The contact process seen from a typical infected site.

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Abstract

This paper considers contact processes on general lattices. Assuming that the expected number of infected sites grows subexponentially, it is shown that the configuration as seen from a typical ('Palmed') infected site at an exponentially distributed time converges, as time tends to infinity, to the upper invariant law conditioned on the origin being infected. The assumption that the expected number of infected sites grows subexponentially is shown to be satisfied if the lattice has subexponential growth and the infection rates satisfy an exponential moment condition.

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Contents

1 Introduction and main results

1.1 Contact processes on countable groups

The aim of this paper is to study contact processes on rather general lattices. In particular, we are interested in the way how a certain property of the lattice, namely subexponential growth, influences the behavior of the process.

To keep things reasonably simple, we assume that the lattice Λ is a countably infinite group with group action $(i,j) \mapsto ij$ and unit element 0, also referred to as the origin. Each site $i \in \Lambda$ can be in one of two states: healthy or infected. Infected sites become healthy with *recovery rate* $\delta \geq 0$. An infected site i infects another site j with *infection rate* $a(i,j) \geq 0$. We assume that the infection rates are invariant with respect to the left action of the group, summable, and statisfy a condition that is a bit stronger than irreducibility:

$$\begin{array}{lll} \text{(i)} & a(i,j) = a(ki,kj) & (i,j,k \in \Lambda), \\ \text{(ii)} & |a| := \displaystyle\sum_i a(0,i) < \infty, & \\ \text{(iii)} & \bigcup_{n\geq 0,\ m\geq 0} A^n A^{-m} = \bigcup_{n\geq 0,\ m\geq 0} A^{-n} A^m = \Lambda, & \\ & \text{where } A := \{i \in \Lambda : a(0,i) > 0\}. & \end{array} \tag{1.1}$$

Here we adopt the convention that sums over i,j,k always run over Λ, unless stated otherwise. For $i \in \Lambda$ and $A, B \subset \Lambda$ we put $AB := \{ij : i \in A,\ j \in B\}$, $iA := \{i\}A$, $Ai := A\{i\}$, $A^{-1} := \{i^{-1} : i \in A\}$, $A^0 := \{0\}$, $A^n := AA^{n-1}$ $(n \geq 1)$, and $A^{-n} := (A^{-1})^n = (A^n)^{-1}$. We let $|A|$ denote the cardinality of A. Note that property (1.1) (iii) is equivalent to the statement that for any two sites i,j there exists a site k from which both i and j can be infected, and a set k' that can be infected both from i and from j.

If Λ has a finite symmetric generating set Δ, then the (left) Cayley graph $\mathcal{G} = \mathcal{G}(\Lambda, \Delta)$ associated with Λ and Δ is the graph with vertex set $\mathcal{V}(\mathcal{G}) := \Lambda$ and edges $\mathcal{E}(\mathcal{G}) := \{\{i,j\} : i^{-1}j \in \Delta\}$. Examples of Cayley graphs are the d-dimensional integer lattice $\mathbb{Z}^d$ $(d \geq 1)$ with edges between points at distance one, or the regular tree T_d $(d \geq 2)$ in which every vertex has $d+1$ neighbors. On Cayley graphs, one often considers *symmetric nearest-neighbor* infection rates of the form $a(i,j) = \lambda 1_{\{i^{-1}j \in \Delta\}}$, with $\lambda > 0$. In this case, λ is simply referred to as 'the' infection rate.

Let η_t be the set of all infected sites at time $t \geq 0$. Then $\eta = (\eta_t)_{t\geq 0}$ is a Markov process in the space $\mathcal{P}(\Lambda) := \{A : A \subset \Lambda\}$ of all subsets of Λ, called the contact process on Λ with infection rates $a = (a(i,j))_{i,j\in\Lambda}$ and recovery rate δ, or shortly the (Λ, a, δ)-*contact process*. If $\delta > 0$, then by rescaling time we may set $\delta = 1$, so it is customary so assume that $\delta = 1$. If $\delta = 0$ then η is a special case of first-passage percolation (see [Kes86]). We equip $\mathcal{P}(\Lambda) \cong \{0,1\}^\Lambda$ with the product topology and the associated Borel-σ-field $\mathcal{B}(\mathcal{P}(\Lambda))$, and let $\mathcal{P}_{\text{fin}}(\Lambda) := \{A \subset \Lambda : |A| < \infty\}$ denote the subspace of finite subsets of Λ.

The contact process can be constructed with the help of Harris' [Har78] *graphical representation.* Let $\omega = (\omega^{\mathrm{r}}, \omega^{\mathrm{i}})$ be a pair of independent, locally finite random subsets of $\Lambda \times \mathbb{R}$ and $\Lambda \times \Lambda \times \mathbb{R}$, respectively, produced by Poisson point processes with intensity δ and local intensity $(j,k,t) \mapsto a(j,k)$, respectively. This is usually visualized by plotting $\Lambda \times \mathbb{R}$ with Λ horizontally and $\mathbb{R}$ vertically. Points $(i,s) \in \omega^{\mathrm{r}}$ and $(j,k,t) \in \omega^{\mathrm{i}}$ are marked with a recovery symbol $*$ at (i,s) and an infection arrow from (j,t) to (k,t), respectively. For $C, D \subset \Lambda \times \mathbb{R}$, say that there is a *path* from C to D, denoted by $C \rightsquigarrow D$, if there exist $n \geq 0$, $i_0, \ldots, i_n \in \Lambda$, and $t_0 \leq \cdots \leq t_{n+1}$ with $(i_0, t_0) \in C$ and $(i_n, t_{n+1}) \in D$, such that $\{i_k\} \times [t_k, t_{k+1}] \cap \omega^{\mathrm{r}} = \emptyset$ for all $k = 0, \ldots, n$ and $(i_{k-1}, i_k, t_k) \in \omega^{\rm i}$ for all $k = 1, \ldots, n$. Thus, a path must walk upwards in time, may follow arrows, and must avoid recoveries. For given $A \in \mathcal{P}(\Lambda)$ and $t_0 \in \mathbb{R}$, put

$$\eta_t^{A\times\{t_0\}} := \{i \in \Lambda : A \times \{t_0\} \rightsquigarrow (i, t_0 + t)\} \qquad (t \geq 0). \tag{1.2}$$

Then $\eta^{A\times\{t_0\}} = (\eta_t^{A\times\{t_0\}})_{t\geq 0}$ is a copy of the (Λ, a, δ)-contact process started in $\eta_0^{A\times\{t_0\}} = A$. For brevity, we put $\eta^A := \eta^{A\times\{0\}}$. The graphical representation couples processes with different initial states in such a way that

$$\eta_t^A \cup \eta_t^B = \eta_t^{A\cup B} \qquad (A, B \in \mathcal{P}(\Lambda),\ t \geq 0). \tag{1.3}$$

Define *reversed infection rates* $a^\dagger$ by $a^\dagger(i, j) := a(j, i)$ $(i, j \in \Lambda)$. Say that a is *symmetric* if $a = a^\dagger$. For $A \in \mathcal{P}(\Lambda)$ and $t_0 \in \mathbb{R}$, put

$$\eta_t^{\dagger\,A\times\{t_0\}} := \{i \in \Lambda : (i, t_0 - t) \rightsquigarrow A \times \{t_0\}\} \qquad (t \geq 0). \tag{1.4}$$

Then $\eta^{\dagger\,A\times\{t_0\}} = (\eta_t^{\dagger\,A\times\{t_0\}})_{t\geq 0}$ is a copy of the $(\Lambda, a^\dagger, \delta)$-contact process started in $\eta_0^{\dagger\,A\times\{t_0\}} = A$. For brevity, we put $\eta^{\dagger\,A} := \eta^{\dagger\,A\times\{0\}}$. Since for any $s \leq t$ and $A, B \in \mathcal{P}(\Lambda)$, the event

$$\big\{\eta_{u-s}^{A\times\{s\}} \cap \eta_{t-u}^{\dagger\,B\times\{t\}} = \emptyset\big\} = \big\{A \times \{s\} \not\rightsquigarrow B \times \{t\}\big\} \tag{1.5}$$

does not depend on $u \in [s, t]$, it follows that the (Λ, a, δ)-contact process and the $(\Lambda, a^\dagger, \delta)$-contact process are dual in the sense that

$$P[\eta_t^A \cap B = \emptyset] = P[A \cap \eta_t^{\dagger\,B} = \emptyset] \qquad (A, B \in \mathcal{P}(\Lambda),\ t \geq 0). \tag{1.6}$$

For any $C \subset \Lambda \times \mathbb{R}$, say that $C \rightsquigarrow \infty$ if there is an infinite path with times $t_k \uparrow \infty$ starting in C, and define $-\infty \rightsquigarrow D$ analogously. Instead of $\{(i, s)\} \rightsquigarrow$ and $\rightsquigarrow \{(j, t)\}$, simply write $(i, s) \rightsquigarrow$ and $\rightsquigarrow (j, t)$. We say that the (Λ, a, δ)-contact process η *survives* if

$$\rho(A) := P\big[\eta_t^A \neq \emptyset\ \forall t \geq 0\big] = P[A \times \{0\} \rightsquigarrow \infty] > 0 \tag{1.7}$$

for some, and hence for all $\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)$. If η does not survive then we say that it *dies out*. Set $\delta_{\rm c} = \delta_{\rm c}(\Lambda, a) := \sup\{\delta \geq 0 :$ the (Λ, a, δ)-contact process survives$\}$. Then the (Λ, a, δ)-contact process survives for $\delta < \delta_{\rm c}$ and dies out for $\delta > \delta_{\rm c}$. One has $\delta_{\rm c} \leq |a|$. If Λ is finitely generated, then moreover $\delta_{\rm c} > 0$ (see Section 3.4).

1.2 Long-time behavior

Since the (Λ, a, δ)-contact process is an attractive spin system, it has an *upper invariant law* $\overline{\nu}$, i.e., an invariant law that is maximal with respect to the stochastic order. It may be constructed as $\overline{\nu} = P[\overline{\eta}_0 \in \cdot\,]$, where

$$\overline{\eta}_t := \{i \in \Lambda : -\infty \rightsquigarrow (i, t)\} \qquad (t \in \mathbb{R}). \tag{1.8}$$

Note that

$$P\big[\overline{\eta}_0 \cap A \neq \emptyset\big] = \rho^\dagger(A) \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda)), \tag{1.9}$$

where $\rho^\dagger$ denotes the survival probability of the $(\Lambda, a^\dagger, \delta)$-contact process. It is easy to see that $\overline{\nu}$ is nontrivial if and only if the $(\Lambda, a^\dagger, \delta)$-contact process survives. Here, we say that a probability law on $\mathcal{P}(\Lambda)$ is *nontrivial* if it gives zero probability to the empty set.

We say that a probability law μ on $\mathcal{P}(\Lambda)$ is *homogeneous* if μ is shift invariant with respect to the left action of the group, i.e., $\mu(\{iA : A \in \mathcal{A}\}) = \mu(\mathcal{A})$ for all $\mathcal{A} \in \mathcal{B}(\mathcal{P}(\Lambda))$. Using duality, it can be shown that

$$\int \mu(\mathrm{d}A) P[\eta_t^A \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \overline{\nu} \tag{1.10}$$

whenever the initial law μ is homogeneous and nontrivial (see [Har76], [Lig85, (VI.2.1)], and [Lig99, (I.1.10)]). Here $\Rightarrow$ denotes weak convergence of probability laws. In particular, (1.10) shows that if $\overline{\nu}$ is nontrivial, then it is the only nontrivial homogeneous invariant law.

The long-time behavior for nonhomogeneous initial laws is more subtle and depends on properties of the lattice Λ and the infection rates a, such as subexponential growth.

