call r the *resampling rate*, s the *selection rate*, and m the *mutation rate*. For $r=0$, our process is a contact process with infection rate s and recovery rate m, while setting $s=m=0$ yields a voter model.

Contact-voter processes fall into the class of Lloyd-Sudbury models described in the last section. Indeed, setting

$$a=0, \quad b=r+s, \quad c=m, \quad d=r+m, \quad e=0 \tag{8}$$

in the generator in (3) yields the generator in (7). We may therefore apply Proposition 2.1 to find duals of contact-voter models. For a given duality parameter η, Proposition 2.1 tells us that the Lloyd-Sudbury model with the parameters

$$\begin{aligned} &a'=2\eta\gamma, \quad b'=\tfrac{1}{1-\eta}s, \quad c'=m-(1+\eta)\gamma, \\ &d'=m+\tfrac{\eta}{1-\eta}s, \quad e'=-\gamma, \end{aligned} \tag{9}$$

where

$$\gamma=\tfrac{\eta}{1-\eta}s-r \tag{10}$$

is dual to the (q,r,s,m)-CVP, provided that the rates a',b',c',d',e' are all nonnegative (and, hence, the LSM with these rates is well-defined). The conditions $a',e' \geq 0$ are equivalent to $\eta \in (-\infty,0] \cup \{r/(r+s)\}$, where $r/(r+s)$ is the value of η that yields a self-duality. The condition $b' \geq 0$ is now trivially fulfilled, while $d' \geq 0$ is equivalent to $-m/(s-m) \leq \eta$ if $m<s$. The condition $c' \geq 0$ leads to somewhat messy conditions on η, but if $-1 \leq \eta$, this is also trivially fulfilled. In the latter case, we can moreover give a nice interpretation to the duals with $\eta \leq 0$.

Let q be as before and let $\varepsilon \in [0,1]$ and $\rho,\beta,\delta \geq 0$ be constants. Let $Y=(Y_t)_{t\geq 0}$ be a Markov process in $\{0,1\}^\Lambda$, defined by the formal generator

$$\begin{aligned} G_{\rm RW}f(y) := \sum_{ij} q(i,j)\Big\{&\rho y(i)\Big((1-y(j))\{f(y+\delta_j-\delta_i)-f(y)\} \\ &+y(j)\{\varepsilon f(y-\delta_j-\delta_i)+(1-\varepsilon)f(y-\delta_i)-f(y)\}\Big) \\ &+\beta y(i)\Big((1-y(j))\{f(y+\delta_j)-f(y)\}+\varepsilon y(j)\{f(y-\delta_j)-f(y)\}\Big)\Big\} \\ +\delta\sum_i y(i)\{f(y-\delta_i)-f(y)\}.& \end{aligned} \tag{11}$$

We may describe the process Y in words as follows. Particles jump from a site i to a site j with rate $\rho q(i,j)$, a particle at the site i gives with rate $\beta q(i,j)$ birth to a particle at the site j, and particles die with rate δ. If in this process, two particles land on the same site, then they annihilate with probability ε, and coalesce with the remaining probability. We call Y a system of random walks with annihilation, branching, and coalescence, with underlying motion kernel q and parameters $\varepsilon, \rho, \beta, \delta$, or shortly the $(q, \varepsilon, \rho, \beta, \delta)$-*RW*. This type of models falls into the class of Lloyd-Sudbury models. Indeed, the $(q, \varepsilon, \rho, \beta, \delta)$-RW is the $(q, 2\rho\varepsilon, \beta, \rho(1-\varepsilon) + \beta\varepsilon + \delta, \delta)$-LSM.

Reinterpreting the duals in (9), we find the following result.

