Introduction to SLE

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Abstract

These are notes for three lectures on SLE held in the seminar on stochastic evolution equations held at the Mathematical Institute of the Czech Academy of Sciences, Žitná 25, Prague, April 2008. The text below is based on G.F. Lawler *Conformally Invariant Processes in the Plane*, Mathematical surveys and monographs 114, AMS, Providence, 2005.

1 Conformal maps

By definition, a *domain* is an open connected set. A function $f : D \to \mathbb{C}$, defined on a domain $D \subset \mathbb{C}$ is *analytic* if the limit

$$f'(z) := \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for each $z \in D$. Analytic functions are infinitely differentiable. Let $B_r(z) := \{w \in \mathbb{C} : |w - z| < r\}$ denote the open ball with radius $r > 0$ around a point $z \in \mathbb{C}$. Then, if f is analytic on D and $B_r(z) \subset D$, the Taylor expansion of f around z converges on $B_r(z)$ and

$$f(w) = f(z) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(z)(w - z)^n \qquad (w \in B_r(z)).$$

If $0 \neq f'(z) = re^{i\theta}$, then f looks locally around $f(z)$ like a rotation by the angle θ and a dilation by r. In particular, analytic functions (viewed as functions from $\mathbb{C} \cong \mathbb{R}^2 \to \mathbb{R}^2$) are angle-preserving transformations.

Let $D, D' \subset \mathbb{C}$ be domains. By definition, a *conformal map* from D to D' is an analytic bijection $f : D \to D'$. It follows that $f'(z) \neq 0$ for all $z \in D$, and $f^{-1} : D' \to D$ is also a conformal map. When discussing conformal maps, it is often useful to consider domains that may contain the point infinity.

The *Riemann sphere* $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is just the one-point compactification of $\mathbb{C}$. A *linear fractional transformation* is a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$f(z) = \frac{az + b}{cz + d} \quad \text{with } a, b, c, d \in \mathbb{C},\ ad - bc \neq 0.$$

For a given linear fractional transformation f, we can always choose the constants a, b, c, d in such a way that $ad - bc = 1$. With that choice, there is a one-to-one correspondence between linear fractional transformations and $\mathrm{SL}_2(\mathbb{C})$, the group of 2×2 complex valued matrices with determinant one. This correspondence is a group isomorphism, i.e., $f_{A_1} f_{A_2} = f_{A_1 A_2}$. We may view $\hat{\mathbb{C}}$ as the surface of a 3-dimensional sphere. In this picture, lines and circles in $\mathbb{C}$ correspond to circles in $\hat{\mathbb{C}}$. Linear fractional transformations map circles in $\hat{\mathbb{C}}$ into circles.

Let $D, D' \subset \mathbb{C}$ be domains such that $D \neq \hat{\mathbb{C}} \neq D'$. Then we can find linear fractional transformations ψ, ϕ such that the images $\phi(D), \psi(D')$ do not contain the point ∞. Extending our earlier definition, we say that a map $f : D \to D'$ is *conformal* if f is a bijection and the map $\tilde{f} : \phi(D) \to \psi(D')$ defined by $\tilde{f} := \phi \circ f \circ \psi^{-1}$ is analytic. Obviously, this definition does not depend on the choice of ϕ, ψ.

Lemma 1.1 (Expansion at infinity) *Let $D, D' \subset \hat{\mathbb{C}}$ be domains such that $D \neq \hat{\mathbb{C}} \neq D'$ and $\infty \in D$, $\infty \in D'$. Let $f : D \to D'$ be a conformal map such that $f(\infty) = \infty$. Set $r := \inf\{|z| : z \notin D\}$. Then there exist complex constants $(b_n)_{n\geq -1}$ such that*

$$f(z) = \sum_{n=-1}^{\infty} b_n z^{-n} \qquad (|z| > r). \tag{1.1}$$

Proof By our definition of a conformal map, the map $y \mapsto 1/f(1/y) =: \tilde{f}(y)$ is analytic on the set $\{1/z : z \in D\}$. Since $f(\infty) = \infty$, we have $\tilde{f}(0) = 0$. Therefore, since f is a bijection, $\tilde{f}(y) \neq 0$ for all $y \neq 0$. It follows that $y \mapsto y/\tilde{f}(y)$ is analytic on $\{1/z : z \in D\}$, hence there exist complex constants $(b_n)_{n\geq -1}$ such that

$$\frac{y}{\tilde{f}(y)} = yf(1/y) = \sum_{n=0}^{\infty} b_{n-1} y^n \qquad (y < 1/r).$$