For the symmetric nearest-neighbor contact process on $\mathbb{Z}^d$ started in a finite initial state, the following picture has been rigorously verified. Either the process dies out in finite time, or in the long run there is a region in space with linearly growing diameter and deterministic limiting shape, such that most of the infected sites lie within this region and there the process is locally in the upper invariant law [BG90]. In particular, it has been shown that the symmetric nearest-neighbor process on $\mathbb{Z}^d$ exhibits *complete convergence*, i.e.,

$$P[\eta_t^A \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \rho(A)\overline{\nu} + (1-\rho(A))\delta_0 \qquad (A \in \mathcal{P}_{\mathrm{fin}}(\Lambda)). \tag{1.11}$$

Note that if complete convergence holds and $\overline{\nu}$ is nontrivial, then by monotonicity, it is the unique nontrivial invariant law. For other contact processes on $\mathbb{Z}^d$ the picture is supposedly similar, provided that the infection rates are symmetric and satisfy an appropriate tail condition. If the infection rates are not symmetric, there is probably still a linearly growing infected region with a limiting shape, but this region may walk out to infinity, so that complete convergence does not hold. (For results in the one-dimensional case, see [Sch86].)

The behavior of the symmetric nearest-neighbor process on regular trees T_d is known to be quite different. Here, there is a second critial value $\delta'_{\mathrm{c}} < \delta_{\mathrm{c}}$ such that for recovery rates $\delta \in [\delta'_{\mathrm{c}}, \delta_{\mathrm{c}})$, the process survives globally but not locally, i.e., $\rho(A) > 0$ but $P[\exists T \geq 0 \text{ s.t. } \eta_t^A \cap \{0\} = \emptyset\ \forall t \geq T] = 1$ for $\emptyset \neq A \in \mathcal{P}_{\mathrm{fin}}(\Lambda)$. In this regime, there is a multitude of nontrivial invariant measures and complete convergence (obviously) does not hold [Lig99, Section I.4].

One would like to understand which properties of the lattices $\mathbb{Z}^d$ and T_d are responsible for the differences in the behavior of the contact process, and which types of behavior are possible on general lattices Λ. The proofs for $\mathbb{Z}^d$ and T_d use the structure of these lattices in an essential way, and are not easily generalized to other lattices.

It is known that (unoriented) percolation has quite different properties on $\mathbb{Z}^d$ and on T_d. Here, the important property of $\mathbb{Z}^d$, that T_d lacks, is *amenability*. For example, the Burton-Keane proof of the uniqueness of the infinite cluster [BK89] works on any amenable graph. Conversely, it is conjectured that on any nonamenable graph, there exists a range of the percolation parameter for which the infinite cluster is not unique. (See [BS01] some partial results in this direction.)

For our main theorem, we will need to assume that the expected number of infected sites in a contact process grows subexponentially. If Λ is finitely generated, then it turns out that the (Λ, a, δ)-contact process grows subexponentially if a satisfies an exponential moment condition and Λ itself has subexponential growth (see Proposition 1 (d) below). Here, by definition, a finitely generated group Λ has *subexponential growth* if

$$\lim_{n\to\infty} \frac{1}{n} \log |\Delta^n| = 0 \tag{1.12}$$

for some, and hence for all finite symmetric generating sets Δ. Observe that $\Delta^n = \{i : |i| \leq n\}$ where $|i|$ denotes the distance of i to the origin in the Cayley graph $\mathcal{G}(\Lambda, \Delta)$. Subexponential growth is stronger than amenability. An example of an amenable finitely generated group that does not have subexponential growth is the lamplighter group. (See [MW89, Section 5] for general facts about amenability and subexponential growth, and [LPP96] for a nice exposition of the lamplighter group.)

1.3 Results

It turns out that every (Λ, a, δ)-contact process has a well-defined exponential growth rate.

Proposition 1 (Exponential growth rate)
(a) *There exists a constant $r = r(\Lambda, a, \delta) \in [-\delta, |a| - \delta]$ such that the (Λ, a, δ)-contact process satisfies*

$$\lim_{t \to \infty} \tfrac{1}{t} \log E\big[|\eta_t^A|\big] = r \qquad (\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)). \tag{1.13}$$

(b) *If the (Λ, a, δ)-contact process survives, then $r \geq 0$.*

(c) $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$.

(d) *Assume that Λ is finitely generated. Let Δ be a finite symmetric generating set and let $|j|$ denote the distance of j to the origin in the Cayley graph $\mathcal{G}(\Lambda, \Delta)$. Then one has $r \leq 0$, provided that $\sum_j a(0,j)e^{\varepsilon|j|} < \infty$ for some $\varepsilon > 0$, and Λ has subexponential growth.*

The proof of Proposition 1 will be given in Sections 2.2–2.3. Part (a) follows from subadditivity, part (b) is trivial, and part (c) is a consequence of duality. Part (d) follows from some basic large deviation estimates. The exponential moment condition on a appearing in part (d) can perhaps be weakened, but we conjecture that it cannot be dropped altogether. Indeed, it seems plausible that even on $\Lambda = \mathbb{Z}$, the exponential growth rate can be positive if a has a sufficiently heavy tail.

To formulate the main results of this paper, we must describe the contact process as seen from a 'typical' infected site at a 'typical' late time. Assume that the exponential growth rate r from Proposition 1 satisfies $r \leq 0$. Recall the graphical construction of the (Λ, a, δ)-contact process (see Section 1.1). Let $(\Omega, \mathcal{F}, P)$ be the probability space on which the Poisson point processes used in the graphical representation are defined. Let τ be an exponentially distributed random variable with mean $\gamma > 0$, constructed on a canonical probability space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \mathcal{E}_\gamma)$. Let $P_\gamma := P \otimes \mathcal{E}_\gamma$ denote product measure and let E_γ denote expectation with respect to P_γ. For $\gamma > 0$ and $\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)$, we define a probability measures $\hat{P}_\gamma^A$ on $\Lambda \times \Omega \times \mathbb{R}_+$ by

$$\hat{P}_\gamma^A(\{i\} \times \mathcal{A}) := \frac{P_\gamma\big(\{i \in \eta_\tau^A\} \cap \mathcal{A}\big)}{E_\gamma\big[|\eta_\tau^A|\big]} \qquad (i \in \Lambda,\ \mathcal{A} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)). \tag{1.14}$$

Using the fact that $r \leq 0$, it is easy to see that $E_\gamma[|\eta_\tau^A|] < \infty$ for all $\gamma > 0$, so that $\hat{P}_\gamma^A$ is well-defined. Let $\iota : \Lambda \times \Omega \times \mathbb{R}_+ \to \Lambda$ denote the projection on Λ. Then, under the law $\hat{P}_\gamma^A$, the random variable η_τ^A describes a contact process observed at an exponentially distributed random time,[1] size-biased on the number of infected sites, and ι is the position of a 'typical' infected site. The law $\hat{P}_\gamma^A[(\iota, \eta_\tau^A) \in \cdot\,]$ is a Campbell law, which is closely related to the more

[1]Note, however, that under the law $\hat{P}_\gamma^A$, the random time τ is no longer exponentially distributed.

widely known Palm laws. (For the relation between Campbell and Palm laws, see [Eth00, Section 6.4].)

Note that $\iota^{-1}\eta_\tau$ is the process η_τ, viewed from the position of the typical infected site ι. The next theorem is the main result of this paper. Recall the definition of $\overline{\eta}$ in (1.8).

Theorem 2 (The process seen from a typical infected site) *Assume that the upper invariant measure of the* (Λ, a, δ)*-contact process is nontrivial and that the exponential growth rate from Proposition 1 satisfies* $r \le 0$. *Let* $\emptyset \neq A \in \mathcal{P}(\Lambda)$. *Then*

(a) *One has*

$$\hat{P}^A_\gamma\big[\iota^{-1}\eta^A_\tau \in \cdot\,\big] \underset{\gamma\to\infty}{\Longrightarrow} P\big[\overline{\eta}_0 \in \cdot\,\big|\,0 \in \overline{\eta}_0\big]. \tag{1.15}$$

(b) *Moreover,*

$$\hat{P}^A_\gamma\big[\iota^{-1}\eta^A_\tau \cap \Delta = \iota^{-1}\overline{\eta}_\tau \cap \Delta\big] \underset{\gamma\to\infty}{\longrightarrow} 1 \qquad (\Delta \in \mathcal{P}_{\rm fin}(\Lambda)), \tag{1.16}$$

and the same holds with $\overline{\eta}_\tau$ *replaced by* η^Λ_τ.

Note that Theorem 2 holds when Λ is a general countable group, but we have only verified that its assumptions are satisfied for certain finitely generated groups (see Proposition 1 (d)). We remark that for fixed $\gamma > 0$, it is not at all obvious (and as far as we know *not true*) that the distribution $\hat{P}^A_\gamma[\iota^{-1}\overline{\eta}_\tau \in \cdot\,]$ should be the same as $P[\overline{\eta}_0 \in \cdot\,|\,0 \in \overline{\eta}_0]$. Thus, none of the statements (1.15) and (1.16) trivially implies the other one.

As a result of our methods, we can also prove the following fact, which is of some interest on its own.

Proposition 3 (Typical particles descend from every surviving site) *Assume that the* (Λ, a, δ)*-contact process survives and that the exponential growth rate from Proposition 1 satisfies* $r \le 0$. *Then*

$$\hat{P}^{\{i\}}_\gamma\big[(j,0) \rightsquigarrow (\iota,\tau)\,\big|\,(j,0) \rightsquigarrow \infty\big] \underset{\gamma\to\infty}{\longrightarrow} 1 \qquad (i,j \in \Lambda). \tag{1.17}$$

One of the original motivations of the present paper was to answer the following question. Assuming survival and subexponential growth, is it true that for any $i, j \in \Lambda$,

$$P\big[\exists (k,t) \text{ s.t. } (i,0) \rightsquigarrow (k,t) \rightsquigarrow \infty \text{ and } (j,0) \rightsquigarrow (k,t)\,\big|\,(i,0) \rightsquigarrow \infty, (j,0) \rightsquigarrow \infty\big] = 1 \quad ? \tag{1.18}$$

This property may be interpreted as some sort of analogue of the uniqueness of the infinite cluster in (unoriented) percolation. Unfortunately, we do not know how to replace the size-biased law in (1.17) by a law conditioned on survival. Question (1.18) has been answered positively for oriented percolation on $\mathbb{Z}^d$ in [GH02]. As a further motivation for (1.18), we note that in the one-dimensional nearest-neighbor case, a considerably stronger statement holds.

Lemma 4 (Coupling of one-dimensional processes) *Consider a* $(\mathbb{Z}, a, \delta)$*-contact process with* $a(i,j) = 0$ *for* $|i-j| \neq 1$. *Assume that the process survives, and assume either* $\delta > 0$ *or* $a(0,1) \wedge a(1,0) > 0$. *Then, for any* $i, j \in \mathbb{Z}$,

$$P\big[\inf\{t \ge 0 : \eta^{\{i\}}_t = \eta^{\{j\}}_t\} < \infty\,\big|\,(i,0) \rightsquigarrow \infty,\ (j,0) \rightsquigarrow \infty\big] = 1. \tag{1.19}$$

1.4 Methods

In this section we describe the main line of our proof of Theorem 2 (a). The first ingredient is a chararacterization of the laws $\hat P^A_\gamma[\iota^{-1}\eta^A_\tau \in \cdot\,]$ and $P[\overline{\eta}_0 \in \cdot\,|\,0 \in \overline{\eta}_0]$ in terms of the dual $(\Lambda, a^\dagger, \delta)$-contact process $\eta^\dagger$. For simplicity, we only present the argument for $\hat P^{\{0\}}_\gamma$. We put

$$\pi_\gamma(A) := E_\gamma\big[|\eta^A_\tau|\big] \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda),\ \gamma > 0). \tag{1.20}$$

Recall that $\pi_\gamma < \infty$ for all $\gamma > 0$ by our assumption that $r \le 0$, and recall the definition of the survival probability ρ in (1.7). We write $\overline{\pi}_\gamma$ and $\overline{\rho}$ for the functions π_γ and ρ normalised to one in the point $\{0\}$:

$$\overline{\rho}(A) := \frac{\rho(A)}{\rho(\{0\})} \quad \text{and} \quad \overline{\pi}_\gamma(A) := \frac{\pi_\gamma(A)}{\pi_\gamma(\{0\})}. \tag{1.21}$$

We let $\rho^\dagger, \pi^\dagger_\gamma, \overline{\rho}^\dagger$, and $\overline{\pi}^\dagger_\gamma$ denote the analogues of $\rho, \pi_\gamma, \overline{\rho}$, and $\overline{\pi}_\gamma$ for the dual $(\Lambda, a^\dagger, \delta)$-contact process.