Proposition 3.1. (Duals of contact-voter processes)

(a) *Each (q, r, s, m)-CVP with $s > 0$ is self-dual with duality parameter $r/(r+s)$.*

(b) *A (q, r, s, m)-CVP is dual, with duality parameter $-\varepsilon$, to the $(q, \varepsilon, r + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW, provided that $m \geq \frac{\varepsilon}{1+\varepsilon}s$. Conversely, a $(q, \varepsilon, \rho, \beta, \delta)$-RW is dual, with duality parameter $\eta = -\varepsilon$, to the $(q, \rho - \varepsilon\beta, (1+\varepsilon)\beta, \delta + \varepsilon\beta)$-CVP, provided that $\rho \geq \varepsilon\beta$.*

Note that the formulas in part (b) simplify a lot if $\varepsilon = 0$, i.e., if there is no annihilation.

By Lemma 2.2, it follows that $(q, \varepsilon, \rho, \beta, \delta)$-RW's are thinnings of each other, and of CVP's.

Lemma 3.2. (Thinnings of contact-voter processes) *Fix q as in (4), $r, m \geq 0$, $s > 0$, and $0 \leq \varepsilon \leq \varepsilon' \leq 1$ such that $m \geq \frac{\varepsilon'}{1+\varepsilon'}s$. Then*

(a) *The $(q, \varepsilon, r + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW is a $(1+\varepsilon)^{-1}(1+\frac{r}{s})^{-1}$-thinning of the (q, r, s, m)-CVP.*

(b) *The $(q, \varepsilon', r + \frac{\varepsilon'}{1+\varepsilon'}s, \frac{1}{1+\varepsilon'}s, m - \frac{\varepsilon'}{1+\varepsilon'}s)$-RW is a $(1+\varepsilon)/(1+\varepsilon')$-thinning of the $(q, \varepsilon, r + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW.*

In particular, it follows that each $(q, \varepsilon, r + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW is a thinning of the $(q, 0, r, s, m)$-RW, i.e., a system of random walks with branching, coalescence, and deaths (with no annihilation).

4 Local mean field limits

4.1 Basic definitions

In this section, we will argue that two types of interacting SDE's, and one particle system, all three with applications in population dynamics, can be obtained as 'local mean field limits' of contact voter processes and their duals. Moreover, we will investigate how the dualities and thinning relations of Proposition 3.1 and Lemma 3.2 behave under these limits. Since our main interest is in dualities and thinnings, we do not give proofs of our limit relations, as this would get too technical.

First, we will consider a *stepping stone model* with underlying motion kernel q and parameters r, s, m, shortly the (q, r, s, m)-*SSM*. This is the Markov process $\mathcal{X}$

in $[0,1]^\Lambda$ defined by the system of stochastic differential equations (SDE)

$$\begin{aligned} \mathrm{d}\mathcal{X}_t(i) = &\sum_j q(j,i)(\mathcal{X}_t(j) - \mathcal{X}_t(i))\,\mathrm{d}t + s\mathcal{X}_t(i)(1-\mathcal{X}_t(i))\,\mathrm{d}t - m\mathcal{X}_t(i)\,\mathrm{d}t \\ &+\sqrt{2r\mathcal{X}_t(i)(1-\mathcal{X}_t(i))}\,\mathrm{d}B_t(i), \end{aligned} \tag{12}$$

driven by independent Brownian motions. This model and generalizations have been considered in [5]. The parameters r, s, m can be interpreted as resampling, selection, and mutation rates.

Second, we will consider a *branching particle system* with additional annihilation and coalescence, with underlying motion kernel q and parameters a, b, c, d, shortly the (q,a,b,c,d)-*BPS*. This is the Markov process $\mathcal{Y}$ in $\mathbb{N}^\Lambda$ defined by the formal generator

$$\begin{aligned} G_{\mathrm{SRW}} f(y) := &\sum_{i\neq j} q(i,j)y(i)\{f(y+\delta_j-\delta_i) - f(y)\} \\ &+a\sum_i y(i)(y(i)-1)\{f(y-2\delta_i)-f(y)\} + b\sum_i y(i)\{f(y+\delta_i)-f(y)\} \\ &+c\sum_i y(i)(y(i)-1)\{f(y-\delta_i)-f(y)\} + d\sum_i y(i)\{f(y-\delta_i)-f(y)\}. \end{aligned} \tag{13}$$

This model, with annihilation rate $a = 0$, has been considered in [1].

