Addendum to:
The contact process seen from a typical infected site.

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Abstract

This is an addendum to the preprint: 'The contact process seen from a typical infected site' by the same author. It is shown how the definition of Campbell-type measures in that paper can be generalized to the case where the exponential growth rate is nonzero. In this more general set-up, it is shown that Lemmas 5 and 6 still hold. Moreover, it is shown that the survival assumption in Lemma 6 is superfluous. However, since the author does no know how to generalize his crucial Proposition 7 to this more general situation, this does not lead to a generalization of the main theorem (Theorem 2).

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1 Results and proof

We start by generalizing the definition of the Campbell-type measures $\hat{P}^A_\gamma$ from [Swa05, formula (1.14)] to arbitrary exponential growth rates r. In this context, it will be convenient to use slightly different notation than in [Swa05]. We will introduce a parameter $\lambda \in \mathbb{R}$ that will play the role of $1/\gamma$ from [Swa05].

As before, we let $(\Omega, \mathcal{F}, P)$ be the probability space on which the Poisson point processes used in the graphical representation are defined. Next, for each $\lambda \in \mathbb{R}$, we introduce a measure space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \mathcal{E}_\lambda)$ where $\mathcal{E}_\lambda(\mathrm{d}x) := e^{-\lambda x}\mathrm{d}x$, and we let $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ denote the identity. Note that $\mathcal{E}_\lambda$ is a finite measure iff $\lambda > 0$, and in this case normalizing $\mathcal{E}_\lambda$ yields the probability measures $\mathcal{E}_\gamma$ with $\gamma := 1/\lambda$ from [Swa05]. We let $\mu_\lambda := P \otimes \mathcal{E}_\lambda$ denote product measure. For $\lambda > r$ and $\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)$, we define probability measures $\hat{P}^A_\lambda$ on $\Lambda \times \Omega \times \mathbb{R}_+$ by

$$\hat{P}^A_\lambda(\{i\} \times \mathcal{A}) := \frac{\mu_\lambda\big(\{i \in \eta^A_\tau\} \cap \mathcal{A}\big)}{\int |\eta^A_\tau| \mathrm{d}\mu_\lambda} \qquad (i \in \Lambda,\ \mathcal{A} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)). \tag{1.1}$$

Note that for $\lambda > 0$, this definition yields the probability measures $\hat{P}^A_{1/\lambda}$ from [Swa05, formula (1.14)]. We next show that the measures $\hat{P}^A_\lambda$ from (1.1) are well-defined for all $\lambda > r$ and $A \neq \emptyset$, and that $\tau \to \infty$ in probability under the laws $\hat{P}^A_\lambda$ as $\lambda \to r$. (The latter statement generalizes [Swa05, formula (3.3) on page 18].) To be precise, we will prove:

Lemma 1 (Campbell measures) *Assume that* $\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)$*. Then* $0 < \int |\eta^A_\tau| \mathrm{d}\mu_\lambda < \infty$ *for all* $\lambda > r$ *and* $\lim_{\lambda \to r} \int |\eta^A_\tau| \mathrm{d}\mu_\lambda = \infty$*. Moreover,*

$$\hat{P}^A_\lambda\big[\tau \geq t\big] \underset{\lambda \to r}{\longrightarrow} 1 \qquad (t > 0). \tag{1.2}$$

Proof We start with the case $A = \{0\}$. Recall that [Swa05, Proposition 1 (a)] is a consequence of the subadditivity of the function $t \mapsto \log E[|\eta_t^{\{0\}}|]$. In fact, subadditivity gives us a little more. By [Lig99, Theorem B.22],

$$\lim_{t\to\infty} \tfrac{1}{t} \log E\big[|\eta_t^{\{0\}}|\big] = \inf_{t>0} \tfrac{1}{t} \log E\big[|\eta_t^{\{0\}}|\big] = r, \tag{1.3}$$

where $r = r(\Lambda, a, \delta) \in [-\delta, |a| - \delta]$ is the exponential growth rate. Thus $E[|\eta_t^{\{0\}}|] = e^{r_t t}$ where $\lim_{t\to\infty} r_t = \inf_{t>0} r_t = r$. Thus, for every $\varepsilon > 0$, there exists a $T_\varepsilon < \infty$ such that

$$e^{rt} \leq E\big[|\eta_t^{\{0\}}|\big] \leq e^{(r+\varepsilon)t} \qquad (t \geq T_\varepsilon). \tag{1.4}$$