Setting $y = 1/z$ and multiplying both sides of the equation by z we arrive at (1.1). ∎

We set

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Lemma 1.2 (Schwartz lemma) *Assume that $f : \mathbb{D} \to \mathbb{D}$ is analytic and $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Moreover, if $|f(z)| = |z|$ for some $0 \neq z \in \mathbb{D}$, then $f(z) = e^{i\theta} z$ $(z \in \mathbb{D})$ for some $\theta \in \mathbb{R}$.*

Proof Since $f(0) = 0$, the function $f(z)/z =: \tilde{f}(z)$ is analytic. The maximum principle (which follows from the Cauchy integral formula) implies that $|\tilde{f}|$ cannot assume a local maximum on $\mathbb{D}$, unless $\tilde{f}$ is constant. ∎

As a consequence of Schwartz lemma, we obtain:

Proposition 1.3 (Conformal maps of the unit disk) *Assume that $f : \mathbb{D} \to \mathbb{D}$ is a conformal map such that $f(0) = 0$. Then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.*

Proof Applying Schwartz lemma to f and its inverse, we obtain $|z| = |f^{-1}(f(z))| \leq |f(z)| \leq |z|$ for all $z \in \mathbb{D}$, hence $f(z) = |z|$ for all $z \in \mathbb{D}$, hence, by Schwartz lemma, $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. ∎

Proposition 1.4 (Möbius transformations) *Assume that $f : \mathbb{D} \to \mathbb{D}$ is a conformal map. Then*

$$f(z) = e^{i\theta} \frac{z - w}{1 - \overline{w} z} \qquad (z \in \mathbb{D}) \tag{1.2}$$

for some $w \in \mathbb{D}$, $\theta \in \mathbb{R}$.

Proof Transformations of the form (1.2) are called Möbius transformations. It is straightforward to check that Möbius transformations are linear fractional transformations that map $\mathbb{D}$ onto itself. Moreover, for each $z, z' \in \mathbb{D}$ we can find a Möbius transformation that maps z into z'. Using these facts, the statement of the proposition can easily be deduced from Proposition 1.3. ∎

By definition, a domain $D \subset \hat{\mathbb{C}}$, $D \neq \hat{\mathbb{C}}$ is called *simply connected* if $\hat{\mathbb{C}} \backslash D$ is connected. We accept the following theorem without proof.

Theorem 1.5 (Riemann mapping theorem) *Let $D \subset \hat{\mathbb{C}}$ be a simply connected domain such that $\hat{\mathbb{C}} \backslash D$ contains at least two points. Then there exists a conformal map $f : D \to \mathbb{D}$.*

We now formulate a few results that allow us to make a unique choice for a conformal map from a domain D to $\mathbb{D}$. Below, if we write $f'(w) > 0$ we mean that $f'(w)$ is real and strictly positive. Note that this says that f is locally 'torsion free'.

Proposition 1.6 (Internal reference point) *Let D be as in Theorem 1.5 and let $z \in D$. Then there exists a unique conformal map $f : D \to \mathbb{D}$ such that $f(z) = 0$ and $f'(z) > 0$.*

Proof Theorem 1.5 gives the existence of at least one conformal map $f : D \to \mathbb{D}$. If $\tilde{f}$ is another such map, then $f \circ \tilde{f}^{-1}$ is a conformal map from $\mathbb{D}$ onto $\mathbb{D}$, so by Proposition 1.4, all conformal maps from D onto $\mathbb{D}$ are of the form $m \circ f$ where $f : D \to \mathbb{D}$ is a fixed conformal map from D onto $\mathbb{D}$ and m is a Möbius transformation. It is straightforward to check that there exists a unique Möbius transformation m such that $m(f(z)) = 0$ and $m'(f(z)) > 0$, hence $(m \circ f)'(z) = m'(f(z))f'(z) > 0$. ∎