Third, we will consider a *super random walk* with quadratic killing, with underlying motion kernel q and parameters α, β, γ, where $\alpha, \gamma > 0$, $\beta \in \mathbb{R}$, shortly the $(q, \alpha, \beta, \gamma)$-*SRW*. This is the Markov process $\mathcal{Z}$ in $[0,\infty)^\Lambda$ defined by the system of stochastic differential equations (SDE)

$$\mathrm{d}\mathcal{Z}_t(i) = \sum_j q(j,i)(\mathcal{Z}_t(j) - \mathcal{Z}_t(i))\,\mathrm{d}t + \sqrt{\alpha\mathcal{Z}_t(i)}\,\mathrm{d}B_t(i) + \beta\mathcal{Z}_t(i)\,\mathrm{d}t - \gamma\mathcal{Z}_t(i)^2\,\mathrm{d}t. \tag{14}$$

This model has been considered in [2].

We will approximate these processes with contact-voter models and their duals, living on a lattice of the form $\Lambda^{(N)} := \Lambda \times \{1,\ldots,N\}$, with underlying motion kernel

$$q^{(N)}\big((i,k),(j,l)\big) := \begin{cases} \frac{\nu^{(N)}}{N} q(i,j) & \text{if } i \neq j, \\ \frac{1-\nu^{(N)}}{N-1} & \text{if } i = j,\ k \neq l. \end{cases} \tag{15}$$

We interpret the collection of sites $\{(i,k) : k = 1,\ldots,N\}$ as a colony of N organisms. Observe that (15) says that our underlying motion jumps with rate $\nu^{(N)}$ to a site in a different colony, and with rate $1-\nu^{(N)}$ to a different site in the same colony. We will be interested in the case that $N \to \infty$ and $\nu^{(N)} \to 0$, i.e., we have large colonies and most of the interaction takes place between organisms living in the same colony.

When considering dualities, it will be useful to assume that the laws of our approximating $\{0,1\}^{\Lambda^{(N)}}$-valued processes $X^{(N)}$ are symmetric with respect to permutations of the individuals within a colony, i.e.,

$$P[\Pi_{i,\pi}(X_t^{(N)}) \in \cdot\,] = P[X_t^{(N)} \in \cdot\,] \qquad (t \geq 0), \tag{16}$$

for each $i \in \Lambda$ and for each permutation π of $\{1, \ldots, N\}$, where

$$\big(\Pi_{i,\pi}(x)\big)(j,k) := \begin{cases} x(i,\pi(k)) & \text{if } j = i, \\ x(j,k) & \text{otherwise,} \end{cases} \tag{17}$$

is an operator that permutes the individuals in colony i. It is easy to see that if (16) holds at $t = 0$, then also for all $t > 0$.

4.2 Stepping stone models

The next proposition says that stepping stone models and branching particle systems can be obtained as local mean field limits of contact voter processes and their duals.

Proposition 4.1. (Stepping stone local mean field limit) *Let $X^{(N)}$ be the $(q^{(N)}, r^{(N)}, s, m)$-CVP and let $Y^{(N)}$ be the $(q^{(N)}, \varepsilon, r^{(N)} + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW, where $q^{(N)}$ is as in (15) and*

$$\nu^{(N)} := 1/(rN), \quad r^{(N)} := rN, \tag{18}$$

for some $\varepsilon \in [0,1]$, $r > 0$, and $s, m \geq 0$. Set

$$\begin{aligned} \overline{X}_t^{(N)}(i) &:= \frac{1}{N}\sum_{k=1}^{N} X_t^{(N)}(i,k), \\ \overline{Y}_t^{(N)}(i) &:= \sum_{k=1}^{N} Y_t^{(N)}(i,k), \end{aligned} \tag{19}$$

