Duals and thinnings of some relatives of the contact process

Jan M. Swart

ÚTIA
Pod vodárenskou věží 4
18208 Praha 8
Czech Republic
e-mail: swart@utia.cas.cz

March 28, 2006

Abstract This paper considers contact processes with additional voter model dynamics. For such models, results of Lloyd and Sudbury can be applied to find a self-duality, as well as dualities and thinning relations with systems of random walks with annihilation, branching, coalescence, and deaths. We show that similar relations, which are known from the literature for certain interacting SDE's, can be derived as local mean field limits of the relations of Lloyd and Sudbury.

MSC 2000. Primary: 82C22; Secondary: 60H10,60J60,60J80.
Keywords. Duality, thinning, Poissonization, contact process, voter model, annihilation, coalescence.
Acknowledgement. Work sponsored by GAČR grant 201/06/1323.

Contents

1 Introduction **2**

2 Lloyd-Sudbury theory **2**
2.1 Lloyd-Sudbury dualities 2
2.2 Thinnings 4

3 Contact-voter models **5**

4 Local mean field limits **7**
4.1 Basic definitions 7
4.2 Stepping stone models 9
4.3 Super random walks 11

A Lloyd-Sudbury dualities **13**

B Local mean field limits **15**
B.1 Contact voter processes 15
B.2 Super random walks 17

1 Introduction

Lloyd and Sudbury [SL95, SL97, Sud00] have studied dualities for general spin systems which have only two-spin interactions and for which the uniform zero configuration is a trap. Based on algebraic considerations coming from quantum theory, they associate dualities with certain linear operators. The requirement that these operators are the product of operators acting only locally on each site then leads them to consider only duality functions of the form

$$\psi_\eta(x,y)=\prod_{i\in\Lambda}\eta^{x(i)y(i)}, \tag{1.1}$$

where Λ is a lattice, $x,y\in\{0,1\}^\Lambda$ are spin configurations, and $\eta\in\mathbb{R}\backslash\{1\}$ is a parameter. They show that there are lots of dualities between the models they consider. Moreover, they show that if two models are dual to the same model (albeit with a different duality parameter), then one of these models is a thinning of the other.

After reviewing some main results of Lloyd and Sudbury in Section 2, in Section 3 we turn our attention to contact processes with additional voter model dynamics. Using Lloyd-Sudbury theory, we show that such models are self-dual, and moreover dual to systems of random walks with annihilation, branching, coalescence, and deaths. We also show that the latter models are thinnings of each other, and of the contact-voter models. In Section 4 we consider systems of interacting SDE's used in population dynamics. More precisely, we consider a version of the stepping stone model with selection and mutation, as well as the super random walk with an additional quadratic killing. We show that such systems can be derived as 'local mean field limits' of contact-voter models, and derive (mostly well-known) dualities, thinnings, and Poissonization relations for such models as limits of the Lloyd-Sudbury relations.

2 Lloyd-Sudbury theory

2.1 Lloyd-Sudbury dualities

Two Markov processes $X=(X_t)_{t\geq 0}$ and $Y=(Y_t)_{t\geq 0}$ with state spaces E_X and E_Y are usually[1] called *dual* to each other with *duality function* $\psi(\cdot,\cdot)$ if

$$E[\psi(X_t,Y_0)]=E[\psi(X_0,Y_t)]\qquad(t\geq 0), \tag{2.1}$$

whenever X and Y are independent, for arbitrary initial laws $P[X_0\in\cdot\,]$ and $P[Y_0\in\cdot\,]$ on E_X and E_Y, respectively. If the functions $\{\psi(\,\cdot\,,y):y\in E_Y\}$ and $\{\psi(x,\cdot\,):y\in E_X\}$ are distribution determining, such a duality is *informative.*

We will in particular be interested in the case that $E_X=E_Y=\{0,1\}^\Lambda$, where Λ is a finite or countably infinite set, and ψ is of the form (1.1). We consider Markov processes in $\{0,1\}^\Lambda$ which have only two-spin interactions and for which the uniform 0 configuration is a trap. More precisely, we consider Markov processes $X=(X_t)_{t\geq 0}$ in $\{0,1\}^\Lambda$ with formal generator

[1] In [SL95, SL97], however, the word duality is used in a much more restricted meaning.

of the form

$$
\begin{aligned}
G_{\rm LS}f(x) := \sum_{i\neq j} q(i,j)\Big\{ & \tfrac{1}{2}ax(i)x(j)\{f(x-\delta_i-\delta_j)-f(x)\} \\
& +bx(i)(1-x(j))\{f(x+\delta_j)-f(x)\} \\
& +cx(i)x(j)\{f(x-\delta_j)-f(x)\} \\
& +dx(i)(1-x(j))\{f(x-\delta_i)-f(x)\} \\
& +ex(i)(1-x(j))\{f(x-\delta_i+\delta_j)-f(x)\}\Big\},
\end{aligned}
\tag{2.2}
$$

where a, b, c, d, e are nonnegative constants and $q : \Lambda \times \Lambda \to \mathbb{R}$ is a nonnegative function such that

$$
q(i,j) = q(j,i) \quad \text{and} \quad \sum_{j:\,j\neq i} q(i,j) = 1. \tag{2.3}
$$

Here $\delta_i(j) := 1$ if $i = j$ and $\delta_i(j) := 0$ otherwise, and we adopt the convention that sums or suprema over i, j always run over Λ, unless stated otherwise. The letters a, b, c, d, e denote annihilation, branching, coalescence, death, and exclusion, respectively. We may interpret q as the jump rates of a continuous time Markov process on Λ, called the *underlying motion* of the interacting particle system X. The processes in (2.2) may be constructed via a graphical representation or via their generator; see [Lig85] as a general reference. We call the Markov process X with formal generator (2.2) the Lloyd-Sudbury model with underlying motion kernel q and parameters a, b, c, d, e, or shortly, the (q, a, b, c, d, e)-*LSM*.

Since there is no spontaneous creation of particles, it is easy to see that a (q, a, b, c, d, e)-LSM X started in

$$
\mathcal{E}_{\rm fin}(\{0,1\}) := \big\{x \in \{0,1\}^\Lambda : \sum_i x(i) < \infty\big\} \tag{2.4}
$$

stays in this space. We will say that two Lloyd-Sudbury models X and X' are dual to each other with duality parameter $\eta \in \mathbb{R}\backslash\{1\}$ if

$$
E\big[\prod_i \eta^{X_t(i)X'_0(i)}\big] = E\big[\prod_i \eta^{X_0(i)X'_t(i)}\big]. \tag{2.5}
$$

for all deterministic initial states X_0 and X'_0. (If X and X' are independent, then integrating over the initial laws we see that (2.5) holds for nondeterministic initial states as well.) Here we assume that $X_0, X'_0 \in \{0,1\}^\Lambda$ are arbitrary if Λ is finite or if $|\eta| < 1$. In case $\eta = -1$ and Λ is infinite, we assume that either $\sum_i X_0(i) < \infty$ or $\sum_i X'_0(i) < \infty$. To avoid convergence problems, in case $|\eta| > 1$, we assume that $|\Lambda| < \infty$.

We cite the next proposition from [Sud00, formula (9)], which is a simplification of [SL95, formula (21)]. While the latter applies only to symmetric models, their formula (20) applies to asymmetric models as well. A simplification of that formula in the spirit of [Sud00, formula (9)], plus an outline of its proof, can be found in Appendix A.