Let E be a compact metric space. For any closed $K \subset E$, we define $\mathrm{diam}(K) := \max\{d(x, y) : x, y \in K\}$. If $K \subset E$ is closed and connected, then for any $x, y \in K$, we define

$$d_K(x, y) := \inf\{\mathrm{diam}(K') : x, y \in K' \subset K,\ K' \text{ closed and connected}\}.$$

Note that $d_K(x, y)$ is a bit similar to what we would call the distance between x and y measured through K (although it is not quite the same). By definition, a closed connected set $K \subset E$ is *locally connected* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $x, y \in K$, $d(x, y) \leq \delta$ implies $d_K(x, y) \leq \varepsilon$.

By definition, a *curve* is a continuous map $\gamma : [a, b) \to \hat{\mathbb{C}}$, where $-\infty \leq a < b \leq \infty$. A curve is *simple* if $a \leq s < t < b$ implies $\gamma(s) \neq \gamma(t)$. A curve is *closed* if $\lim_{t \uparrow b} \gamma(t) = \gamma(a)$. A *Jordan curve* is a simple closed curve. For any $a \leq s < t \leq b$ we write $\gamma[s, t) := \{\gamma(u) : u \in [s, t)\}$ and if $t < b$ we define $\gamma[s, t]$ similarly.

A *Jordan domain* is a domain D such that $\partial D = \gamma[a, b)$ for some Jordan curve γ. Jordan domains are locally connected (but the converse is not true).

Proposition 1.7 (Continuous extension) *Let $f : \mathbb{D} \to D$ be a conformal map. Then f has a continuous extension from $\overline{\mathbb{D}}$ onto $\overline{D}$ if and only if $\hat{\mathbb{C}} \backslash D$ is locally connected. The continuous extension is one-to-one if and only if D is a Jordan domain.*

Remark By the second part of the theorem, we see that first part remains true if $\mathbb{D}$ is replaced by an arbitrary Jordan domain.

Proposition 1.8 (Three points) *If D, D' are Jordan domains and z_1, z_2, z_3 and z'_1, z'_2, z'_3 are points on ∂D and $\partial D'$, respectively, oriented counterclockwise, then there exists a unique conformal map $f : D \to D'$ whose continuous extension satisfies $f(z_i) = z'_i$ $(i = 1, 2, 3)$.*

Proof By the Riemann mapping theorem, it suffices to check the statement for Möbius transformations. ∎

In the next proposition, we are looking for a conformal map that is as close as possible to the identity in the neighborhood of some boundary point.

Proposition 1.9 (Boundary reference point) *Let $A \subset \mathbb{D}$ be a closed subset of $\mathbb{D}$ such that $\mathbb{D} \backslash A$ is simply connected and let $w \in \partial D$, $w \notin \overline{A}$. Then there exists a unique conformal map $g : (\mathbb{D} \backslash A) \to \mathbb{D}$ that can be analytically extended in a neighborhood of w such that $g(w) = w$, $g'(w) = 1$, and $g''(w) = 0$. Moreover, the analytic extension of g satisfies $g(1/\overline{z}) = 1/\overline{g(z)}$.*

Proof Let $I \subset \partial \mathbb{D}$ be a closed interval such that $1 \notin I \supset \overline{A} \backslash A$, and set $\tilde{A} := A \cup \{1/\overline{z} : z \in A\} \cup I$. Let $I' \subset \partial \mathbb{D}$ be a closed interval such that $w \notin I'$. By the Riemann mapping theorem, there exists a unique conformal map $f : (\hat{\mathbb{C}} \backslash \tilde{A}) \to (\hat{\mathbb{C}} \backslash I')$ such that $f(w) = w$ and