let $\mathcal{X}$ be the (q, r, s, m)-SSM, and let $\mathcal{Y}$ be the $(q, \varepsilon r, \frac{1}{1+\varepsilon}s, (1-\varepsilon)r, m - \frac{\varepsilon}{1+\varepsilon})$-BPS. Then, with $\Rightarrow$ denoting weak convergence of measures,

$$P[\overline{X}_0^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{X}_0 \in \cdot\,] \quad \textit{implies} \quad P[\overline{X}_t^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{X}_t \in \cdot\,] \qquad (t \geq 0), \tag{20}$$

and

$$P[\overline{Y}_0^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{Y}_0 \in \cdot\,] \quad \textit{implies} \quad P[\overline{Y}_t^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{Y}_t \in \cdot\,] \qquad (t \geq 0). \tag{21}$$

The convergence in (20) is quite natural, and shows that the stepping stone model is a good model for gene frequencies of an organism living in large colonies, with small migration between these colonies (compare the discussion in [5, Section 1]).

We now investigate what happens to the dualities and thinning relations from Proposition 3.1 and Lemma 3.2 in the local mean field limit from Proposition 4.1. For $\varepsilon = 0$, the dualities below can be found in [1, Theorem 1]; for part (b), see also [5, Lemma 2.1]. The case $\varepsilon > 0$ in part (b) was discovered during work in progress of the present author with Siva Athreya.

Proposition 4.2. (Duals of stepping stone models) *Let q be as in (4), $r, m \geq 0$, $s > 0$, and $\varepsilon \in [0,1]$ such that $m \geq \frac{\varepsilon}{1+\varepsilon}s$. Then*

(a) *The (q,r,s,m)-SSM is self-dual with duality function*

$$\psi(x,x') := e^{-\frac{r}{s}\sum_i x(i)x'(i)} \qquad (x,x' \in [0,1]^\Lambda). \tag{22}$$

(b) *The (q,r,s,m)-SSM is dual to the $(q, \varepsilon r, \frac{1}{1+\varepsilon}s, (1-\varepsilon)r, m - \frac{\varepsilon}{1+\varepsilon})$-BPS, with duality function*

$$\psi(x,y) := \prod_i \big(1-(1+\varepsilon)x(i)\big)^{y(i)} \qquad (x \in [0,1]^\Lambda,\ y \in \mathbb{N}^\Lambda). \tag{23}$$

Proof These dualities can be verified by direct calculation, but the point of this paper is to show that they occur as the limits of the relations in Proposition 3.1, so we follow that road.

We start with self-duality. Let $\mathcal{X}$ and $\mathcal{X}'$ be independent (q,r,s,m)-SSM's. We approximate $\mathcal{X}$ and $\mathcal{X}'$ with independent $(q^{(N)}, r^{(N)}, s, m)$-CVP's $X^{(N)}$ and $X^{'(N)}$, respectively, as in (20). By Proposition 3.1 (a),

$$\begin{aligned} E\Big[\prod_i \prod_{k=1}^N \Big(\frac{r^{(N)}}{r^{(N)}+s}\Big)^{X_t^{(N)}(i,k)X_0^{'(N)}(i,k)}\Big] \\ = E\Big[\prod_i \prod_{k=1}^N \Big(\frac{r^{(N)}}{r^{(N)}+s}\Big)^{X_0^{(N)}(i,k)X_t^{'(N)}(i,k)}\Big] \qquad (t \geq 0). \end{aligned} \tag{24}$$

Using (18), we rewrite the left-hand side of (24) as

$$E\Big[\prod_i \Big(1+N^{-1}\tfrac{s}{r}\Big)^{-NN^{-1}\sum_{k=1}^N X_t^{(N)}(i,k)X_0^{'(N)}(i,k)}\Big]. \tag{25}$$