Lemma 5 (Characterization of laws seen from an infected site)
(a) *One has*

$$\hat P^{\{0\}}_\gamma\big[A \cap \iota^{-1}\eta^{\{0\}}_\tau = \emptyset\big] = \overline{\pi}^\dagger_\gamma(A \cup \{0\}) - \overline{\pi}^\dagger_\gamma(A) \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda),\ \gamma > 0). \tag{1.22}$$

(b) *Moreover,*

$$P\big[A \cap \overline{\eta}_0 = \emptyset\,\big|\,0 \in \overline{\eta}_0\big] = \overline{\rho}^\dagger(A \cup \{0\}) - \overline{\rho}^\dagger(A) \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda)). \tag{1.23}$$

It is not hard to see that the law of a $\mathcal{P}(\Lambda)$-valued random variable η is uniquely characterized by all probabilities of the form $P[A\cap\eta = \emptyset]$ with $A \in \mathcal{P}_{\rm fin}(\Lambda)$. Therefore, by Lemma 5 and the compactness of $\mathcal{P}(\Lambda) \cong \{0,1\}^\Lambda$, in order to prove Theorem 2, it suffices to prove that $\overline{\pi}^\dagger_\gamma \to \overline{\rho}^\dagger$ pointwise as $\gamma \to \infty$. In order to reduce notation, we reverse the role of η and $\eta^\dagger$. Thus, we will prove that pointwise $\lim_{\gamma\to\infty} \overline{\pi}_\gamma = \overline{\rho}$, under the assumptions that the (Λ, a, δ)-contact process survives and its exponential growth rate is zero. (By (1.9) and Proposition 1 (c), this is equivalent to the $(\Lambda, a^\dagger, \delta)$-contact process having a nontrivial upper invariant law and exponential growth rate zero.)

It is not hard to show (see Section 2.1 below) that the (Λ, a, δ)-contact process started in a finite initial state solves the martingale problem for the operator

$$\begin{aligned} Gf(A) := &\sum_{ij} a(i,j) 1_{\{i\in A\}} 1_{\{j\notin A\}} \{f(A \cup \{j\}) - f(A)\} \\ &+\delta \sum_i 1_{\{i\in A\}} \{f(A\backslash\{i\}) - f(A)\}, \end{aligned} \tag{1.24}$$

with domain $\mathcal{D}(G) := \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, where

$$\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda)) := \{f : \mathcal{P}_{\rm fin}(\Lambda) \to \mathbb{R} : |f(A)| \le K|A|^k + M \text{ for some } K, M, k \ge 0\}. \tag{1.25}$$

It can be shown in a few lines that ρ is shift invariant, monotone (i.e., $A \subset B$ implies $\rho(A) \le \rho(B)$), $\rho \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, and

$$G\rho = 0. \tag{1.26}$$

Formula (1.26) says that ρ is a harmonic function for the (Λ,a,δ)-contact process. It is not hard to see that π_γ shift invariant, monotone, $\pi_\gamma\in\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, and

$$G\pi_\gamma(A)=\gamma^{-1}\big(\pi_\gamma(A)-|A|\big)\qquad(A\in\mathcal{P}_{\rm fin}(\Lambda),\ \gamma>0).\tag{1.27}$$

As a consequence, one obtains:

Lemma 6 (Cluster points of the rescaled expected population size) *Assume that the exponential growth rate from Proposition 1 satisfies $r(\Lambda,a,\delta)\leq 0$ and that the (Λ,a,δ)-contact process survives. Then the functions $(\overline{\pi}_\gamma)_{\gamma>0}$ are relatively compact with respect to the product topology on $\mathbb{R}^{\mathcal{P}_{\rm fin}(\Lambda)}$. Each pointwise limit*

$$\overline{\pi}_\infty(A):=\lim_{n\to\infty}\overline{\pi}_{\gamma_n}(A)\qquad(A\in\mathcal{P}_{\rm fin}(\Lambda))\tag{1.28}$$

along a sequence $\gamma_n\to\infty$ is shift invariant, monotone in A, satisfies $\overline{\pi}_\infty\in\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, and

$$G\overline{\pi}_\infty=0.\tag{1.29}$$

It turns out that Lemma 6 gives us enough information to determine $\overline{\pi}_\infty$ uniquely. Combined with the next proposition, Lemma 6 shows that $\overline{\pi}_\gamma\to\overline{\rho}$ pointwise as $\gamma\to\infty$, thereby completing the proof of Theorem 2.

Proposition 7 (Shift invariant monotone harmonic functions) *Assume that the (Λ,a,δ)-contact process survives. Assume that $f:\mathcal{P}_{\rm fin}(\Lambda)\to\mathbb{R}$ is shift invariant, monotone, $f(\emptyset)=0$, $f\in\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, and $Gf=0$. Then there exists a constant $c\geq 0$ such that $f=c\rho$.*

We note that if ν is a homogeneous invariant measure for the $(\Lambda,a^\dagger,\delta)$-contact process, then by duality, $f(A):=\nu(\{A:A\cap B\neq\emptyset\})$ defines a shift invariant, monotone, bounded harmonic function f for the (Λ,a,δ)-contact process. Therefore, in view of (1.9), Proposition 7 is a strengthening of the statement that all homogeneous invariant measures are convex combinations of $\overline{\nu}$ and δ_0.

In order to prove Proposition 7, we need one more lemma.

Lemma 8 (Eventual domination of finite configurations) *Assume that the (Λ,a,δ)-contact process survives. Then*

$$\lim_{t\to\infty}P\big[\exists i\in\Lambda\ s.t.\ \eta^A_t\geq iB\,\big|\,\eta^A_t\neq\emptyset\big]=1\qquad(A,B\in\mathcal{P}_{\rm fin}(\Lambda),\ A\neq\emptyset).\tag{1.30}$$

Formula (1.30) says that η exhibits a form of 'extinction versus unbounded growth'. More precisely, either η_t gets extinct or η_t is eventually larger than a random shift of any finite configuration. We remark that Lemma 8 is no longer true if assumption (1.1) (iii) is replaced by the weaker assumption that $\{i\in\Lambda:a(0,i)>0\}$ generates Λ.

Proof of Proposition 7 Since the (Λ,a,δ)-contact process solves the martingale problem for G, and $Gf=0$, the process $f(\eta^A_t)$ is a martingale. In particular:

$$f(A)=E[f(\eta^A_t)]\qquad(A\in\mathcal{P}_{\rm fin}(\Lambda),\ t\geq 0).\tag{1.31}$$

Equip Λ with an arbitrary linear ordering, and for $A,B\in\mathcal{P}_{\rm fin}(\Lambda)$, put

$$\hat{\imath}_{A,B}:=\begin{cases}\min\{i\in\Lambda:A\geq iB\} & \text{if } \{i\in\Lambda:A\geq iB\}\text{ is nonempty,}\\ 0 & \text{otherwise.}\end{cases}\tag{1.32}$$

Since f is monotone and shift invariant, we have, using Lemma 8,

$$
\begin{aligned}
f(A) &= \lim_{t\to\infty} E[f(\eta_t^A)] \\
&\geq \limsup_{t\to\infty} E[1_{\{\exists i\in\Lambda \text{ s.t. } \eta_t^A \geq iB\}} f(\hat{\imath}_{\eta_t^A,B}B)] \\
&= f(B)\limsup_{t\to\infty} P[\exists i\in\Lambda \text{ s.t. } \eta_t^A \geq iB] \geq f(B)\rho(A) \qquad (A,B\in\mathcal{P}_{\rm fin}(\Lambda)).
\end{aligned}
\tag{1.33}
$$

In particular, this shows that

$$
f(B) \leq \frac{f(\{0\})}{\rho(\{0\})} < \infty \qquad (B\in\mathcal{P}_{\rm fin}(\Lambda)), \tag{1.34}
$$

hence f is bounded. Now let $A_n, B_m \in \mathcal{P}_{\rm fin}(\Lambda)$ be sequences such that $\rho(A_n)\to 1$ and $\rho(B_n)\to 1$. Then, by (1.33),

$$
\liminf_{n\to\infty} f(A_n) \geq \liminf_{n\to\infty} f(B_m)\rho(A_n) = f(B_m) \quad \forall m, \tag{1.35}
$$

and therefore

$$
\liminf_{n\to\infty} f(A_n) \geq \limsup_{m\to\infty} f(B_m). \tag{1.36}
$$

This proves that the limit

$$
\lim_{\rho(A_n)\to 1} f(A_n) =: f(\infty) \tag{1.37}
$$

exists and does not depend on the choice of the sequence A_n with $\rho(A_n)\to 1$. By the Markov property and continuity of the conditional expectation with respect to increasing limits of σ-fields (see Complement 10(b) from [Loe63, Section 29] or [Loe78, Section 32]),

$$
\rho(\eta_t^A) = P\big[\eta_s^A \neq 0\ \forall s\geq 0 \,\big|\, \eta_t^A\big] \to 1_{\{\eta_s^A \neq 0\ \forall s\geq 0\}} \quad \text{a.s.} \quad \text{as } t\to\infty. \tag{1.38}
$$

We conclude that

$$
f(A) = \lim_{t\to\infty} E[f(\eta_t^A)] = \rho(A) f(\infty) \qquad (A\in\mathcal{P}_{\rm fin}(\Lambda)), \tag{1.39}
$$

which shows that f is a scalar multiple of ρ. ∎

1.5 Discussion and open problems

Palm and Campbell laws are standard tools in the study of (critical) spatial branching processes. In this context, they can be described by Kallenberg's backward tree technique; see, for example, [Kal77] or [GW91]. In the context of contact processes, it is less obvious that they should be of any use. For example, size-biasing with $|\eta_t^{\{0\}} \cap \{i\}|$ for fixed i and t is just the same as conditioning on $(0,0) \rightsquigarrow (i,t)$. In this case there seems to be no easy way to prove statements about $i^{-1}\eta_t^{\{0\}}$.

However, by looking at the process seen from a randomly chosen infected site rather than a fixed site, i.e., by looking at Campbell laws rather than Palm laws, we can make a connection with the growth of $E[|\eta_t^{\{0\}}|]$ as $t\to\infty$, and in this way obtain a result. A disadvantage of this approach is that one ends up with statements about size-biased laws, where one would probably be more interested in laws conditioned on survival. Nevertheless, it seems that the statements in Theorem 2 do catch a phenomenon that depends in a crucial way on a property of the underlying lattice, in this case, subexponential growth.

We next state some open problems and questions, and then comment on them.

1. **Problem** Replace the exponentially distributed time τ in (1.14) by a deterministic time t and prove the analogue of Theorem 2 for $t \to \infty$.

2. **Problem** Study the contact process seen from a typical infected site in case the exponential growth rate is positive.