Proposition 1 (Dualities between Lloyd-Sudbury models) *The (q, a, b, c, d, e)-LSM and (q, a', b', c', d', e')-LSM are dual with duality parameter $\eta \in \mathbb{R}\backslash\{1\}$, provided that*

$$
a' = a + 2\eta\gamma, \quad b' = b + \gamma, \quad c' = c - (1+\eta)\gamma, \quad d' = d + \gamma, \quad e' = e - \gamma, \tag{2.6}
$$

where $\gamma := (a+c-d+\eta b)/(1-\eta)$. *In particular, each model with* $b > 0$ *and* $\eta := (d-a-c)/b \neq 1$ *is self-dual with parameter* η.

2.2 Thinnings

In this section, we recall from [SL97] that if two (q,a,b,c,d,e)-LSM's have the same dual (in the sense of Proposition 1), then one is a thinning of the other. Since we will need this further on, we define thinnings here in a somewhat greater generality than is needed at this point.

We may interpret the space $\mathbb{N}^\Lambda$ as the space of all particle configurations x on Λ, where each site $i \in \Lambda$ can be occupied by $x(i) = 0, 1, \ldots$ particles. A *u-thinning* of a particle configuration $x \in \mathbb{N}^\Lambda$ is then obtained by independently throwing away particles, where a particle at $i \in \Lambda$ is kept with probability $u(i)$. More formally, for given $x \in \mathbb{N}^\Lambda$ and $u \in [0,1]^\Lambda$, we can choose $i_n \in \Lambda$ such that $x = \sum_{n=1}^m \delta_{i_n}$ (where m is allowed to be ∞), and we can choose independent $\{0,1\}$-valued random variables χ_n with $P[\chi_n = 1] = u(i_n)$. Setting

$$\mathrm{Thin}_u(x) := \sum_{n=1}^m \chi_n \delta_{i_n} \tag{2.7}$$

then defines a u-thinning of x. We will usually only be interested in the law of $\mathrm{Thin}_u(x)$, and use this symbol for any random variable with the same law as in (2.7). If X, U are random variables with values in $\mathbb{N}^\Lambda$ and $[0,1]^\Lambda$, respectively, then we use the symbol $\mathrm{Thin}_U(X)$ for any random variable with the law

$$P[\mathrm{Thin}_U(X) \in \cdot\,] := \int P[U \in \mathrm{d}u,\ X \in \mathrm{d}x] P[\mathrm{Thin}_u(x) \in \cdot\,]. \tag{2.8}$$

In practice, we will only be interested in the case that U and X are independent. We will sometimes need the elementary relation

$$P[\mathrm{Thin}_v(\mathrm{Thin}_u(x)) \in \cdot\,] = P[\mathrm{Thin}_{vu}(x) \in \cdot\,] \qquad (v, u \in [0,1]^\Lambda,\ x \in \mathbb{N}^\Lambda). \tag{2.9}$$

The next lemma gives suficient conditions for one Lloyd-Sudbury model X^1 to be a v-thinning of another Lloyd-Sudbury model X^2.

Lemma 2 (Thinnings of Lloyd-Sudbury models) *Consider* $(q, a^k, b^k, c^k, d^k, e^k)$*-LSM's* X^k $(k = 0,1,2)$. *Assume that for* $k = 1,2$, X^k *is dual to* X^0 *with the duality function from (1.1) and duality parameter* η^{k0}. *Then*

$$P[X_0^1 \in \cdot\,] = P[\mathrm{Thin}_v(X_0^2) \in \cdot\,] \quad \textit{implies} \quad P[X_t^1 \in \cdot\,] = P[\mathrm{Thin}_v(X_t^2) \in \cdot\,] \qquad (t \geq 0). \tag{2.10}$$

provided that $v := (1-\eta^{20})/(1-\eta^{10}) \in [0,1]$.

Proof This is more or less [Sud00, Theorem 2.1] (although the formulation of thinning is a bit different there), but since the proof is very short, we give it here. We introduce the notation

$$\eta^x := \prod_i \eta^{x(i)} \qquad (\eta \in \mathbb{R},\ x \in \mathbb{N}^\Lambda) \tag{2.11}$$

Setting $\theta^{k0} := 1 - \eta^{k0}$, we see that the duality between X^k and X^0 can be cast in the form

$$E\big[(1-\theta^{k0})^{X_0^0 X_t^k}\big] = E\big[(1-\theta^{k0})^{X_t^0 X_0^k}\big] \qquad (t \geq 0). \tag{2.12}$$

We need the relation

$$E\big[(1-\theta)^{\mathrm{Thin}_{\theta'}(x)}\big] = E\big[(1-\theta\theta')^{x}\big]. \tag{2.13}$$

If $\theta \in [0,1]$, this is the relation $P[\mathrm{Thin}_\theta(\mathrm{Thin}_{\theta'}(x)) = 0] = P[\mathrm{Thin}_{\theta\theta'}(x) = 0]$. It is not hard to show that (2.13) holds more generally for all $\theta \in \mathbb{R}$. Now if the initial laws of X^1 and X^2 are related as in (2.10), then for $t \geq 0$

$$\begin{aligned} & E[(1-\theta^{10})^{X_0^0 \mathrm{Thin}_v(X_t^2)}] = E[(1-\theta^{10})^{\mathrm{Thin}_v(X_0^0 X_t^2)}] = E[(1-v\theta^{10})^{X_0^0 X_t^2}] = E[(1-\theta^{20})^{X_0^0 X_t^2}] \\ & = E[(1-\theta^{20})^{X_t^0 X_0^2}] = E[(1-\theta^{10})^{X_t^0 \mathrm{Thin}_v(X_0^2)}] = E[(1-\theta^{10})^{X_t^0 X_0^1}] = E[(1-\theta^{10})^{X_0^0 X_t^1}]. \end{aligned} \tag{2.14}$$

Since this is true for each X_0^0, it follows that $\mathrm{Thin}_u(X_t^2)$ and X_t^1 are equal in distribution. ∎

Remarks I did not check the answers to the following questions: 1. (Perhaps not too difficult) In Lemma 2, is it sufficient if X^0 is only a formal dual, i.e., is it allowed that some of the rates $a^0, \ldots, e^0$ are negative? 2. (Perhaps somewhat more tricky) Is there a general formula, in the spirit of (2.6), that tells us when one Lloyd-Sudbury model is a thinning of another?

3 Contact-voter models

In this section, we apply the results from the previous section to mixtures of the contact process and the voter model. Let q be as in (2.3) and let $r, s, m \geq 0$ be constants. By definition, the contact-voter process with underlying motion kernel q and parameters r, s, m, shortly the (q, r, s, m)-*CVP*, is the Markov process $X = (X_t)_{t\geq 0}$ in $\{0,1\}^\Lambda$ with formal generator

$$\begin{aligned} G_{\mathrm{CVP}} f(x) := \sum_{i \neq j} q(i,j) \Big\{ & (r+s)x(i)(1-x(j))\{f(x+\delta_j) - f(x)\} \\ & + r(1-x(i))x(j)\{f(x-\delta_j) - f(x)\} \Big\} \\ + m \sum_i & x(i)\{f(x-\delta_i) - f(x)\}. \end{aligned} \tag{3.1}$$

We may interpret $X_t(i) \in \{0,1\}$ as the genetic type of an organism living at time $t \geq 0$ at the site $i \in \Lambda$, where 1 is the fitter type. An organism living at site i invades a site j with rate $rq(i,j)$ if it is of type 0, and with rate $(r+s)q(i,j)$ if it is of type 1. In addition, organisms of type 1 mutate with rate m to organisms of type 0. We call r the *resampling rate*, s the *selection rate*, and m the *mutation rate*. For $r = 0$, our process is a contact process with infection rate s and recovery rate m, while setting $s = m = 0$ yields a voter model.