Let us assume that the laws of $X^{(N)}$ and $X^{'(N)}$ are symmetric with respect to permutations of the individuals within colonies. Then, since $X_t^{(N)}$ and $X_0^{'(N)}$ are moreover independent, it follows that

$$N^{-1}\sum_{k=1}^N X_t^{(N)}(i,k)X_0^{'(N)}(i,k), \tag{26}$$

which is the fraction of the population in colony i that belongs both to $X_t^{(N)}$ and to $X_0^{'(N)}$, converges as $N \to \infty$ in probability to $\mathcal{X}_t(i)\mathcal{X}'_0(i)$. Therefore, we see that the expression in (25) converges, as $N \to \infty$, to

$$E\Big[\prod_i e^{-\frac{s}{r}\mathcal{X}_t(i)\mathcal{X}'_0(i)}\Big]. \tag{27}$$

The same arguments apply to the right-hand side of (24), so we find that $\mathcal{X}$ is self-dual with the duality function from (22).

To prove also part (b) of the proposition, let $\mathcal{X}$ be a (q,r,s,m)-SSM and let $\mathcal{Y}$ be an independent $(q, \varepsilon r, \frac{1}{1+\varepsilon}s, (1-\varepsilon)r, m - \frac{\varepsilon}{1+\varepsilon})$-BPS. We approximate $\mathcal{X}$ and $\mathcal{Y}$ with independent $(q^{(N)}, r^{(N)}, s, m)$-CVP's $X^{(N)}$ and $(q^{(N)}, \varepsilon, r^{(N)} + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW's $Y^{(N)}$, as in (20) and (21). By Proposition 3.1 (b),

$$\begin{aligned} &E\Big[\prod_i (-\varepsilon)^{\sum_{k=1}^N X_t^{(N)}(i,k)Y_0^{(N)}(i,k)}\Big] \\ &\qquad = E\Big[\prod_i (-\varepsilon)^{\sum_{k=1}^N X_t^{(N)}(i,k)Y_0^{(N)}(i,k)}\Big] \quad (t \geq 0). \end{aligned} \tag{28}$$

Assuming that the laws of $X^{(N)}$ and $Y^{(N)}$ are symmetric with respect to permutations of the individuals within a colony, we see that $\sum_{k=1}^N X_t^{(N)}(i,k)Y_0^{(N)}(i,k)$, which is the number of organisms in colony i that belongs both to $X_t^{(N)}$ and to $Y_0^{(N)}$, converges in probability as $N \to \infty$ to $\mathrm{Thin}_{\mathcal{X}_t}(\mathcal{Y}_0)(i)$. Treating the right-hand side of (28) in the same way and taking the limit $N \to \infty$, we find that

$$E\Big[\big(1-(1+\varepsilon)\big)\big)^{\mathrm{Thin}_{\mathcal{X}_t}(\mathcal{Y}_0)}\Big] = E\Big[\big(1-(1+\varepsilon)\big)\big)^{\mathrm{Thin}_{\mathcal{X}_0}(\mathcal{Y}_t)}\Big]. \tag{29}$$

It is not hard to see that this means that $\mathcal{X}$ and $\mathcal{Y}$ are dual with the duality function from (23). ∎

There is also an analogon of Lemma 3.2 for stepping stone models and branching particle systems. We adopt the following notation. For $y \in [0,\infty)^\Lambda$, we write $\mathrm{Pois}(y)$ to denote a Poisson measure with intensity y, i.e., a $\mathbb{N}^\Lambda$-valued random variable whose components are independent Poisson distributed random variables with mean $y(i)$. If Y is random, then we use the symbol $\mathrm{Pois}(Y)$ for any random variable with the law

$$P[\mathrm{Pois}(Y) \in \cdot\,] := \int P[Y \in \mathrm{d}y] P[\mathrm{Pois}(y) \in \cdot\,]. \tag{30}$$