3. **Problem** Study the contact proces seen from a typical infected site chosen from a process conditioned to survive, instead of size-biased on the number of infected sites.

4. **Problem** Prove (1.18) assuming survival and subexponential growth.

5. **Problem** Assuming survival and subexponential growth, prove that conditional on $(i,0) \rightsquigarrow \infty$ and $(j,0) \rightsquigarrow \infty$, eventually most sites in $\eta_t^{\{i\}}$ are also in $\eta_t^{\{j\}}$.

6. **Question** With the same set-up as in the previous problem, is it even true that $\eta_t^{\{i\}}$ and $\eta_t^{\{j\}}$ are eventually equal? (Compare Lemma 4.)

7. **Problem** Prove that $\delta_{\rm c} > 0$ for a contact process on a group Λ that is not finitely generated, for example on the hierarchical group.

8. **Problem** Give an example of a contact process on $\mathbb{Z}$ for which the exponential growth rate is positive.

9. **Question** Assuming that Λ has exponential growth, is it true that the (Λ, a, δ)-contact process survives if and only if $r(\Lambda, a, d) > 0$?

10. **Question** Does survival of the (Λ, a, δ)-contact process imply survival of the $(\Lambda, a^\dagger, \delta)$-contact process?

In order to solve Problem 1 for some lattice Λ, it seems one would have to strengthen Proposition 1 (a) in the sense that

$$\lim_{t\to\infty} \tfrac{\partial}{\partial t} \log E\big[|\eta_t^A|\big] = r \qquad (\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)). \tag{1.40}$$

Then it would follow that each cluster point $\overline{\pi}_\infty$ of the functions $\overline{\pi}_t(A) := E\big[|\eta_t^A|\big]/E\big[|\eta_t^{\{0\}}|\big]$ satisfies $G\overline{\pi}_\infty = r$. However, (1.40) does not simply follow from subadditivity and seems hard to establish in general. Even random times τ that are uniformly distributed on intervals $[0,T]$ seem difficult to treat, since they would require that $\lim_{T\to\infty} \frac{\partial}{\partial T} \log \int_0^T E\big[|\eta_t^A|\big] \mathrm{d}t = r$.

In view of these difficulties, a good approach to Problem 2 is probably to stick to exponentially distributed times and define $\hat{P}_\gamma^A$ and π_γ for $\gamma < 1/r$ as before, but this time let $\gamma \to 1/r$ instead of $\gamma \to \infty$.

Problems 3–5 and Question 6 have been discussed before. The difficulty is to replace size-biased laws by laws conditioned on survival in statements like Proposition 3. Although size-biasing and conditioning are asymptotically equivalent in a 'local' sense (see Proposition 12 below), this does not seem easy. Note that if (1.18) holds for the $(\Lambda, a^\dagger, \delta)$-contact process, then the limit law in Theorem 2 (a) may also be written as $P[\hat{\eta}_0 \in \cdot \mid -\infty \rightsquigarrow (0,0)]$, where $\hat{\eta}_t := \{i \in \Lambda : \exists (j,s) \text{ s.t. } -\infty \rightsquigarrow (j,s) \rightsquigarrow (0,0) \text{ and } (j,s) \rightsquigarrow (i,t)\}$ $(t \in \mathbb{R})$. This construction is similar to Kallenberg's backward tree technique, and also somewhat reminiscent of the construction of the the second lowest extremal invariant measure of the contact process in [SS97, SS99].

Problem 7 seems interesting, since the hierarchical group has found applications in population biology, and the usual comparison with one-dimensional oriented percolation cannot work here.

Problem 8 and Question 9 are naturally motivated by Proposition 1 (d). Related to Question 9 is the more general question: what does the behavior of $E[|\eta_t|]$ for $t \to \infty$ tell us about survival? Especially for critical processes, it seems conceivable that $\lim_{t\to\infty} E[|\eta_t|] = \infty$ while the process dies out. Related to this is Question 10, which has been asked before for branching-coalescing particle systems in [AS05]. For symmetric processes or for processes on abelian groups, the answer is obviously positive, but in general (Λ, a) and $(\Lambda, a^\dagger)$ need not be isomorphic. However, in formula (2.14) below, it is shown that $E[|\eta_t^{\{0\}}|] = E[|\eta_t^{\dagger\,\{0\}}|]$ for all $t \geq 0$.

Related to Question 10 (compare also Question 6) is the following question: is it always true that $\inf\{t \geq 0 : \eta_t^{\{0\}} \subset \overline{\eta}_t\}$ is a.s. finite? Note that if the answer is positive, then extinction of the $(\Lambda, a^\dagger, \delta)$-contact process implies extinction of the (Λ, a, δ)-contact process, since in this case $\overline{\eta} \equiv 0$.

1.6 Outline

Section 2 is devoted to the proof of Theorem 2 (a). In Section 2.1 we prove that contact processes started in finite initial states solve the martingale problem for the operator G in (1.24). We establish Proposition 1 (a)–(c) in Section 2.2, and part (d) in Section 2.3. In Section 2.4, we establish Lemma 5. In Section 2.5, we prove basic facts about the functions ρ and π_γ; in particular, formulas (1.26) and (1.27), and Lemma 6. In Section 2.6, we prove Lemma 8, thereby completing the proof of Theorem 2 in the case $A = \{0\}$. In Section 2.7 we show how the arguments may be generalized to arbitrary $\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)$.

Section 3 contains proofs of all results that are not directly needed for Theorem 2 (a). In Section 3.1, we prove that size-biasing and conditioning on survival are equivalent in a 'local' sense. Section 3.2 contains the proofs of Theorem 2 (b) and Proposition 3. Section 3.3 contains the proof of Lemma 4. For completeness, we prove in Section 3.4 the fact mentioned in the text that $\delta_{\rm c} > 0$ whenever Λ is finitely generated.

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2 The law seen from a typical particle

2.1 A martingale problem

In this section we prove that the (Λ, a, δ)-contact process started in finite initial states solves the martingale problem for the operator G in (1.24)–(1.25).

Proposition 9 (Martingale problem and moment estimate) *For each $f \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ and $A \in \mathcal{P}_{\rm fin}(\Lambda)$, the process*

$$M_t := f(\eta_t^A) - \int_0^t Gf(\eta_s^A)\mathrm{d}s \qquad (t \geq 0) \tag{2.1}$$

is a martingale with respect to the filtration generated by η^A. Moreover, setting $z^{\langle k\rangle} := \prod_{i=0}^{k-1}(z+i)$, one has

$$E\big[|\eta^A_t|^{\langle k\rangle}\big] \leq |A|^{\langle k\rangle} e^{k(|a|-\delta)t} \qquad (A\in\mathcal{P}_{\rm fin}(\Lambda),\ k\geq 1,\ t\geq 0). \tag{2.2}$$

Proof The proof of [AS05, Proposition 8] can in a straightforward way be adapted to the present set-up. Set $f_k(A) := |A|^{\langle k\rangle}$. Then

$$\begin{aligned} Gf_k(A) &= \sum_{ij} a(i,j)1_{\{i\in A\}}1_{\{j\notin A\}}\{(|A|+1)^{\langle k\rangle} - |A|^{\langle k\rangle}\} + \delta\sum_i 1_{\{i\in A\}}\{(|A|-1)^{\langle k\rangle} - |A|^{\langle k\rangle}\},\\ &\leq (|a|-\delta)|A|\{(|A|+1)^{\langle k\rangle} - |A|^{\langle k\rangle}\} = k(|a|-\delta)|A|^{\langle k\rangle}. \end{aligned} \tag{2.3}$$

Define stopping times $\tau_N := \inf\{t\geq 0 : |\eta^A_t|\geq N\}$. The stopped process $(\eta^A_{t\wedge\tau_N})_{t\geq 0}$ has bounded jump rates, and therefore standard theory tells us that for each $N\geq 1$ and $f\in\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, the process

$$M^N_t := f(\eta^A_{t\wedge\tau_N}) - \int_0^{t\wedge\tau_N} Gf(\eta^A_s)\mathrm{d}s \qquad (t\geq 0) \tag{2.4}$$

is a martingale. Moreover, it easily follows from (2.3) that

$$E\big[|\eta^A_{t\wedge\tau_N}|^{\langle k\rangle}\big] \leq |A|^{\langle k\rangle} e^{k(|a|-\delta)t} \qquad (k\geq 1,\ t\geq 0). \tag{2.5}$$

It is easy to see that $f\in\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$ implies $Gf\in\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$. Using this fact and (2.5) for some sufficiently high k (depending on f), one can show that for fixed $t\geq 0$, the random variables $(M^N_t)_{N\geq 1}$ are uniformly integrable. Therefore, letting $N\to\infty$ in (2.4), one finds that the process in (2.1) is a martingale. Letting $N\to\infty$ in (2.5) yields (2.2). ∎

2.2 The exponential growth rate

In this section we prove Proposition 1 (a)–(c).

Proof of Proposition 1 (a) By a slight abuse of notation, let us write (compare (1.20))

$$\pi_t(A) := E\big[|\eta^A_t|\big] \qquad (A\in\mathcal{P}_{\rm fin}(\Lambda),\ t\geq 0). \tag{2.6}$$

We start by showing that

$$\pi_{s+t}(\{0\}) \leq \pi_s(\{0\})\pi_t(\{0\}) \qquad (s,t\geq 0). \tag{2.7}$$

By (1.3),

$$E\big[|\eta^A_t|\big] = E\Big[\big|\bigcup_{i\in A}\eta^{\{i\}}_t\big|\Big] \leq \sum_{i\in A} E\big[|\eta^{\{i\}}_t|\big] = |A|E\big[|\eta^{\{0\}}_t|\big], \tag{2.8}$$

where in the last step we have used shift invariance. As a consequence,

$$\pi_{s+t}(\{0\}) = \int P[\eta^{\{0\}}_s\in\mathrm{d}A]E\big[|\eta^A_t|\big] \leq \int P[\eta^{\{0\}}_s\in\mathrm{d}A]|A|E\big[|\eta^{\{0\}}_t|\big] = \pi_s(\{0\})\pi_t(\{0\}). \tag{2.9}$$

This proves (2.7). It follows that $t\mapsto\log\pi_t(\{0\})$ is subadditive and therefore, by [Lig99, Theorem B.22], the limit

$$\lim_{t\to\infty}\tfrac{1}{t}\log\pi_t(\{0\}) =: r\in[-\infty,\infty] \tag{2.10}$$

exists. By monotonicity and (2.8),

$$\pi_t(\{0\}) \leq \pi_t(A) \leq |A|\pi_t(\{0\}) \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda)). \tag{2.11}$$