Contact-voter processes fall into the class of Lloyd-Sudbury models described in the last section. Indeed, setting

$$a = 0, \quad b = r+s, \quad c = m, \quad d = r+m, \quad e = 0 \tag{3.2}$$

in the generator in (2.2) yields the generator in (3.1). We may therefore apply Proposition 1 to find duals of contact-voter models. For a given duality parameter η, Proposition 1 tells us that the Lloyd-Sudbury model with the parameters

$$\begin{aligned} &a' = 2\eta\gamma, \quad b' = \tfrac{1}{1-\eta}s, \quad c' = m - (1+\eta)\gamma, \\ &d' = m + \tfrac{\eta}{1-\eta}s, \quad e' = -\gamma, \end{aligned} \tag{3.3}$$

where

$$\gamma = \tfrac{\eta}{1-\eta}s - r \tag{3.4}$$

is dual to the (q,r,s,m)-CVP, provided that the rates a',b',c',d',e' are all nonnegative (and, hence, the LSM with these rates is well-defined). The conditions $a',e' \geq 0$ are equivalent to $\eta \in (-\infty,0] \cup \{r/(r+s)\}$, where $r/(r+s)$ is the value of η that yields a self-duality. The condition $b' \geq 0$ is now trivially fulfilled, while $d' \geq 0$ is equivalent to $-m/(s-m) \leq \eta$ if $m < s$. The condition $c' \geq 0$ leads to somewhat messy conditions on η, but if $-1 \leq \eta$, this is also trivially fulfilled. In the latter case, we can moreover give a nice interpretation to the duals with $\eta \leq 0$.

Let q be as before and let $\varepsilon \in [0,1]$ and $\rho,\beta,\delta \geq 0$ be constants. Let $Y = (Y_t)_{t\geq 0}$ be a Markov process in $\{0,1\}^\Lambda$, defined by the formal generator

$$\begin{aligned} G_{\rm RW} f(y) := \sum_{ij} q(i,j)\Big\{&\rho y(i)\Big((1-y(j))\{f(y+\delta_j-\delta_i)-f(y)\} \\ &+y(j)\{\varepsilon f(y-\delta_j-\delta_i)+(1-\varepsilon)f(y-\delta_i)-f(y)\}\Big) \\ &+\beta y(i)\Big((1-y(j))\{f(y+\delta_j)-f(y)\}+\varepsilon y(j)\{f(y-\delta_j)-f(y)\}\Big)\Big\} \\ +\delta\sum_i y(i)\{f(y-\delta_i)-f(y)\}.& \end{aligned} \tag{3.5}$$

We may describe the process Y in words as follows. Particles jump from a site i to a site j with rate $\rho q(i,j)$, a particle at the site i gives with rate $\beta q(i,j)$ birth to a particle at the site j, and particles die with rate δ. If in this process, two particles land on the same site, then they annihilate with probability ε, and coalesce with the remaining probability. We call Y a system of random walks with annihilation, branching, and coalescence, with underlying motion kernel q and parameters $\varepsilon,\rho,\beta,\delta$, or shortly the $(q,\varepsilon,\rho,\beta,\delta)$-*RW*. This type of models falls into the class of Lloyd-Sudbury models. Indeed, the $(q,\varepsilon,\rho,\beta,\delta)$-RW is the $(q,2\rho\varepsilon,\beta,\rho(1-\varepsilon)+\beta\varepsilon+\delta,\delta)$-LSM.

Reinterpreting the duals in (3.3), we find the following result.

Proposition 3 (Duals of contact-voter processes)

(a) *Each (q,r,s,m)-CVP with $s>0$ is self-dual with duality parameter $r/(r+s)$.*

(b) *A (q,r,s,m)-CVP is dual, with duality parameter $-\varepsilon$, to the $(q,\varepsilon,r+\frac{\varepsilon}{1+\varepsilon}s,\frac{1}{1+\varepsilon}s,m-\frac{\varepsilon}{1+\varepsilon}s)$-RW, provided that $m \geq \frac{\varepsilon}{1+\varepsilon}s$. Conversely, a $(q,\varepsilon,\rho,\beta,\delta)$-RW is dual, with duality parameter $\eta=-\varepsilon$, to the $(q,\rho-\varepsilon\beta,(1+\varepsilon)\beta,\delta+\varepsilon\beta)$-CVP, provided that $\rho \geq \varepsilon\beta$.*

Note that the formulas in part (b) simplify a lot if $\varepsilon = 0$, i.e., if there is no annihilation.

By Lemma 2, it follows that $(q,\varepsilon,\rho,\beta,\delta)$-RW's are thinnings of each other, and of CVP's.

Lemma 4 (Thinnings of contact-voter processes) *Fix q as in (2.3), $r, m \geq 0$, $s > 0$, and $0 \leq \varepsilon \leq \varepsilon' \leq 1$ such that $m \geq \frac{\varepsilon'}{1+\varepsilon'}s$. Then*

(a) *The $(q, \varepsilon, r + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW is a $(1+\varepsilon)^{-1}(1+\frac{r}{s})^{-1}$-thinning of the (q, r, s, m)-CVP.*

(b) *The $(q, \varepsilon', r + \frac{\varepsilon'}{1+\varepsilon'}s, \frac{1}{1+\varepsilon'}s, m - \frac{\varepsilon'}{1+\varepsilon'}s)$-RW is a $(1+\varepsilon)/(1+\varepsilon')$-thinning of the $(q, \varepsilon, r + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW.*

In particular, it follows that each $(q, \varepsilon, r + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW is a thinning of the $(q, 0, r, s, m)$-RW, i.e., a system of random walks with branching, coalescence, and deaths (with no annihilation).

4 Local mean field limits

4.1 Basic definitions

In this section, we will argue that two types of interacting SDE's, and one particle system, all three with applications in population dynamics, can be obtained as 'local mean field limits' of contact voter processes and their duals. Moreover, we will investigate how the dualities and thinning relations of Proposition 3 and Lemma 4 behave under these limits. Since our main interest is in dualities and thinnings, we do not give detailed proofs of our limit relations, as this would get too technical, in particular for infinite initial states. Instead, we give the main calculations, which for finite Λ are almost a proof, and roughly indicate what needs to be done to make this precise.

In order to properly define our systems and their duals, we need a few definitions. Let q be as in (2.3). It is possible to choose strictly positive constants $(w_i)_{i \in \Lambda}$ in such a way that

$$\sum_i w_i < \infty \quad \text{and} \quad \sum_j q(i,j) w_j \leq K w_i \quad \forall i \in \Lambda \tag{4.1}$$

for some $K < \infty$. We fix such constants from now on and define a *Liggett-Spitzer norm* and L_1-norm

$$\|x\|_w := \sum_i w_i |x(i)|, \quad |x| := \sum_i |x(i)| \qquad (x \in \mathbb{R}^\Lambda). \tag{4.2}$$

Our processes of interest will be Markov processes taking values in the spaces

$$\mathcal{E}(S) := \{x \in S^\Lambda : \|x\|_w < \infty\} \quad \text{and} \quad \mathcal{E}_{\text{fin}}(S) := \{x \in S^\Lambda : |x| < \infty\}, \tag{4.3}$$

where $S = \mathbb{N}, [0,1]$, or $[0, \infty)$.

First, we will consider a *stepping stone model* with underlying motion kernel q and parameters r, s, m, shortly the (q, r, s, m)-*SSM*. This is the Markov process $\mathcal{X}$ in $\mathcal{E}([0,1])$ or $\mathcal{E}_{\text{fin}}([0,1])$ defined by the system of stochastic differential equations (SDE)

$$\begin{aligned} \mathrm{d}\mathcal{X}_t(i) = &\sum_j q(j,i)(\mathcal{X}_t(j) - \mathcal{X}_t(i))\,\mathrm{d}t + s\mathcal{X}_t(i)(1 - \mathcal{X}_t(i))\,\mathrm{d}t - m\mathcal{X}_t(i)\,\mathrm{d}t \\ &+ \sqrt{2r\mathcal{X}_t(i)(1 - \mathcal{X}_t(i))}\,\mathrm{d}B_t(i). \end{aligned} \tag{4.4}$$

This model and generalizations have been considered in [SU86]. The parameters r, s, m can be interpreted as resampling, selection, and mutation rates.