Lemma 4.3. (Poissonizations of stepping stone models) *Let q be as in (4), $r, m \geq 0$, $s > 0$, and $0 \leq \varepsilon \leq \varepsilon' \leq 1$ such that $m \geq \frac{\varepsilon'}{1+\varepsilon'}s$. Then*

(a) *The* $(q, \varepsilon r, \frac{1}{1+\varepsilon}s, (1-\varepsilon)r, m-\frac{\varepsilon}{1+\varepsilon})$*-BPS* $\mathcal{Y}$ *is a Poissonization of the* (q,r,s,m)*-SSM* $\mathcal{X}$*, weighted with* $(1+\varepsilon)^{-1}\frac{r}{s}$*, i.e.,*

$$\begin{aligned} &P[\mathcal{Y}_0 \in \cdot\,] = P[\mathrm{Pois}((1+\varepsilon)^{-1}\tfrac{r}{s}\mathcal{X}_0) \in \cdot\,] \\ &\quad \textit{implies} \quad P[\mathcal{Y}_t \in \cdot\,] = P[\mathrm{Pois}((1+\varepsilon)^{-1}\tfrac{r}{s}\mathcal{X}_t) \in \cdot\,]. \end{aligned} \tag{31}$$

(b) *The* $(q, \varepsilon' r, \frac{1}{1+\varepsilon'}s, (1-\varepsilon')r, m-\frac{\varepsilon'}{1+\varepsilon'})$*-BPS is a* $(1+\varepsilon)/(1+\varepsilon')$*-thinning of the* $(q, \varepsilon r, \frac{1}{1+\varepsilon}s, (1-\varepsilon)r, m-\frac{\varepsilon}{1+\varepsilon})$*-BPS.*

This can be proved by taking the local mean field limits from Proposition 4.1 in the thinning relations of Lemma 3.2. Alternatively, these relations can be proved directly using the dualities from Proposition 4.2. Since this is very straightforward, we skip the proof.

4.3 Super random walks

The next proposition says that super random walks with quadratic killing can be obtained as local mean field limits of contact processes. (Note that in this case, we do not need additional voter model dynamics.)

Proposition 4.4. (Super-RW local mean field limit) *Let* $\alpha, \gamma > 0$*,* $\beta \in \mathbb{R}$*, and let* $X^{(N)}$ *be the* $(q^{(N)}, 0, s^{(N)}, m^{(N)})$*-CVP, where* $q^{(N)}$ *is as in (15), and*

$$s^{(N)} := \sqrt{\alpha\gamma N}, \quad \nu^{(N)} := 1/s^{(N)}, \quad m^{(N)} := s^{(N)} - \beta. \tag{32}$$

Set $\phi := \sqrt{\alpha/\gamma}$ *and*

$$\overline{X}_t^{(N)}(i) := \frac{\phi}{\sqrt{N}} \sum_{k=1}^{N} X_t^{(N)}(i,k). \tag{33}$$

Let $\mathcal{Z}$ *be the* (q,α,β,γ)*-SRW. Then*

$$P[\overline{X}_0^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{Z}_0^0 \in \cdot\,] \quad \textit{implies} \quad P[\overline{X}_t^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{Z}_t^0 \in \cdot\,] \qquad (t \geq 0). \tag{34}$$

The proposition below shows what happens to the well-known self-duality of the contact process in the local mean field limit from Proposition 4.4.