Taking logarithms, dividing by t, and letting $t \to \infty$ we arrive at (1.13). Since η can be bounded from below by a simple death process and from above by a branching process (see (2.15) below), one has

$$e^{-\delta t} \leq E\big[|\eta_t^{\{0\}}|\big] \leq e^{(|a|-\delta)t} \qquad (t \geq 0), \tag{2.12}$$

which implies that $-\delta \leq r \leq |a| - \delta$. ∎

Proof of Proposition 1 (b) If the (Λ, a, δ)-contact process survives, then

$$\pi_t(\{0\}) \geq P[\eta_t^{\{0\}} \neq 0] \underset{t\to\infty}{\longrightarrow} P[\eta_s^{\{0\}} \neq 0 \ \forall s \geq 0] > 0, \tag{2.13}$$

which implies that $r \geq 0$. ∎

Proof of Proposition 1 (c) By duality (formula (1.6)) and shift invariance,

$$\begin{aligned} E\big[|\eta_t^{\{0\}}|\big] &= \sum_i P\big[\eta_t^{\{0\}} \cap \{i\} \neq \emptyset\big] = \sum_i P\big[\{0\} \cap \eta_t^{\dagger\,\{i\}} \neq \emptyset\big] \\ &= \sum_i P\big[\{i^{-1}\} \cap \eta_t^{\dagger\,\{0\}} \neq \emptyset\big] = E\big[|\eta_t^{\dagger\,\{0\}}|\big], \end{aligned} \tag{2.14}$$

which implies that $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$. ∎

2.3 Subexponential growth

Proof of Proposition 1 (d) Consider a branching process on Λ, started with one particle in the origin, where a particle at i produces a new particle at j with rate $a(i,j)$, and each particle dies with rate δ. Let $B_t(i)$ denote the number of particles at site $i \in \Lambda$ and time $t \geq 0$. It is not hard to see that $\eta^{\{0\}}$ and B may be coupled such that

$$1_{\eta_t^{\{0\}}} \leq B_t \qquad (t \geq 0). \tag{2.15}$$

Let $(\xi_t)_{t\geq 0}$ be a random walk on Λ that jumps from i to j with rate $a(i,j)$, started in $\xi_0 = 0$. Then it is not hard to see that (compare [Lig99, Proposition I.1.21])

$$E[B_t(i)] = P[\xi_t = i]e^{(|a|-\delta)t} \qquad (i \in \Lambda,\ t \geq 0). \tag{2.16}$$

Let $\lambda > 0$ be a constant, to determined later. It follows from (2.15) and (2.16) that

$$\begin{aligned} E\big[|\eta_t^{\{0\}}|\big] &\leq \sum_i \big(1 \wedge P[\xi_t = i]e^{(|a|-\delta)t}\big) \\ &= |\{i \in \Lambda : |i| \leq \lambda t\}| + P[|\xi_t| > \lambda t]e^{(|a|-\delta)t} \qquad (t \geq 0). \end{aligned} \tag{2.17}$$

Let $(Y_i)_{i\geq 1}$ be i.i.d. $\mathbb{N}$-valued random variables with $P[Y_i = k] = \frac{1}{|a|}\sum_{j:\ |j|=k} a(0,j)$ $(k \geq 0)$, let N be a Poisson-distributed random variable with mean $|a|$, independent of the $(Y_i)_{i\geq 1}$, and let $(X_m)_{m\geq 1}$ be i.i.d. random variables with law $P[X_m \in \cdot\,] = P[\sum_{i=1}^N Y_i \in \cdot\,]$. Since the random walk ξ makes jumps whose sizes are distributed in the same way as the Y_i, and the number of jumps per unit of time is Poisson distributed with mean $|a|$, it follows that

$$P[|\xi_t| > \lambda t] \leq P\Big[\frac{1}{\lceil t\rceil}\sum_{m=1}^{\lceil t\rceil} X_m > \lambda\frac{t}{\lceil t\rceil}\Big] \qquad (t > 0), \tag{2.18}$$

where $\lceil t\rceil$ denotes t rounded up to the next integer. By our assumptions,

$$E\big[e^{\varepsilon X_m}\big] = E\big[e^{\varepsilon\sum_{i=1}^N Y_k}\big] = e^{-|a|}\sum_{n=0}^{\infty}\frac{|a|^n}{n!}E\big[e^{\varepsilon Y_1}\big]^n = e^{-|a|(1-E[e^{\varepsilon Y_1}])} < \infty, \tag{2.19}$$

for some $\varepsilon > 0$. Therefore, by [DZ98, Theorem 2.2.3 and Lemma 2.2.20], for each $R > 0$ there exists a $\lambda > 0$ and $K < \infty$ such that

$$P\Big[\frac{1}{n}\sum_{m=1}^{n} X_m > \lambda\Big] \leq Ke^{-nR} \qquad (n \geq 1). \tag{2.20}$$

Choosing λ such that (2.20) holds for some $R > |a| - \delta$ yields, by (2.18)

$$\lim_{t\to\infty} P\big[|\xi_t| > \lambda t\big]e^{(|a|-\delta)t} = 0. \tag{2.21}$$

Inserting this into (2.17) we find that the exponential growth rate $r = r(\Lambda, a, \delta)$ satisfies

$$r \leq \limsup_{t\to\infty}\frac{1}{t}\log|\{i \in \Lambda : |i| \leq \lambda t\}| = 0, \tag{2.22}$$

where we have used that Λ is subexponential. ∎

2.4 Duality and Campbell laws

Proof of Lemma 5 (a) This follows by writing

$$\begin{aligned}
\hat P^{\{0\}}_\gamma\big[A \cap \iota^{-1}\eta^{\{0\}}_\tau = \emptyset\big] &\overset{(1)}{=} E_\gamma\big[|\eta^{\{0\}}_\tau|\big]^{-1}\sum_i P_\gamma\big[i \in \eta^{\{0\}}_\tau,\ A \cap i^{-1}\eta^{\{0\}}_\tau = \emptyset\big]\\
&\overset{(2)}{=} E_\gamma\big[|\eta^{\{0\}}_\tau|\big]^{-1}\sum_i P_\gamma\big[0 \in \eta^{\{i^{-1}\}}_\tau,\ A \cap \eta^{\{i^{-1}\}}_\tau = \emptyset\big]\\
&\overset{(3)}{=} E_\gamma\big[|\eta^{\{0\}}_\tau|\big]^{-1}\sum_j \Big\{P_\gamma\big[(A\cup\{0\}) \cap \eta^{\{j\}}_\tau \neq \emptyset\big] - P_\gamma\big[A \cap \eta^{\{j\}}_\tau \neq \emptyset\big]\Big\}\\
&\overset{(4)}{=} E_\gamma\big[|\eta^{\dagger\,\{0\}}_\tau|\big]^{-1}\sum_j \Big\{P_\gamma\big[\eta^{\dagger\,A\cup\{0\}}_\tau \cap \{j\} \neq \emptyset\big] - P_\gamma\big[\eta^{\dagger\,A}_\tau \cap \{j\} \neq \emptyset\big]\Big\}\\
&\overset{(5)}{=} E_\gamma\big[|\eta^{\dagger\,\{0\}}_\tau|\big]^{-1}\Big\{E_\gamma\big[|\eta^{\dagger\,A\cup\{0\}}_\tau|\big] - E_\gamma\big[|\eta^{\dagger\,A}_\tau|\big]\Big\}\\
&\overset{(6)}{=} \pi^\dagger_\gamma(\{0\})^{-1}\big\{\pi^\dagger_\gamma(A\cup\{0\}) - \pi^\dagger_\gamma(A)\big\} \overset{(7)}{=} \overline{\pi}^\dagger_\gamma(A\cup\{0\}) - \overline{\pi}^\dagger_\gamma(A).
\end{aligned} \tag{2.23}$$

Here, in step (2) we have used shift invariance, in step (3) we have changed the summation order and used that $\{0 \in \eta^{\{j\}}_\tau,\ A \cap \eta^{\{j\}}_\tau = \emptyset\} = \{(A\cup\{0\}) \cap \eta^{\{j\}}_\tau \neq \emptyset\}\backslash\{A \cap \eta^{\{j\}}_\tau \neq \emptyset\}$, and in step (4) we have used duality (formula (1.6)) and formula (2.14). ∎

Proof of Lemma 5 (b) We have

$$
\begin{aligned}
P\big[A\cap\overline{\eta}_0=\emptyset\,\big|\,0\in\overline{\eta}_0\big] &\stackrel{(1)}{=} P\big[0\in\overline{\eta}_0\big]^{-1}P\big[0\in\overline{\eta}_0,\ A\cap\overline{\eta}_0=\emptyset\big]\\
&\stackrel{(2)}{=} P\big[\{0\}\cap\overline{\eta}_0\neq\emptyset\big]^{-1}\big\{P\big[(A\cup\{0\})\cap\overline{\eta}_0\neq\emptyset\big]-P\big[A\cap\overline{\eta}_0\neq\emptyset\big]\big\}\\
&\stackrel{(3)}{=} \rho^\dagger(\{0\})\big\{\rho^\dagger(A\cup\{0\})-\rho^\dagger(A)\big\}\stackrel{(4)}{=}\overline{\rho}^\dagger(A\cup\{0\})-\overline{\rho}^\dagger(A),
\end{aligned}
\tag{2.24}
$$

where in step (3) we have used (1.9). ∎

2.5 Harmonic functions

In this section we prove formulas (1.26) and (1.27), and Lemma 6.

Proof of (1.26) The shift invariance and monotonicity of ρ follow from the corresponding properties of the contact process. Since ρ is bounded, obviously $\rho\in\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$. Since η^A solves the martingale problem for G, for any $f\in\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, one has

$$
\int_0^t E[Gf(\eta^A_s)]\mathrm{d}s=E[f(\eta^A_t)]-f(A)\qquad(A\in\mathcal{P}_{\rm fin}(\Lambda)),
\tag{2.25}
$$

and therefore

$$
Gf(A)=\lim_{t\to0}t^{-1}\big\{E[f(\eta^A_t)]-f(A)\big\}\qquad(A\in\mathcal{P}_{\rm fin}(\Lambda)).
\tag{2.26}
$$

By the Markov property,

$$
\rho(\eta^A_t)=E\big[\eta^A_s\neq0\ \forall s\geq0\,\big|\,\eta^A_t\big]=E\big[\eta^A_s\neq0\ \forall s\geq0\,\big|\,\mathcal{F}^A_t\big],
\tag{2.27}
$$

where $(\mathcal{F}^A_t)_{t\geq0}$ denotes the filtration generated by η^A. It follows that $\rho(\eta^A_t)$ is a martingale, and therefore, by (2.26), $G\rho=0$. ∎

Proof of (1.27) The shift invariance and monotonicity of π_γ follow from the corresponding properties of the contact process. It follows from (1.3) that $\pi_\gamma(A)\leq\pi_\gamma(\{0\})|A|$, which shows that $\pi_\gamma\in\mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$. Moreover,

$$
\begin{aligned}
&t^{-1}\big\{E[\pi_\gamma(\eta^A_t)]-\pi_\gamma(A)\big\}\\
&\quad=t^{-1}\Big\{E_\gamma\big[|\eta^A_{t+\tau}|\big]-E_\gamma\big[|\eta^A_\tau|\big]\Big\}\\
&\quad=t^{-1}\Big\{\int_t^\infty E\big[|\eta^A_s|\big]\tfrac{1}{\gamma}e^{-(s-t)/\gamma}\,\mathrm{d}s-\int_0^\infty E\big[|\eta^A_s|\big]\tfrac{1}{\gamma}e^{-s/\gamma}\,\mathrm{d}s\Big\}\\
&\quad=t^{-1}(e^{t/\gamma}-1)\int_0^\infty E\big[|\eta^A_s|\big]\tfrac{1}{\gamma}e^{-s/\gamma}\,\mathrm{d}s-t^{-1}e^{t/\gamma}\int_0^t E\big[|\eta^A_s|\big]\tfrac{1}{\gamma}e^{s/\gamma}\,\mathrm{d}s\Big\}.
\end{aligned}
\tag{2.28}
$$

Letting $t\to0$, using (2.26), it follows that

$$
G\pi_\gamma(A)=\tfrac{1}{\gamma}\pi_\gamma(A)-\tfrac{1}{\gamma}|A|\qquad(A\in\mathcal{P}_{\rm fin}(\Lambda),\ \gamma>0),
\tag{2.29}
$$

as desired. ∎

Proof of Lemma 6 It follows from (1.22) that $\overline{\pi}_\gamma(A)\leq|A|$, which shows that the functions $(\overline{\pi}_\gamma)_{\gamma>0}$ are relatively compact, and each pointwise limit $\overline{\pi}_\infty$ along a sequence $\gamma_n\to\infty$ satisfies