It follows from the upper bound in (1.4) that

$$\int |\eta_\tau^{\{0\}}|\,\mathrm{d}\mu_\lambda = \int_0^\infty E\big[|\eta_t^{\{0\}}|\big]\, e^{-\lambda t}\,\mathrm{d}t < \infty \tag{1.5}$$

for all $\lambda > r$. The lower bound in (1.4) and monotone convergence moreover show that

$$\lim_{\lambda\downarrow r} \int |\eta_\tau^{\{0\}}|\,\mathrm{d}\mu_\lambda = \int_0^\infty E\big[|\eta_t^{\{0\}}|\big]\, e^{-rt}\,\mathrm{d}t = \infty, \tag{1.6}$$

and therefore, for $\lambda > r$,

$$\hat{P}_\lambda^{\{0\}}\big[\tau < t\big] = \frac{\int_0^t E\big[|\eta_s^{\{0\}}|\big]\, e^{-\lambda s}\,\mathrm{d}s}{\int_0^\infty E\big[|\eta_s^{\{0\}}|\big]\, e^{-\lambda s}\,\mathrm{d}s} \leq \frac{\int_0^t E\big[|\eta_s^{\{0\}}|\big]\, e^{-rs}\,\mathrm{d}s}{\int_0^\infty E\big[|\eta_s^{\{0\}}|\big]\, e^{-\lambda s}\,\mathrm{d}s} \underset{\lambda\to r}{\longrightarrow} 0. \tag{1.7}$$

The generalization from $\{0\}$ to general $\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)$ is straightforward, by [Swa05, formula (2.11)]. ∎

Adapting the notation of [Swa05] to our present, more general set-up, we define

$$\pi_\lambda(A) := \int |\eta_\tau^A|\,\mathrm{d}\mu_\lambda \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda),\ \lambda > r). \tag{1.8}$$

Note that for $\lambda > 0$, our present π_λ is up to a multiplicative constant (which will be irrelevant later on) equal to $\pi_{1/\lambda}$ from [Swa05]. By Lemma 1, $\pi_\lambda < \infty$ for all $\lambda > r$. We write $\overline{\pi}_\lambda$ for the function π_λ normalised to one in the point $\{0\}$:

$$\overline{\pi}_\lambda(A) := \frac{\pi_\lambda(A)}{\pi_\lambda(\{0\})}. \tag{1.9}$$

We let $\pi_\lambda^\dagger$ and $\overline{\pi}_\lambda^\dagger$ denote the analogues of π_λ and $\overline{\pi}_\lambda$ for the dual $(\Lambda, a^\dagger, \delta)$-contact process. With these definitions, [Swa05, Lemma 5 (a)] generalizes in a straightforward way to:

$$\hat{P}_\lambda^{\{0\}}\big[A \cap \iota^{-1}\eta_\tau^{\{0\}} = \emptyset\big] = \overline{\pi}_\lambda^\dagger(A \cup \{0\}) - \overline{\pi}_\lambda^\dagger(A) \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda),\ \gamma > 0). \tag{1.10}$$

Here is a generalisation of [Swa05, Lemma 6] to the present context. We also remove the condition from [Swa05, Lemma 6] that the (Λ, a, δ)-contact process survives, and add an (obvious) statement about subadditivity.

Lemma 2 Cluster points of the rescaled expected population size) *The functions* $(\overline{\pi}_\lambda)_{\lambda>r}$ *are relatively compact with respect to the product topology on* $\mathbb{R}^{\mathcal{P}_{\rm fin}(\Lambda)}$. *Each pointwise limit*

$$\overline{\pi}_r(A) := \lim_{n\to\infty} \overline{\pi}_{\lambda_n}(A) \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda)) \tag{1.11}$$

along a sequence $\lambda_n \to r$ *is shift invariant, monotone, and subadditive in* A, *satisfies* $\overline{\pi}_r \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, *and*

$$G\overline{\pi}_r = r\overline{\pi}_r. \tag{1.12}$$

Proof By a straightforward generalization of [Swa05, formula (1.27)], we obtain

$$G\pi_\lambda(A) = \lambda\pi_\lambda(A) - |A| \qquad (A \in \mathcal{P}_{\rm fim}(\Lambda),\ \lambda > r)). \tag{1.13}$$

The relative compactness of $(\overline{\pi}_\lambda)_{\lambda>r}$ is proved in exactly the same way as the relative compactness of $(\overline{\pi}_\gamma)_{\gamma>0}$ in [Swa05, Lemma 6]. Since each $\overline{\pi}_\lambda$ is shift invariant, monotone, and subadditive (the latter fact was not mentioned in [Swa05] but follows immediately from [Swa05, formula (1.3)]), so is $\overline{\pi}_r$. To prove also (1.12), we observe that in analogy with [Swa05, formula (2.31)],

$$G\overline{\pi}_r(A) = \lim_{\lambda\to r} \frac{\lambda\pi_\lambda(A) - |A|}{\pi_\lambda(\{0\})} = \lim_{\lambda\to r}\ \lambda\overline{\pi}_\lambda(A) + \frac{|A|}{\pi_\lambda(\{0\})} = r\overline{\pi}_r(A), \tag{1.14}$$

where we have used that $\pi_\lambda(\{0\}) = \int |\eta_\tau^{\{0\}}|\mathrm{d}\mu_\lambda \to \infty$ as $\lambda \to r$ by Lemma 1. ∎

References

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