$f'(w) > 0$. Define $\tilde{f}(z) := 1/\overline{f(1/\overline{z})}$. Then $\tilde{f}$ satisfies the same properties as f and hence, since the latter is uniquely defined, $\tilde{f} = f$. It follows that f maps $\mathbb{D}\backslash A$ into $\mathbb{D}$, and f can be analytically extended to an open neighborhood of w. It is straightforward to check that there exists a unique Möbius transformation m such that $(m \circ f)'(w) = m'(w)f'(w) = 1$ and $(m \circ f)''(w) = m''(w)f'(w)^2 + m'(w)f''(w) = 0$, hence $g := m \circ f$ is the unique map we are looking for. Note that g can be analytically extended to a map from $\hat{\mathbb{C}}\backslash\tilde{A}$ onto $I'' := m(I')$. In particular, since $g'(w) > 0$, the map g is 'built in the same way' as f (but for one particular choice of I'), hence the analytic extension of g satisfies $g(1/\overline{z}) = 1/\overline{g(z)}$. ∎

The next lemma shows that the conditions $g(w) = w$, $g'(w) = 1$ and $g''(w) = 0$ are invariant under conformal transformations.

Lemma 1.10 (Conformal invariance) *Let $D, D' \subset \hat{\mathbb{C}}$ be a domains, let $f : D \to D'$ be a conformal map, let $w \in D$, $w' := f(w)$, and assume that $w, w' \in \mathbb{C}$. Let $g : D \to \mathbb{C}$ be analytic and define $\tilde{g} : D' \to D'$ by $\tilde{g} := f \circ g \circ f^{-1}$. Assume that $g(w) = w$, $g'(w) = 1$, and $g''(w) = 0$. Then $\tilde{g}(w') = w'$, $\tilde{g}'(w') = 1$, and $\tilde{g}''(w') = 0$.*

Proof Obviously $\tilde{g}(w') = w'$. Define $h := f^{-1}$ and observe that $1 = \frac{\partial}{\partial z}z = \frac{\partial}{\partial z}f(h(z)) = f'(h(z))h'(z)$ and $0 = \frac{\partial^2}{\partial z^2}z = \frac{\partial^2}{\partial z^2}f(h(z)) = f''(h(z))h'(z)^2 + f'(h(z))h''(z)$. We calculate $\frac{\partial}{\partial z}f(g(h(z))) = f'(g(h(z)))g'(h(z))h'(z)$ and $\frac{\partial^2}{\partial z^2}f(g(h(z))) = f''(g(h(z)))g'(h(z))^2h'(z)^2 + f'(g(h(z)))g''(h(z))h'(z)^2 + f'(g(h(z)))g'(h(z))h''(z)$. Therefore $\tilde{g}'(w') = f'(w)g'(w)h'(w') = f'(w)h'(w') = 1$ and $\tilde{g}''(w') = f''(w)g'(w)^2h'(w')^2 + f'(w)g''(w)h'(w')^2 + f'(w)g'(w)h''(w') = f''(w)h'(w')^2 + f'(w)h''(w') = 0$. ∎

2 Half-plane capacity

We set

$$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

Consider the linear fractional transformations f and f^{-1} given by

$$f(z) := \frac{z-i}{z+i} \quad \text{and} \quad f^{-1}(z) := i\frac{1+z}{1-z}.$$

It is straightforward to check that f and f^{-1} are each other's inverse and that $f : \mathbb{H} \to \mathbb{D}$ is a bijection such that $f(\infty) = 1$, $f(0) = -1$, and $f(i) = 0$. Moreover,

$$\overline{f(z)} = 1/f(\overline{z}) \tag{2.1}$$

Applying this transformation of space to Proposition 1.4, we obtain:

Proposition 2.1 (Möbius transformations) *Assume that $f : \mathbb{H} \to \mathbb{H}$ is a conformal map. Then*

$$f(z) = \frac{az+b}{cz+d} \qquad (z \in \mathbb{D}) \tag{2.2}$$

for some $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$.