Second, we will consider a *branching particle system* with additional annihilation and coalescence, with underlying motion kernel q and parameters a, b, c, d, shortly the (q, a, b, c, d)-*BPS*. This is the Markov process $\mathcal{Y}$ in $\mathcal{E}(\mathbb{N})$ or $\mathcal{E}_{\rm fin}(\mathbb{N})$ defined by the formal generator

$$\begin{aligned} G_{\rm SRW} f(y) := &\sum_{i\neq j} q(i,j) y(i)\{f(y+\delta_j-\delta_i)-f(y)\} \\ &+a\sum_i y(i)(y(i)-1))\{f(y-2\delta_i)-f(y)\}+b\sum_i y(i)\{f(y+\delta_i)-f(y)\} \\ &+c\sum_i y(i)(y(i)-1)\{f(y-\delta_i)-f(y)\}+d\sum_i y(i)\{f(y-\delta_i)-f(y)\}. \end{aligned} \tag{4.5}$$

This model, with annihilation rate $a = 0$, has been considered in [AS05].

Third, we will consider a *super random walk* with quadratic killing, with underlying motion kernel q and parameters α, β, γ, where $\alpha, \gamma > 0$, $\beta \in \mathbb{R}$, shortly the $(q, \alpha, \beta, \gamma)$-*SRW*. This is the Markov process $\mathcal{Z}$ in $\mathcal{E}([0, \infty))$ or $\mathcal{E}_{\rm fin}([0, \infty))$ defined by the system of stochastic differential equations (SDE)

$$\mathrm{d}\mathcal{Z}_t(i) = \sum_j q(j,i)(\mathcal{Z}_t(j)-\mathcal{Z}_t(i))\,\mathrm{d}t+\sqrt{\alpha\mathcal{Z}_t(i)}\,\mathrm{d}B_t(i)+\beta\mathcal{Z}_t(i)\,\mathrm{d}t-\gamma\mathcal{Z}_t(i)^2\,\mathrm{d}t. \tag{4.6}$$

This model has been considered in [HWS05].

We will approximate these processes with contact-voter models and their duals, living on a lattice of the form $\Lambda^{(N)} := \Lambda \times \{1, \ldots, N\}$, with underlying motion kernel

$$q^{(N)}\big((i,k),(j,l)\big) := \begin{cases} \frac{\nu^{(N)}}{N}q(i,j) & \text{if } i\neq j, \\ \frac{1-\nu^{(N)}}{N-1} & \text{if } i=j,\ k\neq l. \end{cases} \tag{4.7}$$

We interpret the collection of sites $\{(i, k) : k = 1, \ldots, N\}$ as a colony of N organisms. Observe that (4.7) says that our underlying motion jumps with rate $\nu^{(N)}$ to a site in a different colony, and with rate $1 - \nu^{(N)}$ to a different site in the same colony. We will be interested in the case that $N \to \infty$ and $\nu^{(N)} \to 0$, i.e., we have large colonies and most of the interaction takes place between organisms living in the same colony.

When considering dualities, it will be useful to assume that the laws of our approximating $\{0, 1\}^{\Lambda^{(N)}}$-valued processes $X^{(N)}$ are symmetric with respect to permutations of the individuals within a colony, i.e.,

$$P[\Pi_{i,\pi}(X^{(N)}_t)\in\cdot\,] = P[X^{(N)}_t\in\cdot\,] \qquad (t\geq 0), \tag{4.8}$$

for each $i \in \Lambda$ and for each permutation π of $\{1, \ldots, N\}$, where

$$\big(\Pi_{i,\pi}(x)\big)(j,k) := \begin{cases} x(i,\pi(k)) & \text{if } j=i, \\ x(j,k) & \text{otherwise,} \end{cases} \tag{4.9}$$

is an operator that permutes the individuals in colony i. It is easy to see that if (4.8) holds at $t = 0$, then also for all $t > 0$.

4.2 Stepping stone models

We claim that stepping stone models and branching particle systems can be obtained as local mean field limits of contact voter processes and their duals.

Claim 5 (Stepping stone local mean field limit) *Let $X^{(N)}$ be the $(q^{(N)}, r^{(N)}, s, m)$-CVP and let $Y^{(N)}$ be the $(q^{(N)}, \varepsilon, r^{(N)} + \frac{\varepsilon}{1+\varepsilon}s, \frac{1}{1+\varepsilon}s, m - \frac{\varepsilon}{1+\varepsilon}s)$-RW, where $q^{(N)}$ is as in (4.7) and*

$$\nu^{(N)} := 1/(rN), \quad r^{(N)} := rN. \tag{4.10}$$

for some $\varepsilon \in [0,1]$, $r > 0$, and $s, m \geq 0$. Set

$$\begin{aligned} \overline{X}_t^{(N)}(i) &:= \frac{1}{N}\sum_{k=1}^{N} X_t^{(N)}(i,k), \\ \overline{Y}_t^{(N)}(i) &:= \sum_{k=1}^{N} Y_t^{(N)}(i,k), \end{aligned} \tag{4.11}$$

let $\mathcal{X}$ be the (q, r, s, m)-SSM, and let $\mathcal{Y}$ be the $(q, \varepsilon r, \frac{1}{1+\varepsilon}s, (1-\varepsilon)r, m - \frac{\varepsilon}{1+\varepsilon})$-BPS. Then

$$P[\overline{X}_0^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{X}_0 \in \cdot\,] \quad \textit{implies} \quad P[\overline{X}_t^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{X}_t \in \cdot\,] \qquad (t \geq 0), \tag{4.12}$$

and

$$P[\overline{Y}_0^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{Y}_0 \in \cdot\,] \quad \textit{implies} \quad P[\overline{Y}_t^{(N)} \in \cdot\,] \underset{N\to\infty}{\Longrightarrow} P[\mathcal{Y}_t \in \cdot\,] \qquad (t \geq 0). \tag{4.13}$$

The main ingredients for a proof of Claim 5 can be found in Appendix B.1. The convergence in (4.12) is quite natural, and shows that the stepping stone model is a good model for gene frequencies of an organism living in large colonies, with small migration between these colonies (compare the discussion in [SU86, Section 1]).

We now investigate what happens to the dualities and thinning relations from Proposition 3 and Lemma 4 in the local mean field limit from Claim 5. For $\varepsilon = 0$, the dualities below can be found in [AS05, Theorem 1]; for part (b), see also [SU86, Lemma 2.1]. The case $\varepsilon > 0$ in part (b) was discovered during work in progress of the present author with Siva Athreya.

Proposition 6 (Duals of stepping stone models) *Let q be as in (2.3), $r, m \geq 0$, $s > 0$, and $\varepsilon \in [0,1]$ such that $m \geq \frac{\varepsilon}{1+\varepsilon}s$. Then*

(a) *The (q, r, s, m)-SSM is self-dual with duality function*

$$\psi(x, x') := e^{-\frac{r}{s}\sum_i x(i)x'(i)} \qquad (x, x' \in \mathcal{E}([0,1])). \tag{4.14}$$

(b) *The (q, r, s, m)-SSM is dual to the $(q, \varepsilon r, \frac{1}{1+\varepsilon}s, (1-\varepsilon)r, m - \frac{\varepsilon}{1+\varepsilon})$-BPS, with duality function*

$$\psi(x, y) := \prod_i \big(1 - (1+\varepsilon)x(i)\big)^{y(i)} \qquad (x \in \mathcal{E}([0,1]),\ y \in \mathcal{E}(\mathbb{N})). \tag{4.15}$$

where we assume that either $|x| < \infty$ or $|y| < \infty$ in case $\varepsilon = 1$.