$\overline{\pi}_\infty \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$. Since each $\overline{\pi}_{\gamma_n}$ is shift invariant an monotone, the same is true for $\overline{\pi}_\infty$. If $f_n, f \in \mathcal{S}(\mathcal{P}_{\rm fin}(\Lambda))$, $f_n \to f$ pointwise, and the f_n are uniformly bounded on sets of the form $\{A \in \mathcal{P}_{\rm fin}(\Lambda) : |A| \leq K\}$, then it is not hard to see that pointwise

$$\lim_{n\to\infty} Gf_n = Gf. \tag{2.30}$$

Applying this to the functions $\overline{\pi}_{\gamma_n}$, which satisfy the uniform bound $\overline{\pi}_{\gamma_n}(A) \leq |A|$, using (1.27), we find that

$$G\overline{\pi}_\infty(A) = \lim_{n\to\infty} \frac{\pi_{\gamma_n}(A) - |A|}{\gamma_n \pi_{\gamma_n}(\{0\})} = \lim_{n\to\infty} \Big(\frac{\overline{\pi}_{\gamma_n}(A)}{\gamma_n} - \frac{|A|}{\gamma_n \pi_{\gamma_n}(\{0\})}\Big) = 0 \qquad (A \in \mathcal{P}_{\rm fin}(\Lambda)), \tag{2.31}$$

where we have used that

$$\liminf_{n\to\infty} \pi_{\gamma_n}(\{0\}) = \liminf_{n\to\infty} E_{\gamma_n}\big[|\eta^{\{0\}}_\tau|\big] \geq \liminf_{n\to\infty} P_{\gamma_n}\big[\eta^{\{0\}}_\tau \neq \emptyset\big] > 0 \tag{2.32}$$

since we are assuming survival. ∎

2.6 Eventual domination of finite configurations

In this section we prove Lemma 8. We start with two preparatory lemmas.

Lemma 10 (Local creation of finite configurations) *For each $B \in \mathcal{P}_{\rm fin}(\Lambda)$ and $t > 0$, there exists a finite $\Delta \subset \Lambda$ and $j \in \Lambda$ such that*

$$\varepsilon := P\big[\eta^{\{0\}}_t \supset jB \ \textit{and}\ \eta^{\{0\}}_s \subset \Delta\ \forall 0 \leq s \leq t\big] > 0. \tag{2.33}$$

Proof It follows from assumption (1.1) (iii) that there exists a site $j^{-1} \in \Lambda$ with $P\big[\eta^{\{j^{-1}\}}_t \supset B\big] > 0$, and therefore $P\big[\eta^{\{0\}}_t \supset jB\big] > 0$. Since $\bigcup_{0\leq s\leq t} \eta^{\{0\}}_s$ is a.s. finite, we can choose a finite but large enough Δ such that (2.33) holds. ∎

Lemma 11 (Domination of finite configurations) *For each $B \in \mathcal{P}_{\rm fin}(\Lambda)$, $t > 0$, and $A_n \in \mathcal{P}_{\rm fin}(\Lambda)$ satisfying $\lim_{n\to\infty} |A_n| = \infty$, one has*

$$\lim_{n\to\infty} P[\exists i \in \Lambda \textit{ s.t. } \eta^{A_n}_t \geq iB] = 1. \tag{2.34}$$

Proof Let Δ, j, and ε be as in Lemma 10. We can find $\tilde{A}_n \subset A_n$ such that $|\tilde{A}_n| \to \infty$ as $n \to \infty$, and for fixed n, the sets $(k\Delta)_{k\in\tilde{A}_n}$ are disjoint. It follows that

$$\begin{aligned} &P[\exists i \in \Lambda \text{ s.t. } \eta^{A_n}_t \geq iB] \\ &\quad \geq 1 - \prod_{k\in\tilde{A}_n} \big(1 - P\big[\eta^{\{k\}}_t \supset kjB \text{ and } \eta^{\{k\}}_s \subset k\Delta\ \forall 0 \leq s \leq t\big]\big) \\ &\quad = 1 - (1-\varepsilon)^{|\tilde{A}_n|} \underset{n\to\infty}{\longrightarrow} 1, \end{aligned} \tag{2.35}$$

where we have used (2.33) and the fact that events concerning the graphical representation in disjoint parts of space are independent. ∎

Proof of Lemma 8 If $\delta = 0$, then obviously $\lim_{t\to\infty} |\eta^A_t| = \infty$ a.s. If $\delta > 0$, then it is easy to see that $\inf\{\rho(A) : |A| \leq M\} < 1$ for all $M < \infty$. Therefore, by (1.38),

$$\eta^A_t = \emptyset \text{ for some } t \geq 0 \quad \text{or} \quad |\eta^A_t| \underset{t\to\infty}{\longrightarrow} \infty \qquad \text{a.s.} \tag{2.36}$$

Fix $\emptyset \neq B \in \mathcal{P}_{\rm fin}(\Lambda)$ and set $\psi_t(A) := P[\exists i \in \Lambda \text{ s.t. } \eta^A_t \geq iB] \quad (A \in \mathcal{P}_{\rm fin}(\Lambda),\ t \geq 0)$. Then, for each $t > 0$,

$$\lim_{T\to\infty} P[\exists i \in \Lambda \text{ s.t. } \eta^A_T \supset iB] = \lim_{T\to\infty} E[\psi_t(\eta^A_{T-t})] = \rho(A), \tag{2.37}$$

where we have used Lemma 11 and (2.36). ∎

2.7 Generalization to arbitrary initial states

In this section, we show how the proof of Theorem 2 (a) must be adapted to cover general initial states $\emptyset \neq A \in \mathcal{P}(\Lambda)$.

Proof of Theorem 2 (a) for general initial states For $A, B \in \mathcal{P}_{\rm fin}(\Lambda)$ with $A \neq \emptyset$, we observe that $i \in BA^{-1} \Leftrightarrow B \cap iA \neq \emptyset$, and therefore

$$|BA^{-1}| = \sum_i 1_{\{B \cap iA \neq \emptyset\}}. \tag{2.38}$$

We define

$$\pi_{A,\gamma}(B) := E_\gamma\big[|\eta_\tau A^{-1}|\big] \quad \text{and} \quad \overline{\pi}_{A,\gamma}(B) := \frac{\pi_{A,\gamma}(B)}{\pi_{A,\gamma}(\{0\})}, \tag{2.39}$$

and let $\pi^\dagger_{A,\gamma}$ and $\overline{\pi}^\dagger_{A,\gamma}$ denote the analogues of $\pi_{A,\gamma}$ and $\overline{\pi}_{A,\gamma}$ for the $(\Lambda, a^\dagger, \delta)$-contact process. Generalizing the proof of Lemma 5 (a), we find that

$$\hat{P}^A_\gamma\big[B \cap \iota^{-1}\eta^A_\tau = \emptyset\big] = \overline{\pi}^\dagger_{A,\gamma}(B \cup \{0\}) - \overline{\pi}^\dagger_{A,\gamma}(B). \tag{2.40}$$

Since $|B| \leq |BA^{-1}| \leq |A|\,|B|$ for any $A, B \in \mathcal{P}_{\rm fin}(\Lambda)$ with $A \neq \emptyset$, it follows that

$$\lim_{t\to\infty} \tfrac{1}{t} \log E\big[|\eta^B_t A^{-1}|\big] = r \qquad (\emptyset \neq B \in \mathcal{P}_{\rm fin}(\Lambda)), \tag{2.41}$$

where r is the exponential growth rate from Proposition 1. The proofs of (1.27) and Lemma 6 now carry over to the functions $(\overline{\pi}_{A,\gamma})_{\gamma>0}$ without a change, and therefore the arguments in Section 1.4 show that Theorem 2 (a) holds for general initial states $\emptyset \neq A \in \mathcal{P}(\Lambda)$. ∎

3 Proofs of further results

Recall that $\omega = (\omega^{\rm r}, \omega^{\rm i})$, the pair of Poisson point processes used in the graphical representation, is a random variable taking values in the space $\Gamma := \mathcal{P}_{\rm loc}(\Lambda \times \mathbb{R}) \times \mathcal{P}_{\rm loc}(\Lambda \times \Lambda \times \mathbb{R})$, where $\mathcal{P}_{\rm loc}(\Lambda \times \mathbb{R})$ and $\mathcal{P}_{\rm loc}(\Lambda \times \Lambda \times \mathbb{R})$ denote the spaces of locally finite subsets of $\Lambda \times \mathbb{R}$ and $\Lambda \times \Lambda \times \mathbb{R}$, respectively. These spaces can in a natural way be identified with subspaces of the spaces of locally finite counting measures on $\Lambda \times \mathbb{R}$ and $\Lambda \times \Lambda \times \mathbb{R}$, respectively. Using this identification, we equip $\mathcal{P}_{\rm loc}(\Lambda \times \mathbb{R})$ and $\mathcal{P}_{\rm loc}(\Lambda \times \Lambda \times \mathbb{R})$ with the vague topology. We equip Γ with the product topology and let $\mathcal{B}(\Gamma)$ denote the associated Borel-σ-field.

We equip $\Lambda \times \mathbb{R}$ and $\Lambda \times \Lambda \times \mathbb{R}$ with a group structure by putting $(i,s)(j,t) := (ij, s+t)$ and $(i,j,s)(k,l,t) := (ik, jl, s+t)$, respectively. In line with our earlier notation, for any subset $\alpha \subset \Lambda \times \mathbb{R}$, we write $(i,s)\alpha := \{(ij, s+t) : (j,t) \in \alpha\}$. For $\beta \subset \Lambda \times \Lambda \times \mathbb{R}$, we define $(i,j,s)\beta$ analogously. We define shift operators $\theta_{i,t} : \Gamma \to \Gamma$ by

$$\theta_{i,t}(\alpha, \beta) := ((i,t)\alpha, (i,i,t)\beta) \tag{3.1}$$

$(i \in \Lambda,\ t \in \mathbb{R},\ (\alpha,\beta) \in \Gamma)$. Thus, $\theta_{i,t}$ shifts a graphical representation by left-multiplication with i and increasing all times by t.

3.1 Conditioning and size-biasing

In this section, we prove that size-biasing and conditioning on survival are asymptotically equivalent in a 'local' sense. Let

$$\omega_t := \big(\omega^{\rm r} \cap \Lambda \times (-\infty, t]\,,\, \omega^{\rm i} \cap \Lambda \times \Lambda \times (-\infty, t]\big) \tag{3.2}$$

denote the restriction of the Poisson point processes used in the graphical representation to the time interval $(-\infty, t]$.