By definition, a *compact $\mathbb{H}$-hull* is a set $A \subset \mathbb{H}$ such that A is a bounded closed subset of $\mathbb{H}$, and $\mathbb{H}\backslash A$ is simply connected.

Proposition 2.2 (Half-plane capacity) *Let A be a compact $\mathbb{H}$-hull and $r := \sup\{|z| : z \in A\}$. Then there exists a unique conformal transformation $g_A : (\mathbb{H}\backslash A) \to \mathbb{H}$ such that*

$$g_A(z) = z + \sum_{n=1}^{\infty} b_n z^{-n} \qquad (|z| > r), \tag{2.3}$$

where $(b_n)_{n\geq1}$ are real constants.

Proof This is just a reformulation of Proposition 1.9. Indeed, we have just seen that there exists a linear fractional transformation $f : \mathbb{H} \to \mathbb{D}$ with $f(\infty) = 1$. By Proposition 1.9, there exists a unique conformal map $g : (\mathbb{D}\backslash f(A)) \to \mathbb{D}$ such that $g(1) = 1$, $g'(1) = 1$, $g''(1) = 0$. Define $g_A := f^{-1} \circ g \circ f$. Then g_A is a conformal map from $\mathbb{H}\backslash A$ onto $\mathbb{H}$ that can be extended to an open neighborhood of ∞ in such a way that $g_A(\infty) = \infty$. By Lemma 1.1, g_A has an expansion at infinity of the form

$$g_A(z) = b_{-1}z + \sum_{n=0}^{\infty} b_n z^{-n} \qquad (|z| > r),$$

where $(b_n)_{n\geq -1}$ are constants. By Proposition 1.9 and (2.1), the function g_A can be analytically extended in such a way that $g_A(\overline{z}) = \overline{g_A(z)}$. It follows that the constants $(b_n)_{n\geq -1}$ are real. Define $\tilde{g}(z) := 1/g_A(1/z)$. Then, by Lemma 1.10, $\tilde{g}'(0) = 1$, and $\tilde{g}''(0) = 0$. We check that

$$\begin{aligned}
\tilde{g}'(z) &= \tfrac{\partial}{\partial z}\Big(b_{-1}z^{-1} + \sum_{n=0}^{\infty} b_n z^n\Big)^{-1} \\
&= -\Big(b_{-1}z^{-1} + \sum_{n=0}^{\infty} b_n z^n\Big)^{-2}\Big(-b_{-1}z^{-2} + \sum_{n=0}^{\infty} n b_n z^{n-1}\Big)\Big|_{z=0} \\
&= \frac{b_{-1} - \sum_{n=0}^{\infty} n b_n z^{n+1}}{(b_{-1} + \sum_{n=0}^{\infty} b_n z^{n+1})^2} = \frac{b_{-1} + O(z^2)}{b_{-1}^2 + 2b_{-1}b_0 z + O(z^2)} \quad \text{as } z \to 0,
\end{aligned}$$

from which it is easy to see that $\tilde{g}'(0) = 1$ and $\tilde{g}''(0) = 0$ if and only if $b_{-1} = 1$ and $b_0 = 0$. ∎

In (2.3), the (real) coefficient b_1 is called the *half-plane capacity (from infinity)* of A, denoted by $\mathrm{hcap}(A)$. One can check that $\mathrm{hcap}(A) > 0$ if $A \neq \emptyset$, and $\mathrm{hcap}(rA + x) = r^2\mathrm{hcap}(A)$ for all $r > 0$, $x \in \mathbb{R}$.

3 The chordal Loewner equation

Let $\gamma : [0, \infty) \to \mathbb{C}$ be a simple curve with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$. Let $H_t := \mathbb{H}\backslash\gamma(0, t]$, which is a simply connected domain, and let $g_t := g_{\gamma(0,t]} : H_t \to \mathbb{H}$ be the unique conformal map in (2.3).