Proof These dualities can be verified by direct calculation, but the point of this paper is to show that they occur as the limits of the relations in Proposition 3, so we follow that road.

We start with self-duality. Let $\mathcal{X}$ and $\mathcal{X}'$ be independent (q,r,s,m)-SSM's. We approximate $\mathcal{X}$ and $\mathcal{X}'$ with independent $(q^{(N)},r^{(N)},s,m)$-CVP's $X^{(N)}$ and $X'^{(N)}$, respectively, as in (4.12). By Proposition 3 (a),

$$\begin{aligned} E\Big[\prod_i\prod_{k=1}^N\Big(\frac{r^{(N)}}{r^{(N)}+s}\Big)^{X_t^{(N)}(i,k)X_0^{'(N)}(i,k)}\Big] \\ = E\Big[\prod_i\prod_{k=1}^N\Big(\frac{r^{(N)}}{r^{(N)}+s}\Big)^{X_0^{(N)}(i,k)X_t^{'(N)}(i,k)}\Big] \qquad (t\geq 0). \end{aligned} \tag{4.16}$$

Using (4.10), e rewrite the left-hand side of (4.16) as

$$E\Big[\prod_i\Big(1+N^{-1}\tfrac{s}{r}\Big)^{-NN^{-1}\sum_{k=1}^N X_t^{(N)}(i,k)X_0^{'(N)}(i,k)}\Big]. \tag{4.17}$$

Let us assume that the laws of $X^{(N)}$ and $X'^{(N)}$ are symmetric with respect to permutations of the individuals within colonies. Then, since $X_t^{(N)}$ and $X_0^{'(N)}$ are moreover independent, it follows that

$$N^{-1}\sum_{k=1}^N X_t^{(N)}(i,k)X_0^{'(N)}(i,k), \tag{4.18}$$

which is the fraction of the population in colony i that belongs both to $X_t^{(N)}$ and to $X_0^{'(N)}$, converges as $N\to\infty$ in probability to $\mathcal{X}_t(i)\mathcal{X}_0'(i)$. Therefore, we see that the expression in (4.17) converges, as $N\to\infty$, to

$$E\Big[\prod_i e^{-\frac{s}{r}\mathcal{X}_t(i)\mathcal{X}_0'(i)}\Big]. \tag{4.19}$$

The same arguments apply to the right-hand side of (4.16), so we find that $\mathcal{X}$ is self-dual with the duality function from (4.14).

To prove also part (b) of the proposition, let $\mathcal{X}$ be a (q,r,s,m)-SSM and let $\mathcal{Y}$ be an independent $(q,\varepsilon r,\frac{1}{1+\varepsilon}s,(1-\varepsilon)r,m-\frac{\varepsilon}{1+\varepsilon})$-BPS. We approximate $\mathcal{X}$ and $\mathcal{Y}$ with independent $(q^{(N)},r^{(N)},s,m)$-CVP's $X^{(N)}$ and $(q^{(N)},\varepsilon,r^{(N)}+\frac{\varepsilon}{1+\varepsilon}s,\frac{1}{1+\varepsilon}s,m-\frac{\varepsilon}{1+\varepsilon}s)$-RW's $Y^{(N)}$, as in (4.12) and (4.13). By Proposition 3 (b),

$$\begin{aligned} E\Big[\prod_i(-\varepsilon)^{\sum_{k=1}^N X_t^{(N)}(i,k)Y_0^{(N)}(i,k)}\Big] \\ = E\Big[\prod_i(-\varepsilon)^{\sum_{k=1}^N X_t^{(N)}(i,k)Y_0^{(N)}(i,k)}\Big] \quad (t\geq 0). \end{aligned} \tag{4.20}$$

Assuming that the laws of $X^{(N)}$ and $Y^{(N)}$ are symmetric with respect to permutations of the individuals within a colony, we see that $\sum_{k=1}^N X_t^{(N)}(i,k)Y_0^{(N)}(i,k)$, which is the number of organisms in colony i that belongs both to $X_t^{(N)}$ and to $Y_0^{(N)}$, converges in probability as $N\to\infty$ to $\mathrm{Thin}_{\mathcal{X}_t}(\mathcal{Y}_0)(i)$. Treating the right-hand side of (4.20) in the same way and taking the limit $N\to\infty$, we find that

$$E\Big[\big(1-(1+\varepsilon)\big)\big)^{\mathrm{Thin}_{\mathcal{X}_t}(\mathcal{Y}_0)}\Big]=E\Big[\big(1-(1+\varepsilon)\big)\big)^{\mathrm{Thin}_{\mathcal{X}_0}(\mathcal{Y}_t)}\Big]. \tag{4.21}$$

By (2.13), this means that $\mathcal{X}$ and $\mathcal{Y}$ are dual with the duality function from (4.15). ∎

There is also an analogon of Lemma 4 for stepping stone models and branching particle systems. We adopt the following notation. For $y\in[0,\infty)^\Lambda$, we write $\mathrm{Pois}(y)$ to denote a Poisson measure with intensity y, i.e., a $\mathbb{N}^\Lambda$-valued random variable whose components are independent Poisson distributed random variables with mean $y(i)$. If Y is random, then we use the symbol $\mathrm{Pois}(Y)$ for any random variable with the law

$$P[\mathrm{Pois}(Y)\in\cdot\,]:=\int P[Y\in\mathrm{d}y]P[\mathrm{Pois}(y)\in\cdot\,]. \tag{4.22}$$

Lemma 7 (Poissonizations of stepping stone models) *Let q be as in (2.3), $r,m\geq 0$, $s>0$, and $0\leq\varepsilon\leq\varepsilon'\leq 1$ such that $m\geq\frac{\varepsilon'}{1+\varepsilon'}s$. Then*

(a) *The $(q,\varepsilon r,\frac{1}{1+\varepsilon}s,(1-\varepsilon)r,m-\frac{\varepsilon}{1+\varepsilon})$-BPS $\mathcal{Y}$ is a Poissonization of the (q,r,s,m)-SSM $\mathcal{X}$, weighted with $(1+\varepsilon)^{-1}\frac{r}{s}$, i.e.,*

$$P[\mathcal{Y}_0\in\cdot\,]=P[\mathrm{Pois}((1+\varepsilon)^{-1}\tfrac{r}{s}\mathcal{X}_0)\in\cdot\,]\quad\textit{implies}\quad P[\mathcal{Y}_t\in\cdot\,]=P[\mathrm{Pois}((1+\varepsilon)^{-1}\tfrac{r}{s}\mathcal{X}_t)\in\cdot\,]. \tag{4.23}$$

(b) *The $(q,\varepsilon' r,\frac{1}{1+\varepsilon'}s,(1-\varepsilon')r,m-\frac{\varepsilon'}{1+\varepsilon'})$-BPS is a $(1+\varepsilon)/(1+\varepsilon')$-thinning of the $(q,\varepsilon r,\frac{1}{1+\varepsilon}s,(1-\varepsilon)r,m-\frac{\varepsilon}{1+\varepsilon})$-BPS.*

This can be proved by taking the local mean field limits from Claim 5 in the thinning relations of Lemma 4. Alternatively, these relations can be proved directly using the dualities from Proposition 6. Since this is very straightforward, we skip the proof.

4.3 Super random walks

We claim that super random walks with quadratic killing can be obtained as local mean field limits of contact processes. (Note that in this case, we do not need additional voter model dynamics.)