Proposition 12 (Conditioning and size-biasing) *Assume that the (Λ, a, δ)-contact process survives and that the exponential growth rate satisfies $r(\Lambda, a, \delta) \le 0$. Then, for any $\emptyset \ne A \in \mathcal{P}_{\rm fin}(\Lambda)$,*

$$\hat P^A_\gamma\big[\tau \ge t\big] \underset{\gamma\to\infty}{\longrightarrow} 1 \qquad (t > 0), \tag{3.3}$$

and

$$\hat P^A_\gamma\big[\omega_t \in \mathcal{A}\big] \underset{\gamma\to\infty}{\longrightarrow} P_\gamma\big[\omega_t \in \mathcal{A}\,\big|\, A \times \{0\} \rightsquigarrow \infty\big] \qquad (t \in \mathbb{R},\ \mathcal{A} \in \mathcal{B}(\Gamma)). \tag{3.4}$$

Proof It suffices to prove the claims for $t > 0$. Formula (3.3) follows by writing

$$\begin{aligned} \hat P^A_\gamma\big[\tau < t\big] &:= \frac{\int_0^t E[|\eta^A_s|]\frac{1}{\gamma}e^{-s/\gamma}\mathrm{d}s}{\int_0^\infty E[|\eta^A_s|]\frac{1}{\gamma}e^{-s/\gamma}\mathrm{d}s} \\ &\le \frac{\frac{1}{\gamma}\int_0^t E[|\eta^A_s|]\mathrm{d}s}{\int_0^\infty P[\eta^A_s \ne \emptyset]\frac{1}{\gamma}e^{-s/\gamma}\mathrm{d}s} \le \frac{\int_0^t E[|\eta^A_s|]\mathrm{d}s}{\gamma\rho(A)} \underset{\gamma\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.5}$$

To prove (3.4), write

$$\hat P^A_\gamma\big[\omega_t \in \mathcal{A}\big] = \hat P^A_\gamma\big[\omega_t \in \mathcal{A}\,\big|\, \tau \ge t\big]\,\hat P^A_\gamma\big[\tau \ge t\big] + \hat P^A_\gamma\big[\omega_t \in \mathcal{A},\ \tau < t\big], \tag{3.6}$$

and observe that

$$\begin{aligned} \hat P^A_\gamma\big[\omega_t \in \mathcal{A}\,\big|\, \tau \ge t\big] &= \frac{E_\gamma[|\eta^A_{t+\tau}|1_{\{\omega_t\in\mathcal{A}\}}]}{E_\gamma[|\eta^A_{t+\tau}|]} = \frac{E[E_\gamma[|\eta^A_{t+\tau}|]\,|\,\omega_t]1_{\{\omega_t\in\mathcal{A}\}}]}{E[E_\gamma[|\eta^A_{t+\tau}|\,|\,\omega_t]]} \\ &= \frac{E[\pi_\gamma(\eta^A_t)1_{\{\omega_t\in\mathcal{A}\}}]}{E[\pi_\gamma(\eta^A_t)]} = \frac{E[\overline{\pi}_\gamma(\eta^A_t)1_{\{\omega_t\in\mathcal{A}\}}]}{E[\overline{\pi}_\gamma(\eta^A_t)]} \underset{\gamma\to\infty}{\longrightarrow} \frac{E[\overline{\rho}(\eta^A_t)1_{\{\omega_t\in\mathcal{A}\}}]}{E[\overline{\rho}(\eta^A_t)]}, \end{aligned} \tag{3.7}$$

where we have used that $\overline{\pi}_\gamma \to \overline{\rho}$ pointwise as $\gamma \to \infty$ by Lemma 6 and Proposition 7, and bounded convergence, using the uniform bound $\pi_\gamma \le |\cdot|$. Since

$$\begin{aligned} \frac{E[\overline{\rho}(\eta^A_t)1_{\{\omega_t\in\mathcal{A}\}}]}{E[\overline{\rho}(\eta^A_t)]} &= \frac{E[\rho(\eta^A_t)1_{\{\omega_t\in\mathcal{A}\}}]}{E[\rho(\eta^A_t)]} \\ &= \frac{E[P[A\times\{0\} \rightsquigarrow \infty\,|\,\omega_t]1_{\{\omega_t\in\mathcal{A}\}}]}{E[P[A\times\{0\} \rightsquigarrow \infty\,|\,\omega_t]]} = P\big[\omega_t \in \mathcal{A}\,\big|\, A\times\{0\} \rightsquigarrow \infty\big], \end{aligned} \tag{3.8}$$

formula (3.4) follows from (3.3), (3.6), and (3.7). ∎

3.2 Coupling to the maximal process

In this section we prove Theorem 2 (b) and Proposition 3. In analogy with (1.14), we put

$$\hat{P}^{\dagger A}_\gamma(\{i\}\times\mathcal{A}) := \frac{P_\gamma\big(\{i\in\eta^{\dagger A}_\tau\}\cap\mathcal{A}\big)}{E_\gamma\big[|\eta^{\dagger A}_\tau|\big]} \qquad (i\in\Lambda,\ \mathcal{A}\in\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)), \tag{3.9}$$

which is well-defined for any $\emptyset\neq A\in\mathcal{P}_{\rm fin}(\Lambda)$ and $\gamma>0$, provided that $r(\Lambda,a,\delta)\leq 0$. Recall that $\eta^{\dagger A}_\tau=\{i\in\Lambda:(i,-\tau)\rightsquigarrow A\times\{0\}\}$. We can view $\eta^{\dagger A}_\tau$ as the set of all 'ancestors' at time $-\tau$ of the set A at time 0. As before, let $\iota:\Lambda\times\Omega\times\mathbb{R}_+\to\Lambda$ denote the projection on Λ. Then, under the law $\hat{P}^{\dagger A}_\gamma$, the random variable ι describes the position of a 'typical' ancestor of A.

In the next lemma, we shift the graphical representation ω in such a way that the 'typical' infected site and time (ι,τ), chosen with respect to $\hat{P}^{\{0\}}_\gamma$, are mapped to the point $(0,0)$. Note that under such a shift, the origin is mapped to ι^{-1}. Thus, the next lemma can be described by saying that if we start the contact process with only the origin infected, then seen from a typical infected site, the origin is a typical ancestor.

Lemma 13 (Origin seen from a typical infected site) *Assume that $r(\Lambda,a,\delta)\leq 0$. Then*

$$\hat{P}^{\{0\}}_\gamma\big[(\iota^{-1},\theta_{\iota^{-1},-\tau}\omega,\tau)\in\cdot\,\big]=\hat{P}^{\dagger\{0\}}_\gamma\big[(\iota,\omega,\tau)\in\cdot\,\big]. \tag{3.10}$$

Proof Let us write $(i,s)\overset{\omega}{\rightsquigarrow}(j,t)$ when (i,s) can be connected to (j,t) along a path in the graphical representation ω. Then

$$\begin{aligned}
&\hat{P}^{\{0\}}_\gamma\big[\iota^{-1}=j,\ (\theta_{\iota^{-1},-\tau}\omega,\tau)\in\mathcal{A}\big]=\hat{P}^{\{0\}}_\gamma\big[\iota=j^{-1},\ (\theta_{j,-\tau}\omega,\tau)\in\mathcal{A}\big]\\
&\quad=E_\gamma\big[|\eta^{\{0\}}_\tau|\big]^{-1}P_\gamma\big[j^{-1}\in\eta^{\{0\}}_\tau,\ (\theta_{j,-\tau}\omega,\tau)\in\mathcal{A}\big]\\
&\quad=E_\gamma\big[|\eta^{\{0\}}_\tau|\big]^{-1}P_\gamma\big[(0,0)\overset{\omega}{\rightsquigarrow}(j^{-1},\tau),\ (\theta_{j,-\tau}\omega,\tau)\in\mathcal{A}\big]\\
&\quad=E_\gamma\big[|\eta^{\{0\}}_\tau|\big]^{-1}P_\gamma\big[(j,-\tau)\overset{\theta_{j,-\tau}\omega}{\rightsquigarrow}(0,0),\ (\theta_{j,-\tau}\omega,\tau)\in\mathcal{A}\big]\\
&\quad=E_\gamma\big[|\eta^{\{0\}}_\tau|\big]^{-1}P_\gamma\big[(j,-\tau)\overset{\omega}{\rightsquigarrow}(0,0),\ (\omega,\tau)\in\mathcal{A}\big]\\
&\quad=E_\gamma\big[|\eta^{\dagger\{0\}}_\tau|\big]^{-1}P_\gamma\big[j\in\eta^{\dagger\{0\}}_\tau,\ (\omega,\tau)\in\mathcal{A}\big]=\hat{P}^{\dagger\{0\}}_\gamma\big[\iota=j,\ (\omega,\tau)\in\mathcal{A}\big],
\end{aligned}\tag{3.11}$$

where we have used (2.14). ∎

In order to prove Theorem 2 (b), we need two more lemmas.

Lemma 14 (Large populations) *Assume that the (Λ,a,δ)-contact process survives and that the exponential growth rate satisfies $r(\Lambda,a,\delta)\leq 0$. Then, for any $\emptyset\neq A\in\mathcal{P}_{\rm fin}(\Lambda)$,*

$$\hat{P}^A_\gamma\big[|\eta^A_\tau|\geq K\big]\underset{\gamma\to\infty}{\longrightarrow}1 \qquad (K<\infty). \tag{3.12}$$

Proof This follows by writing

$$\begin{aligned}
\hat{P}^A_\gamma\big[|\eta^A_\tau|\geq K\big]&=\frac{E_\gamma\big[|\eta^A_\tau|1_{\{|\eta^A_\tau|\geq K\}}\big]}{E_\gamma\big[|\eta^A_\tau|\big]}\\
&=\frac{E_\gamma\big[|\eta^A_\tau|1_{\{|\eta^A_\tau|\geq K\}}\,\big|\,\eta^A_\tau\neq\emptyset\big]}{E_\gamma\big[|\eta^A_\tau|\,\big|\,\eta^A_\tau\neq\emptyset\big]}\geq E_\gamma\big[1_{\{|\eta^A_\tau|\geq K\}}\,\big|\,\eta^A_\tau\neq\emptyset\big]\underset{\gamma\to\infty}{\longrightarrow}1,
\end{aligned}\tag{3.13}$$

where we have used (2.36), and the fact that $|\eta^A_\tau|$ and $1_{\{\eta^A_\tau \geq K\}}$ are positively correlated since the functions $z \mapsto z$ and $z \mapsto 1_{\{z\geq K\}}$ are nondecreasing. ∎

Recall that in the proof (in Section 1.4) of Proposition 7, sequences $A_n \in \mathcal{P}_{\rm fin}(\Lambda)$ such that $\rho(A_n) \to 1$ played an important role. Although we did not need this fact there, the next lemma shows that for $\delta > 0$, actually $\rho(A_n) \to 1$ if and only if $|A_n| \to \infty$.

Lemma 15 (High survival probabilities) *Assume that the* (Λ, a, δ)*-contact process survives, and* $A_n \in \mathcal{P}_{\rm fin}(\Lambda)$*. Then* $|A_n| \to \infty$ *implies* $\rho(A_n) \to 1$*.*

Proof By (1.38) there exist $B_m \in \mathcal{P}_{\rm fin}(\Lambda)$ with $\rho(B_m) \to 1$. Now if $A_n \in \mathcal{P}_{\rm fin}(\Lambda)$ satisfy $|A_n| \to \infty$, then by Lemma 11,

$$\begin{aligned}
&\liminf_{n\to\infty} \rho(A_n) \\
&\quad \geq \liminf_{n\to\infty} P\big[\eta^{A_n}_s \neq \emptyset \ \forall s \geq t \,\big|\, \exists i \in \Lambda \text{ s.t. } \eta^{A_n}_t \geq iB_m\big] P\big[\exists i \in \Lambda \text{ s.t. } \eta^{A_n}_t \geq iB_m\big] \\
&\quad \geq \rho(B_m),
\end{aligned} \tag{3.14}$$

for each $t > 0$ and m. Letting $m \to \infty$ yields the claim. ∎

We now first prove Theorem 2 (b) in the case $A = \{0\}$, and then indicate how the arguments may be generalised to $\emptyset \neq A \in \mathcal{P}_{\rm fin}(\Lambda)$. We will obtain Proposition 3 as a corollary to our proofs in the case $A = \{0\}$.