Lemma 3.1 (Loewner transform) *There exists a continuous function $U : [0, \infty) \to \mathbb{R}$ such that*

$$\lim_{z\to\gamma(t)} g_t(z) = U_t \qquad (t \geq 0). \tag{3.1}$$

Remark It is not possible to extend g_t continuously to $\overline{H}_t$ since the limit depends on whether we approach $\gamma[0, t]$ from the left or from the right. However, Lemma 3.1 says that at the tip of the curve, there is no ambiguity, and g_t (continuously extended) maps $\gamma(t)$ to a unique point U_t.

The idea of the Loewner equation is that given U_t, it should be possible to reconstruct γ. Let $b_t := \mathrm{hcap}(\gamma(0, t])$.

Lemma 3.2 (Continuity of capacity) *The function $t \mapsto b_t$ is continuous, strictly increasing, and satisfies $b_0 = 0$.*

Usually, we are not interested in the precise parametrization of γ. Therefore, we can *reparametrize γ by half-plane capacity*, i.e., let $\tilde{\gamma}(t) := \gamma(b^{-1}(2t))$. Then $\tilde{\gamma} : [0, b_\infty) \to \mathbb{C}$ is a simple curve with $\mathrm{hcap}(\tilde{\gamma}(0, t]) = 2t$, where $b_\infty := \lim_{t\to\infty} b_t$. Thus, 'without loss of generality', we may restrict ourselves to the case $b_t = 2t$.

In view of this, assume that γ is as before, that $t \mapsto b_t$ is once continuously differentiable, and that $b_\infty = \infty$. Let $b'_t := \frac{\partial}{\partial t} b_t$, and for any $z \in \mathbb{H}$, set

$$\tau_z := \inf\{t > 0 : z \in \gamma(0, t]\},$$

which may be infinite.

Proposition 3.3 (Loewner equation) *Let U_t be as in Lemma 3.1. Then, for each $z \in \mathbb{H}$, the function $[0, \tau_z) \ni t \mapsto g_t(z) \in \mathbb{H}$ is continuously differentiable and solves*

$$\frac{\partial}{\partial t} g_t(z) = \frac{b'_t}{g_t(z) - U_t} \qquad (0 \le t < \tau_z). \tag{3.2}$$

If $\tau_z < \infty$, then $\lim_{t \uparrow \tau_z} g_t(z) = U_{\tau_z}$.

Proof (sketch) For $0 \le s \le t$, define $H^s_t := g_s(H_t)$ and define $g_{s,t} : H^s_t \to \mathbb{H}$ by $g_{s,t} := g_t \circ g_s^{-1}$ so that $g_{s,t} \circ g_s = g_t$. Set $A^s_t := \mathbb{H} \backslash H^s_t$ and $b_{s,t} := \mathrm{hcap}(A^s_t)$. Then

$$g_{s,t}(z) = z + b_{s,t} z^{-1} + \sum_{n=2}^{\infty} b_{n,s,t} z^{-n} \quad \text{as } z \to \infty.$$

Since

$$g_s(z) = z + b_s z^{-1} + \sum_{n=2}^{\infty} b_{n,s} z^{-n} \quad \text{as } z \to \infty$$

we see that

$$g_t(z) = g_{s,t} \circ g_s(z) = z + (b_s + b_{s,t}) z^{-1} + O(z^{-2}) \quad \text{as } z \to \infty,$$

hence $b_{s,t} = b_t - b_s$. Now, if $\varepsilon > 0$ is small, it can be shown that $A^t_{t+\varepsilon}$ is a small set, located approximately at the position U_t. It follows that

$$g_{t,t+\varepsilon}(z) = z + (b_{t+\varepsilon} - b_t)(z - U_t)^{-1} + o(\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

Inserting $z = g_t(z')$ and using the differentiability of b_t, we obtain

$$g_{t+\varepsilon}(z') = g_t(z') + \frac{\varepsilon b'_t}{g_t(z) - U_t} + o(\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

∎

Using the Loewner equation (3.2), we can recover the curve γ from its Loewner transform U.