Claim 8 (Super-RW local mean field limit) *Let $\alpha,\gamma>0$, $\beta\in\mathbb{R}$, and let $X^{(N)}$ be the $(q^{(N)},0,s^{(N)},m^{(N)})$-CVP, where $q^{(N)}$ is as in (4.7), and*

$$s^{(N)}:=\sqrt{\alpha\gamma N},\quad \nu^{(N)}:=1/s^{(N)},\quad m^{(N)}:=s^{(N)}-\beta. \tag{4.24}$$

Set $\phi:=\sqrt{\alpha/\gamma}$ and

$$\overline{X}_t^{(N)}(i):=\frac{\phi}{\sqrt{N}}\sum_{k=1}^{N}X_t^{(N)}(i,k). \tag{4.25}$$

Let $\mathcal{Z}$ be the (q,α,β,γ)-SRW. Then

$$P[\overline{X}_0^{(N)}\in\cdot\,]\underset{N\to\infty}{\Longrightarrow}P[\mathcal{Z}_0^0\in\cdot\,]\quad\textit{implies}\quad P[\overline{X}_t^{(N)}\in\cdot\,]\underset{N\to\infty}{\Longrightarrow}P[\mathcal{Z}_t^0\in\cdot\,]\qquad(t\geq 0). \tag{4.26}$$

The main idea of the proof of Claim 8 can be found in Appendix B.2.

The next proposition shows what happens to the well-known self-duality of the contact process in the local mean field limit from Claim 8.

Proposition 9 (Self-duality of SRW) *The (q,α,β,γ)-SRW is self-dual with duality function*

$$\psi(z,z') := e^{-\frac{\gamma}{\alpha}\sum_i z(i)z'(i)}. \tag{4.27}$$

Proof We will show that this duality is the local mean field limit of the contact process self-duality. Let $\mathcal{Z}$ and $\mathcal{Z}'$ be independent (q,α,β,γ)-SRW's. Approximate $\mathcal{Z}$ and $\mathcal{Z}'$ with independent $(q^{(N)},0,s^{(N)},m^{(N)})$-CVP's $X^{(N)}$ and $X'^{(N)}$ as in Claim 8. By Proposition 3 (a), $X^{(N)}$ and $X'^{(N)}$ are dual with duality parameter 0, i.e., with duality function

$$\psi(x,x') = \prod_i \prod_{k=1}^{N} 0^{x(i,k)x'(i,k)} = \prod_i 1_{\{\sum_{k=1}^N x(i,k)x'(i,k)=0\}}, \tag{4.28}$$

i.e.,

$$\begin{aligned} &E\Big[\prod_i 1_{\{\sum_{k=1}^N X_t^{(N)}(i,k)X_0'^{(N)}(i,k)\,=\,0\}}\Big] \\ &\quad = E\Big[\prod_i 1_{\{\sum_{k=1}^N X_0^{(N)}(i,k)X_t'^{(N)}(i,k)\,=\,0\}}\Big] \qquad (t\geq 0). \end{aligned} \tag{4.29}$$

Assuming that the laws of $X^{(N)}$ and $X'^{(N)}$ are symmetric with respect to permutations of the individuals within colonies, we observe that

$$\sum_{k=1}^{N} X_0^{(N)}(i,k)X_t'^{(N)}(i,k) \tag{4.30}$$

converges in probability to $\mathrm{Pois}(\frac{\gamma}{\alpha}\mathcal{Z}_0\mathcal{Z}_t')(i)$, since for large N, the first process contains approximately $\sqrt{\frac{\gamma}{\alpha}}\mathcal{Z}_0(i)N^{1/2}$ individuals, each having an approximate probability $\sqrt{\frac{\gamma}{\alpha}}\mathcal{Z}_t'(i)N^{-1/2}$ to belong to the dual process as well. It follows that the left-hand side of (4.29) converges, as $N\to\infty$, to

$$P[\mathrm{Pois}(\tfrac{\gamma}{\alpha}\mathcal{Z}_0\mathcal{Z}_t')=0] = E\big[e^{-\frac{\gamma}{\alpha}\sum_i \mathcal{Z}_0(i)\mathcal{Z}_t'(i)}\big]. \tag{4.31}$$

Treating the right-hand side of (4.29) in the same way, we find that $\mathcal{Z}$ is self-dual with the duality function from (4.14). ∎

We have just seen that Proposition 9 follows by taking the local mean field limit of Proposition 3 (a). One may wonder if taking the same limit in Proposition 3 (b), one may discover other duals of super random walks. It turns out that this is not the case. Rather, the dual processes from Proposition 3 (b) also converge to super random walks, and one finds duality relations that, using scaling, can all be reduced to the self-duality in Proposition 9. Likewise, the thinning relations from Lemma 4 converge, under the local mean field limit from Claim 8, to trivial scaling relations. Since this is quite lengthy and does not yield anything new, we omit the details.

A Lloyd-Sudbury dualities

In this appendix, we outline the proof of a generalization of Proposition 1. Generalizing (2.2), we look at models with generators of the form

$$\begin{aligned}
G_{\mathrm{LS}}f(x) := \sum_{i\neq j} \Big\{ & \tfrac{1}{2}a(i,j)x(i)x(j)\{f(x-\delta_i-\delta_j)-f(x)\} \\
&+b(i,j)x(i)(1-x(j))\{f(x+\delta_j)-f(x)\} \\
&+c(i,j)x(i)x(j)\{f(x-\delta_j)-f(x)\} \\
&+d(i,j)x(i)(1-x(j))\{f(x-\delta_i)-f(x)\} \\
&+e(i,j)x(i)(1-x(j))\{f(x-\delta_i+\delta_j)-f(x)\}\Big\},
\end{aligned} \tag{A.1}$$

where $a,b,c,d,e:\Lambda\times\Lambda\to\mathbb{R}$ are nonnegative functions such that

$$\sup_i \sum_j g(i,j) < \infty, \quad \sup_j \sum_i g(i,j) < \infty \qquad (g=a,b,c,d,e). \tag{A.2}$$

Without loss of generality, we may assume (as we do from now on) that $a(i,j) = a(j,i)$. Generalizing our earlier definition, within this appendix, let us call the Markov process with generator G_{LS} the (a,b,c,d,e)-*LSM*. We will outline a proof of the following generalization of Proposition 1.

Proposition 10 (Dualities between Lloyd-Sudbury models) *The* (a,b,c,d,e)*-LSM and* (a',b',c',d',e')*-LSM are dual with duality parameter* $\eta\in\mathbb{R}\backslash\{1\}$ *provided that*

$$\begin{aligned}
a'(i,j) &= a(i,j) + \eta\big(\gamma(i,j)+\gamma(j,i)\big), \\
b'(i,j) &= b(j,i) + \gamma(i,j), \\
c'(i,j) &= c(i,j) - \gamma(j,i) - \eta\gamma(i,j) + \big(e(i,j)-e(j,i)\big) + \eta\big(b(i,j)-b(j,i)\big), \\
d'(i,j) &= d(i,j) + \gamma(i,j) + \big(e(i,j)-e(j,i)\big), \\
e'(i,j) &= e(j,i) - \gamma(i,j),
\end{aligned} \tag{A.3}$$

where

$$\gamma(i,j) := \big(a(i,j)+c(j,i)-d(i,j)+\eta b(j,i)-e(i,j)+e(j,i)\big)/(1-\eta) \tag{A.4}$$

Observe that these formulas simplify to (2.6) if the functions $a,\ldots,e$ are symmetric. We note that an equivalent set of equations for (A.3) is

$$\begin{aligned}
a'(i,j) + \eta\big(e'(i,j)+e'(j,i)\big) &= a(i,j) + \eta\big(e(i,j)+e(j,i)\big), \\
b'(i,j) + e'(i,j) &= b(j,i) + e(j,i), \\
\gamma'(i,j) &= -\gamma(j,i), \\
d'(i,j) + e'(i,j) &= d(i,j) + e(i,j), \\
e'(i,j) - \tfrac{1}{2}\gamma'(j,i) &= e(j,i) - \tfrac{1}{2}\gamma(i,j),
\end{aligned} \tag{A.5}$$

where $\gamma'(i,j)$ is defined as in (A.4), with the rates $a,\ldots,e$ replaced by $a',\ldots,e'$.