Proposition 4.5. (Self-duality of SRW) *The* (q,α,β,γ)*-SRW is self-dual with duality function*

$$\psi(z,z') := e^{-\frac{\gamma}{\alpha}\sum_i z(i)z'(i)}. \tag{35}$$

Proof We will show that this duality is the local mean field limit of the contact process self-duality. Let $\mathcal{Z}$ and $\mathcal{Z}'$ be independent (q,α,β,γ)-SRW's. Approximate $\mathcal{Z}$ and $\mathcal{Z}'$ with independent $(q^{(N)}, 0, s^{(N)}, m^{(N)})$-CVP's $X^{(N)}$ and $X'^{(N)}$ as in Proposition 4.4. By Proposition 3.1 (a), $X^{(N)}$ and $X'^{(N)}$ are dual with duality parameter 0, i.e., with duality function

$$\psi(x, x') = \prod_i \prod_{k=1}^{N} 0^{x(i,k)x'(i,k)} = \prod_i 1_{\{\sum_{k=1}^N x(i,k)x'(i,k)=0\}}, \tag{36}$$

i.e.,

$$\begin{aligned} E\Big[\prod_i 1_{\{\sum_{k=1}^N X_t^{(N)}(i,k) X_0'^{(N)}(i,k) = 0\}}\Big] \\ = E\Big[\prod_i 1_{\{\sum_{k=1}^N X_0^{(N)}(i,k) X_t'^{(N)}(i,k) = 0\}}\Big] \qquad (t \geq 0). \end{aligned} \tag{37}$$

Assuming that the laws of $X^{(N)}$ and $X'^{(N)}$ are symmetric with respect to permutations of the individuals within colonies, we observe that

$$\sum_{k=1}^{N} X_0^{(N)}(i,k) X_t'^{(N)}(i,k) \tag{38}$$

converges in probability to $\mathrm{Pois}(\frac{\gamma}{\alpha}\mathcal{Z}_0\mathcal{Z}_t')(i)$, since for large N, the first process contains approximately $\sqrt{\frac{\gamma}{\alpha}}\mathcal{Z}_0(i)N^{1/2}$ individuals, each having an approximate probability $\sqrt{\frac{\gamma}{\alpha}}\mathcal{Z}_t'(i)N^{-1/2}$ to belong to the dual process as well. It follows that the left-hand side of (37) converges, as $N \to \infty$, to

$$P[\mathrm{Pois}(\tfrac{\gamma}{\alpha}\mathcal{Z}_0\mathcal{Z}_t') = 0] = E\big[e^{-\frac{\gamma}{\alpha}\sum_i \mathcal{Z}_0(i)\mathcal{Z}_t'(i)}\big]. \tag{39}$$

Treating the right-hand side of (37) in the same way, we find that $\mathcal{Z}$ is self-dual with the duality function from (22). ∎

We have just seen that Proposition 4.5 follows by taking the local mean field limit of Proposition 3.1 (a). One may wonder if taking the same limit in Proposition 3.1 (b), one may discover other duals of super random walks. It turns out that this is not the case. Rather, the dual processes from Proposition 3.1 (b) also converge to super random walks, and one finds duality relations that, using scaling, can all be reduced to the self-duality in Proposition 4.5. Likewise, the thinning relations from Lemma 3.2 converge, under the local mean field limit from Proposition 4.4, to trivial scaling relations. Since this is quite lengthy and does not yield anything new, we omit the details.

References

[1] S.R. Athreya and J.M. Swart. Branching-coalescing particle systems. *Prob. Theory Relat. Fields* 131(3): 376–414, 2005.

[2] M. Hutzenthaler and A. Wakolbinger. Ergodic behaviour of locally regulated branching populations. ArXiv math.PR/0509612.

[3] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. II: The concept of duality in interacting particle systems. *Ann. Probab.* 23(4): 1816–1830, 1995.

[4] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. IV: Quasi-duality and thinnings of interacting particle systems *Ann. Probab.* 25(1): 96–114, 1997.

[5] T. Shiga and K. Uchiyama. Stationary states and their stability of the stepping stone model involving mutation and selection. *Probab. Theory Relat. Fields* 73: 87–117, 1986.

[6] A. Sudbury. Dual families of interacting particle systems on graphs. *J. Theor. Probab.* 13(3): 695–716, 2000.

[7] J.M. Swart. Duals and thinnings of some relatives of the contact process. arXiv:math.PR/0604335, 19 pages, 2006.

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