Theorem 3.4 (Reconstructing the curve) *Let $U : [0, \infty) \to \mathbb{R}$ be a continuous function. Then there exists a unique family of compact $\mathbb{H}$-hulls $(A_t)_{t \ge 0}$ such that $A_0 = \emptyset$, $\mathrm{hcap}(A_t) = 2t$, $A_s \subset A_t$ for $s \le t$, and the functions g_{A_t} from (2.3) satisfy*

$$\frac{\partial}{\partial t} g_{A_t}(z) = \frac{2}{g_{A_t}(z) - U_t} \qquad (0 \le t < \tau_z), \tag{3.3}$$

where $\tau_z := \inf\{t > 0 : z \in A_t\}$ (which may be infinite). If $t \mapsto U_t$ is α-Hölder continuous for some $\alpha > 1/2$, then there exists a simple curve $\gamma : [0, \infty) \to \mathbb{C}$ with $\gamma(0) = U_0$ and $\gamma(0, \infty) \subset \mathbb{H}$ such that $\gamma(0, t] = A_t$ for all $t \ge 0$.

4 The Schramm-Loewner equation

While the Loewner equation was invented (by Löwner) in 1923, it lasted till the year 1999 until someone had the idea of inserting Brownian motion as the driving function U_t in (3.3). That someone was Oded Schramm and the resulting Stochastic Loewner Equation is called the Schramm-Loewner Equation, or SLE, in his honor. By definition, the *chordal Schramm-Loewner equation* with parameter $\kappa \geq 0$ is the equation

$$\tfrac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t} \qquad (0 \leq t < \tau_z), \tag{4.1}$$

where $(B_t)_{t \geq 0}$ is standard Brownian motion. By Theorem 3.4, solutions to (4.1) are random collections of conformal maps $g_t : \mathbb{H} \backslash A_t \to \mathbb{H}$ $(t \geq 0)$, where $(A_t)_{t \geq 0}$ is a growing family of compact $\mathbb{H}$-hulls. Since Brownian motion is *not* α-Hölder continuous for any $\alpha > 1/2$, it is not clear from Theorem 3.4 whether the hulls $(A_t)_{t \geq 0}$ are generated by a simple curve. It turns out that they *are* generated by a curve, but not necessarily a simple one. (Below, $\overline{\mathbb{H}}$ denotes the closure of $\mathbb{H}$ in $\mathbb{C}$.)

Theorem 4.1 (Solutions to SLE) *Let $(A_t)_{t \geq 0}$ be the random growing family of compact $\mathbb{H}$-hulls defined by (4.1). Then there exists a random curve $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma[0, \infty) \subset \overline{\mathbb{H}}$ and such that $\mathbb{H} \backslash A_t$ is the unique connected component of $\mathbb{H} \backslash \gamma(0, t]$ containing ∞ as a limit point. For $0 \leq \kappa \leq 4$, the curve γ is a.s. simple. For $4 < \kappa < 8$, one has a.s. $\bigcup_{t>0} A_t = \overline{\mathbb{H}}$ but $\gamma[0, \infty) \neq \overline{\mathbb{H}}$. For $8 \leq \kappa$, one has a.s. $\gamma[0, \infty) = \overline{\mathbb{H}}$.*

Note that for $8 \leq \kappa$, the curve γ is *space filling*! (Try to think of a deterministic curve with this property -the Peano curve is an example.)

Remark The proof that solutions of (4.1) generate a curve is difficult. It is not too difficult, however, to see where the critical value $\kappa = 4$ comes from. For any $x \in \mathbb{R}$, we may ask whether $\tau_x := \inf\{t > 0 : x \in A_t\}$ is a.s. finite or infinite. For $z \in \mathbb{H} \backslash \{0\}$, set

$$Z_t^z := (g_t(z) - \sqrt{\kappa} B_t)/\sqrt{\kappa}.$$

Then Z^z solves the SDE

$$\mathrm{d}Z_t^z = \frac{2\mathrm{d}t}{\kappa Z_t^z} - \mathrm{d}B_t \qquad (0 \leq t < \tau_z), \tag{4.2}$$

where

$$\tau_z := \inf\{t > 0 : Z_t^z = 0\}.$$

We may also consider solutions Z^x to (4.2) with $x \in \mathbb{R}$. Then τ_x is the first time the boundary point x is 'swallowed' by the curve $\gamma[0, t]$, i.e., the first time γ hits a point in $(-\infty, x]$. If $z = x$ is real, then solutions to (4.2) are Bessel processes. It is well-known (and not hard to prove) for such Bessel processes that the point 0 is accessible if and only $\kappa \geq 4$.