Proof of Proposition 10 By standard theory [EK86, Section 4.4], we need to check that

$$G_{\mathrm{LS}}\psi_\eta(\,\cdot\,,y)(x) = G'_{\mathrm{LS}}\psi_\eta(x,\cdot\,)(y) \qquad (x,y\in\{0,1\}^\Lambda), \tag{A.6}$$

where $G'_{\rm LS}$ is defined as in (A.1), with the rates $a,\ldots,e$ replaced by $a',\ldots,e'$. Diligent calculation reveals that

$$
\begin{aligned}
&G_{\rm LS}\psi_\eta(\,\cdot\,,y)(x)\\
&=(\eta^{-1}-1)\sum_{i\neq j}\Big\{\tfrac{1}{2}\Big((\eta^{-1}-1)a(i,j)+(1-\eta)\big(e(i,j)+e(j,i)\big)\Big)x(i)x(j)y(i)y(j)\\
&\quad+\Big(a(i,j)-d(i,j)-e(i,j)+\eta b(j,i)+c(j,i)+\eta e(j,i)\Big)x(i)x(j)y(i)\\
&\quad-(1-\eta)e(i,j)x(i)y(i)y(j)\\
&\quad+\Big(d(i,j)+e(i,j)\Big)x(i)y(i)\\
&\quad-\eta\Big(b(i,j)+e(i,j)\Big)x(i)y(j)\Big\}\psi_\eta(x,y).
\end{aligned}
\tag{A.7}
$$

and

$$
\begin{aligned}
&G'_{\rm LS}\psi_\eta(x,\cdot\,)(y)\\
&=(\eta^{-1}-1)\sum_{i\neq j}\Big\{\tfrac{1}{2}\Big((\eta^{-1}-1)a'(i,j)+(1-\eta)\big(e'(i,j)+e'(j,i)\big)\Big)x(i)x(j)y(i)y(j)\\
&\quad-(1-\eta)e'(i,j)x(i)x(j)y(i)\\
&\quad+\Big(a'(i,j)-d'(i,j)-e'(i,j)+\eta b'(j,i)+c'(j,i)+\eta e'(j,i)\Big)x(i)y(i)y(j)\\
&\quad+\Big(d'(i,j)+e'(i,j)\Big)x(i)y(i)\\
&\quad-\eta\Big(b'(j,i)+e'(j,i)\Big)x(i)y(j)\Big\}\psi_\eta(x,y).
\end{aligned}
\tag{A.8}
$$

Comparing (A.7) and (A.8), we see that (A.6) is satisfied provided that

$$
\begin{aligned}
&a'(i,j)+\eta\big(e'(i,j)+e'(j,i)\big)=a(i,j)+\eta\big(e(i,j)+e(j,i)\big),\\
&-(1-\eta)e'(i,j)=a(i,j)-d(i,j)-e(i,j)+\eta b(j,i)+c(j,i)+\eta e(j,i),\\
&a'(i,j)-d'(i,j)-e'(i,j)+\eta b'(j,i)+c'(j,i)+\eta e'(j,i)=-(1-\eta)e(i,j),\\
&d'(i,j)+e'(i,j)=d(i,j)+e(i,j),\\
&b'(j,i)+e'(j,i)=b(i,j)+e(i,j).
\end{aligned}
\tag{A.9}
$$

which can be solved to yield (A.3). ∎

Remark The original motivation for the calculations in this appendix was to generalize the self-duality in Proposition 3 (a) to the case that $q(i,j)\neq q(j,i)$. Surprisingly, this does not work, unless $r=0$. In particular, if $m=0$ but $r,s>0$, one can check from (A.3) that a $(q,r,s,0)$-CVP can only be dual to another $(q',r',s',0)$-CVP if $q=q'$ and $q(i,j)=q(j,i)$.

B Local mean field limits

B.1 Contact voter processes

Idea of proof of Claim 5 The process $\overline{X}^{(N)}$ jumps as

$$
\begin{aligned}
x \to x + \tfrac{1}{N}\delta_i \quad & \text{with rate} \quad \nu^{(N)} N^{-1}(r^{(N)} + s) \sum_{j:j\neq i} q(j,i)x(j)N(N - x(i)N) \\
& \qquad + (1-\nu^{(N)})(N-1)^{-1}(r^{(N)} + s)x(i)N(N - x(i)N) \\
x \to x - \tfrac{1}{N}\delta_i \quad & \text{with rate} \quad \nu^{(N)} N^{-1} r^{(N)} \sum_{j:j\neq i} q(j,i)(N - x(j)N)x(i)N \\
& \qquad + (1-\nu^{(N)})(N-1)^{-1} r^{(N)} (N - x(i)N)x(i)N \\
& \qquad + m x(i)N.
\end{aligned} \tag{B.1}
$$

It follows that the process started in $\overline{X}^{(N)}_0 = x$ satisfies

$$
\begin{aligned}
\lim_{t\to 0} t^{-1} E[\overline{X}^{(N)}_t(i) - x(i)] = {} & \nu^{(N)} r^{(N)} \sum_{j:j\neq i} q(j,i)(x(j) - x(i)) \\
& + \nu^{(N)} s \sum_{j:j\neq i} q(j,i)x(j)(1 - x(i)) \\
& + (1-\nu^{(N)}) s (1 - N^{-1})^{-1} x(i)(1 - x(i)) \\
& - m x(i) \\
\lim_{t\to 0} t^{-1} E[(\overline{X}^{(N)}_t(i) - x(i))^2] = {} & \nu^{(N)}(2r^{(N)} + s) N^{-1} \sum_{j:j\neq i} q(j,i)x(j)(1 - x(i)) \\
& + (1-\nu^{(N)})(2r^{(N)} + s)(N-1)^{-1} x(i)(1 - x(i)) \\
& + m x(i) N^{-1}.
\end{aligned} \tag{B.2}
$$

Under the assumptions (4.10), this simplifies to

$$
\begin{aligned}
\lim_{t\to 0} t^{-1} E[\overline{X}^{(N)}_t(i) - x(i)] = {} & \sum_{j:j\neq i} q(j,i)(x(j) - x(i)) \\
& + s x(i)(1 - x(i)) - m x(i) + O(N^{-1}), \\
\lim_{t\to 0} t^{-1} E[(\overline{X}^{(N)}_t(i) - x(i))^2] = {} & 2r x(i)(1 - x(i)) + O(N^{-1})
\end{aligned} \tag{B.3}
$$

as $N \to \infty$. This, plus an estimate on a higher moment, can be used to show that the limiting process $\mathcal{X}$ started in $\mathcal{X}_0 = x$ satisfies

$$
\begin{aligned}
\lim_{t\to 0} t^{-1}\big(E[f(\mathcal{X}_t)] - f(x)\big) = {} & \sum_{i\neq j} q(j,i)(x(j) - x(i)) \tfrac{\partial}{\partial x(i)} f(x) + s \sum_i x(i)(1 - x(i)) \tfrac{\partial}{\partial x(i)} f(x) \\
& - m \sum_i x(i) \tfrac{\partial}{\partial x(i)} f(x) + r \sum_i x(i)(1 - x(i)) \tfrac{\partial^2}{\partial x(i)^2} f(x)
\end{aligned} \tag{B.4}
$$

for all twice continuously differentiable f depending on finitely many coordinates. In the right-hand side of (B.4) we reckognize the generator of the diffusion process in (4.4). More formally, the calculations above can be used to show that the processes $\overline{X}^{(N)}$ are tight, and each weak limit point satisfies the martingale problem for the operator in (B.4). Standard theory then yields the convergence in (4.12).

To prove the convergence in (4.13), set

$$\begin{aligned} \rho^{(N)} &:= r^{(N)} + \tfrac{\varepsilon}{1+\varepsilon}s, \\ \beta &:= \tfrac{1}{1+\varepsilon}s, \\ \delta &:= m - \tfrac{\varepsilon}{1+\varepsilon}s, \end{aligned} \tag{B.5}$$

and observe that the assumptions (4.10) imply that

$$\begin{aligned} \rho^{(N)}\nu^{(N)} &\to 1, \\ \rho^{(N)}N^{-1} &\to r. \end{aligned} \tag{B.6}$$

The process $\overline{Y}^{(N)}$ jumps as

$$\begin{array}{lll} y \to y - \delta_i + \delta_j & \text{with rate} & \nu^{(N)}N^{-1}\rho^{(N)}q(i,j)y(i)(N-y(j)), \\[2mm] y \to y - \delta_i - \delta_j & \text{with rate} & \varepsilon\nu^{(N)}N^{-1}\rho^{(N)}q(i,j)y(i)y(j), \\[2mm] y \to y + \delta_i & \text{with rate} & \nu^{(N)}N^{-1}\beta \sum_{j:j\neq i} q(j,i)y(j)(N-y(i)) \\ & & +(1-\nu^{(N)})(N-1)^{-1}\beta y(i)(N-y(i)), \\[2mm] y \to y - \delta_i & \text{with rate} & (1-\varepsilon)\nu^{(N)}N^{-1}\rho^{(N)} \sum_{j:j\neq i} q(i,j)y(i)y(j) \\ & & +(1-\varepsilon)(1-\nu^{(N)})(N-1)^{-1}\rho^{(N)}y(i)(y(i)-1) \\ & & +\varepsilon\nu^{(N)}N^{-1}\beta \sum_{j:j\neq i} q(j,i)y(j)y(i) \\ & & +\varepsilon(1-\nu^{(N)})(N-1)^{-1}\beta y(i)(y(i)-1) \\ & & +\delta y(i), \\[2mm] y \to y - 2\delta_i & \text{with rate} & \varepsilon(1-\nu^{(N)})(N-1)^{-1}\rho^{(N)}y(i)(y(i)-1). \end{array} \tag{B.7}$$

In the limit $N \to \infty$, this simplifies to

$$\begin{array}{lll} y \to y - \delta_i + \delta_j & \text{with rate} & q(i,j)y(i), \\[2mm] y \to y + \delta_i & \text{with rate} & \tfrac{1}{1+\varepsilon}sy(i), \\[2mm] y \to y - \delta_i & \text{with rate} & (1-\varepsilon)ry(i)(y(i)-1) + (m - \tfrac{\varepsilon}{1+\varepsilon}s)y(i), \\[2mm] y \to y - 2\delta_i & \text{with rate} & \varepsilon ry(i)(y(i)-1), \end{array} \tag{B.8}$$

plus terms of order N^{-1}. ∎

B.2 Super random walks

Idea of proof of Claim 5 The proof of (4.26) is very similar to the proof of (4.12). Indeed, the process $\overline{X}^{(N)}$ jumps as

$$\begin{aligned}
x \to x + \tfrac{\phi}{\sqrt{N}}\delta_i \qquad &\text{with rate} \qquad \nu^{(N)} N^{-1} s^{(N)} \sum_{j:\, j\neq i} q(j,i) x(j) \phi^{-1}\sqrt{N}(N - x(i)\phi^{-1}\sqrt{N}) \\
&\qquad\qquad\qquad +(1-\nu^{(N)})(N-1)^{-1} s^{(N)} x(i)\phi^{-1}\sqrt{N}(N - x(i)\phi^{-1}\sqrt{N}) \\
x \to x - \tfrac{\phi}{\sqrt{N}}\delta_i \qquad &\text{with rate} \qquad m^{(N)} x(i)\phi^{-1}\sqrt{N}.
\end{aligned} \tag{B.9}$$

It follows that the process started in $\overline{X}_0^{(N)} = x$ satisfies

$$\begin{aligned}
\lim_{t\to 0} t^{-1} E[\overline{X}_t^{(N)}(i) - x(i)] = {}& \nu^{(N)} s^{(N)} \sum_{j:\, j\neq i} q(j,i) x(j)(1 - x(i)\phi^{-1} N^{-1/2}) \\
&+(1-\nu^{(N)}) s^{(N)} (x(i) - \phi^{-1} N^{-1/2} x(i)^2)(1-N^{-1})^{-1} \\
&-m^{(N)} x(i) \\
\lim_{t\to 0} t^{-1} E[(\overline{X}_t^{(N)}(i) - x(i))^2] = {}& \phi N^{-1/2} \nu^{(N)} s^{(N)} \sum_{j:\, j\neq i} q(j,i) x(j)(1 - x(i)\phi^{-1} N^{-1/2}) \\
&+\phi N^{-1/2}(1-\nu^{(N)}) s^{(N)} (x(i) - \phi^{-1} N^{-1/2} x(i)^2)(1-N^{-1})^{-1} \\
&+\phi N^{-1/2} m^{(N)} x(i).
\end{aligned} \tag{B.10}$$

By (4.24), for large N this simplifies to

$$\begin{aligned}
\lim_{t\to 0} t^{-1} E[\overline{X}_t^{(N)}(i) - x(i)] &= \sum_{j:\, j\neq i} q(j,i) x(j) + (\beta - 1) x(i) - \gamma x(i)^2 \\
&= \sum_{j:\, j\neq i} q(j,i)(x(j) - x(i)) + \beta x(i) - \gamma x(i)^2, \qquad \text{(B.11)} \\
\lim_{t\to 0} t^{-1} E[(\overline{X}_t^{(N)}(i) - x(i))^2] &= \alpha x(i),
\end{aligned}$$

plus terms of order $N^{-1/2}$. (Here we have used that $\sum_{j:\, j\neq i} q(j,i) = 1$.) It follows that $\overline{X}^{(N)}$ converges to the process with generator

$$\begin{aligned}
Gf(z) = {}& \sum_{i\neq j} q(j,i)(z(j) - z(i)) \tfrac{\partial}{\partial z(i)} + \tfrac{1}{2}\alpha \sum_i z(i) \tfrac{\partial^2}{\partial z(i)^2} \\
&+\beta \sum_i z(i) \tfrac{\partial}{\partial z(i)} - \gamma \sum_i z(i)^2 \tfrac{\partial}{\partial z(i)},
\end{aligned} \tag{B.12}$$

i.e., the process in (4.6). ∎

References

[AS05] S.R. Athreya and J.M. Swart. Branching-coalescing particle systems. *Prob. Theory Relat. Fields* 131(3), 376–414, 2005.

[EK86] S.N. Ethier and T.G. Kurtz. *Markov Processes; Characterization and Convergence.* John Wiley & Sons, New York, 1986.

[HWS05] M. Hutzenthaler and A. Wakolbinger. Ergodic behaviour of locally regulated branching populations. ArXiv math.PR/0509612.

[Lig85] T.M. Liggett. *Interacting Particle Systems.* Springer, New York, 1985.

[SL95] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. II: The concept of duality in interacting particle systems. *Ann. Probab.* 23(4), 1816–1830, 1995.

[SL97] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. IV: Quasi-duality and thinnings of interacting particle systems *Ann. Probab.* 25(1), 96–114, 1997.

[SU86] T. Shiga and K. Uchiyama. Stationary states and their stability of the stepping stone model involving mutation and selection. *Probab. Theory Relat. Fields* 73, 87–117, 1986.

[Sud00] A. Sudbury. Dual families of interacting particle systems on graphs. *J. Theor. Probab.* 13(3), 695–716, 2000.