Further analysis shows that $\tau_x < \tau_y$ a.s. for all $x < y$ iff $\kappa \geq 8$, which leads to the second critical value.

We mention that SLE is *scale invariant*, in the sense that if γ is the random curve generated by (4.1), then

$$\big(r\gamma(r^{-2}t)\big)_{t \geq 0} \quad \text{is equally distributed with} \quad (\gamma(t))_{t \geq 0}.$$

In particular, $\{\gamma(t) : t \in [0, \infty)\}$ and $\{r\gamma(t) : t \in [0, \infty)\}$ are equally distributed (although this does not carry much information if $\kappa \geq 8$!).

Moreover, SLE is *conformally invariant*. Let $f : \mathbb{H} \to D$ be a conformal map from $\mathbb{H}$ to a simply connected domain D, and assume that ∂D is sufficiently smooth at $w := f(\infty)$. Let g_t

be the solution to (4.1) and set $\tilde{g}_t := f \circ g_t \circ f^{-1}$. Then $\tilde{g}_t$ is the unique conformal map from $D \backslash f(A_t)$ to D such that $\tilde{g}_t'(w) = 1$ and $\tilde{g}_t''(w) = 0$. Now $f \circ \gamma$ is a random curve starting at $z := f(0)$ and growing towards the point w. We call $f \circ \gamma$ the SLE-curve in D started at z and growing towards w.

It is also possible to define SLE curves growing towards an interior point $w \in D$. This leads to the so-called radial SLE, which is a stochastic version of the radial Loewner equation. In particular, for curves in $\mathbb{D}$ growing towards 0 one considers functions $g_t : (\mathbb{D} \backslash A_t) \to \mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$. There exists also a whole-plane SLE for curves starting at 0 and growing towards ∞.

5 Locality and the restriction property

SLE provides the rigorous framework for studying phenomena known for many years in theoretical physics as 'conformal field theory'. It is believed that solutions to (4.1) for different values of κ correspond to scaling limits of some well-known models. More precisely, the following relations are conjectured (and in some cases proved):

κ	model
2	loop-erased random walk
$2\frac{2}{3}$	self-avoiding random walk
6	critical percolation

These relations (that we do not detail here) have been made rigorous for the loop-erased random walk and for critical site percolation on the triangular lattice. Motivation for the values $\kappa = 2\frac{2}{3}$ and $\kappa = 6$ in case of self-avoiding random walk and critical percolation come from the following two propositions.

Proposition 5.1 (Restriction property) *Let γ be the random curve generated by (4.1). Assume that A is a convex $\mathbb{H}$-hull such that $0 \notin \overline{A}$ and let $g_A : \mathbb{H} \backslash A \to \mathbb{H}$ be as in (2.3). Then $\gamma[0, \infty)$, conditioned on the event $\gamma[0, \infty) \cap A = \emptyset$, is equally distributed with $g_A^{-1}(\gamma[0, \infty))$ if and only if $\kappa = 2\frac{2}{3}$.*

Proposition 5.2 (Locality) *Let $\mathbb{D}$ be the unit disc and let z, w, w' be distinct points on ∂D. Let $I \subset \mathbb{D}$ be the closed interval with endpoints w, w' such that $z \notin I$. Let γ be the SLE-curve with $\kappa = 6$ starting at z and growing towards w, and let γ' be defined similarly but growing towards w'. Then γ and γ', stopped at the first time they hit I, are equally distributed up to reparametrization.*

We conclude this note with six pictures made by Vincent Beffara, showing SLE for $\kappa = 1, 3, 4, 5, 6, 8$.