Proof of Theorem 2 (b) in the case $A = \{0\}$ By Lemma 13, we must show that for fixed $\Delta \in \mathcal{P}_{\rm fin}$, the sets $\{j \in \Delta : (\iota, -\tau) \rightsquigarrow (j, 0)\}$, $\{j \in \Delta : -\infty \rightsquigarrow (j, 0)\}$, and $\{j \in \Delta : \Lambda \times \{-\tau\} \rightsquigarrow (j, 0)\}$ are asymptotically equal under the laws $\hat{P}^{\dagger\,\{0\}}_\gamma$ as $\gamma \to \infty$. It suffices to show that for any $j \in \Lambda$,

$$\hat{P}^{\dagger\,\{0\}}_\gamma \big[(\iota, -\tau) \rightsquigarrow (j, 0) \,\big|\, \Lambda \times \{-\tau\} \rightsquigarrow (j, 0)\big] \underset{\gamma\to\infty}{\longrightarrow} 1. \tag{3.15}$$

and

$$\hat{P}^{\dagger\,\{0\}}_\gamma \big[-\infty \rightsquigarrow (j, 0) \,\big|\, \Lambda \times \{-\tau\} \rightsquigarrow (j, 0)\big] \underset{\gamma\to\infty}{\longrightarrow} 1. \tag{3.16}$$

Reversing the direction of time and interchanging the roles of η and $\eta^\dagger$, this then yields Proposition 3 as a corollary.

For any $t > 0$, by Proposition 12,

$$\begin{aligned}
&\hat{P}^{\dagger\,\{0\}}_\gamma \big[\Lambda \times \{-\tau\} \rightsquigarrow (j, 0)\big] = \hat{P}^{\dagger\,\{0\}}_\gamma \big[\eta^{\dagger\,\{j\}}_\tau \neq \emptyset\big] \\
&\quad \leq \hat{P}^{\dagger\,\{0\}}_\gamma \big[\eta^{\dagger\,\{j\}}_t \neq \emptyset\big] + \hat{P}^{\dagger\,\{0\}}_\gamma \big[\tau < t\big] \underset{\gamma\to\infty}{\longrightarrow} P\big[\eta^{\dagger\,\{j\}}_t \neq \emptyset \,\big|\, -\infty \rightsquigarrow (0, 0)\big].
\end{aligned} \tag{3.17}$$

Letting $t \to \infty$ yields

$$\limsup_{\gamma\to\infty} \hat{P}^{\dagger\,\{0\}}_\gamma \big[\Lambda \times \{-\tau\} \rightsquigarrow (j, 0)\big] \leq P\big[-\infty \rightsquigarrow (j, 0) \,\big|\, -\infty \rightsquigarrow (0, 0)\big] =: \phi(j). \tag{3.18}$$

By Lemma 13 and Theorem 2 (a),

$$\lim_{\gamma\to\infty} \hat{P}^{\dagger\,\{0\}}_\gamma \big[(\iota, -\tau) \rightsquigarrow (j, 0)\big] = \lim_{\gamma\to\infty} \hat{P}^{\{0\}}_\gamma \big[j \in \iota^{-1}\eta^{\{0\}}_\tau\big] = P\big[j \in \overline{\eta}_0 \,\big|\, 0 \in \overline{\eta}_0\big] = \phi(j). \tag{3.19}$$

Combining (3.18) and (3.19) we arrive at (3.15).

Since conditional on $\eta_\tau^{\dagger\,\{0\}}$, the typical site ι is chosen with equal probabilities from the sites in $\eta_\tau^{\dagger\,\{0\}}$,

$$\hat{P}_\gamma^{\dagger\,\{0\}}\big[(\iota,-\tau)\rightsquigarrow(j,0)\,\big|\,\Lambda\times\{-\tau\}\rightsquigarrow(j,0)\big]=\hat{E}_\gamma^{\dagger\,\{0\}}\Big[\frac{|\eta_\tau^{\dagger\,\{j\}}\cap\eta_\tau^{\dagger\,\{0\}}|}{|\eta_\tau^{\dagger\,\{0\}}|}\,\Big|\,\Lambda\times\{-\tau\}\rightsquigarrow(j,0)\Big]. \tag{3.20}$$

Therefore, (3.15) and Lemma 14 imply that

$$\lim_{\gamma\to\infty}\hat{P}_\gamma^{\dagger\,\{0\}}\big[|\eta_\tau^{\dagger\,\{j\}}|\geq K\,\big|\,\Lambda\times\{-\tau\}\rightsquigarrow(j,0)\big]=1\qquad(K<\infty), \tag{3.21}$$

which by Lemma 15 implies (3.16). ∎

Generalization to arbitrary initial states In analogy with (3.9), we define, for any $\emptyset\neq A,B\in\mathcal{P}_{\rm fin}(\Lambda)$,

$$\hat{P}_{A,\gamma}^{\dagger\,B}(\{i\}\times\mathcal{A}):=\frac{P_\gamma\big(\{\eta_\tau^{\dagger\,B}\cap iA\neq\emptyset\}\cap\mathcal{A}\big)}{E_\gamma\big[|\eta_\tau^{\dagger\,B}A^{-1}|\big]}\qquad(i\in\Lambda,\ \mathcal{A}\in\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)). \tag{3.22}$$

Note that this is a probability measure by (2.38). As before, let $\iota:\Lambda\times\Omega\times\mathbb{R}_+\to\Lambda$ denote the projection on Λ. Then, under the law $\hat{P}_{A,\gamma}^{\dagger\,B}$, the random variable ι describes a 'typical' site such that $\iota A\times\{-\tau\}\rightsquigarrow B\times\{0\}$. By an obvious analogue of Lemma 13, we must prove the following generalisations of (3.15) and (3.16):

$$\begin{array}{ll}\text{(i)} & \hat{P}_{A,\gamma}^{\dagger\,\{0\}}\big[\iota A\times\{-\tau\}\rightsquigarrow(j,0)\,\big|\,\Lambda\times\{-\tau\}\rightsquigarrow(j,0)\big]\underset{\gamma\to\infty}{\longrightarrow}1,\\ \text{(ii)} & \quad\hat{P}_{A,\gamma}^{\dagger\,\{0\}}\big[-\infty\rightsquigarrow(j,0)\,\big|\,\Lambda\times\{-\tau\}\rightsquigarrow(j,0)\big]\underset{\gamma\to\infty}{\longrightarrow}1.\end{array} \tag{3.23}$$

Define a measure $\tilde{P}_{A,\gamma}^{\dagger\,\{0\}}$ on $\Lambda\times\Lambda\times\Omega\times\mathbb{R}_+$ by

$$\tilde{P}_{A,\gamma}^{\dagger\,\{0\}}(\{k\}\times\{i\}\times\mathcal{A}):=\hat{E}_{A,\gamma}^{\dagger\,\{0\}}\Big[|\eta_\tau^{\dagger\,\{0\}}\cap iA|^{-1}1_{\{k\,\in\,\eta_\tau^{\dagger\,\{0\}}\,\cap\,iA\}}1_{\{i\}\,\times\,\mathcal{A}}\Big]. \tag{3.24}$$

Let $\kappa,\iota:\Lambda\times\Lambda\times\Omega\times\mathbb{R}_+\to\Lambda$ denote the projections on the first and second coordinate, respectively. Then, under the law $\tilde{P}_{A,\gamma}^{\dagger\,\{0\}}$, the random variable κ describes a site chosen with equal probabilities from $\eta_\tau^{\dagger\,\{0\}}\cap\iota A$. Therefore, in order to prove (3.23), it suffices to prove:

$$\begin{array}{ll}\text{(i)} & \tilde{P}_{A,\gamma}^{\dagger\,\{0\}}\big[(\kappa,-\tau)\rightsquigarrow(j,0)\,\big|\,\Lambda\times\{-\tau\}\rightsquigarrow(j,0)\big]\underset{\gamma\to\infty}{\longrightarrow}1,\\ \text{(ii)} & \quad\tilde{P}_{A,\gamma}^{\dagger\,\{0\}}\big[-\infty\rightsquigarrow(j,0)\,\big|\,\Lambda\times\{-\tau\}\rightsquigarrow(j,0)\big]\underset{\gamma\to\infty}{\longrightarrow}1.\end{array} \tag{3.25}$$

We claim that $\tilde{P}_{A,\gamma}^{\dagger\,\{0\}}\big[(\kappa,\omega,\tau)\in\cdot\,\big]$ has a density with respect to $\hat{P}_\gamma^{\dagger\,\{0\}}\big[(\iota,\omega,\tau)\in\cdot\,\big]$ that is uniformly bounded away from 0 and ∞, and therefore (3.25) follows from (3.15) and (3.16). Indeed, by (3.22) and (3.24),

$$\begin{aligned}&\tilde{P}_{A,\gamma}^{\dagger\,\{0\}}(\{k\}\times\Lambda\times\mathcal{A})\\ &\quad=E_\gamma\big[|\eta_\tau^{\dagger\,\{0\}}A^{-1}|\big]^{-1}\sum_i E_\gamma\big[|\eta_\tau^{\dagger\,\{0\}}\cap iA|^{-1}1_{\{k\,\in\,\eta_\tau^{\dagger\,\{0\}}\,\cap\,iA\}}1_{\{\eta_\tau^{\dagger\,\{0\}}\,\cap\,iA\,\neq\,\emptyset\}}1_{\mathcal{A}}\big]\\ &\quad=ZE_\gamma\big[|\eta_\tau^{\dagger\,\{0\}}|\big]^{-1}E_\gamma\big[F1_{\{k\,\in\,\eta_\tau^{\dagger\,\{0\}}\}}1_{\mathcal{A}}\big]=\hat{E}_\gamma^{\dagger\,\{0\}}\big[ZF(k)1_{\{k\}\,\times\,\mathcal{A}}\big],\end{aligned} \tag{3.26}$$

where $Z := E_\gamma[|\eta_\tau^{\dagger\{0\}}|]/E_\gamma[|\eta_\tau^{\dagger\{0\}}A^{-1}|]$ satisfies $|A|^{-1} \leq Z \leq 1$ and

$$F(k) := \sum_i |\eta_\tau^{\dagger\{0\}} \cap iA|^{-1} 1_{\{k \in iA\}} = \sum_{i \in kA^{-1}} |\eta_\tau^{\dagger\{0\}} \cap iA|^{-1} \tag{3.27}$$

satisfies $1 \leq F(k) \leq |A|$. ∎

3.3 Coupling of one-dimensional processes

Proof of Lemma 4 For any point (i, s) such that $(i, s) \rightsquigarrow \infty$, set

$$r_{s,t}(i) := \max\{j \in \mathbb{Z} : (i, s) \rightsquigarrow (j, t) \rightsquigarrow \infty\} \qquad (t \geq s). \tag{3.28}$$

Then $(r_{s,t}(i))_{t \geq s}$ is the right-most path to infinity starting at (i, s). By symmetry and the nearest-neighbor property, it suffices to show that for any (i, s) and (j, s) such that $(i, s) \rightsquigarrow \infty$ and $(j, s) \rightsquigarrow \infty$, there exists a $t \geq s$ such that $r_{s,t}(i) = r_{s,t}(j)$. Imagine that this is not the case. Then, for any $i \in \mathbb{Z}$ and $s \leq t$, the maximum

$$R_{s,t}(i) := \max\{j \in \mathbb{Z} : r_{t,u}(j) = r_{s,u}(i) \text{ for some } u \geq t\} \tag{3.29}$$

exists. Set

$$\chi_s := \{i \in \mathbb{Z} : R_{s,s}(i) = i\} \qquad (s \in \mathbb{R}). \tag{3.30}$$

It is not hard to see that $R_{s,t}$ maps $\mathbb{Z}$ into χ_t and that $R_{s,t} : \chi_s \to \chi_t$ is one-to-one. We claim that $R_{s,t} : \chi_s \to \chi_t$ is with positive probability not surjective if $s < t$. Indeed, since we are assuming that $\delta > 0$ or $a(0, 1) \wedge a(1, 0) > 0$, it is easy to see that with positive probability there exist $i, j, k \in \chi_t$ with $i < j < k$ such that

$$\max\{i' \in \mathbb{Z} : (0, s) \rightsquigarrow (i', t)\} = i \quad \text{and} \quad \max\{k' \in \mathbb{Z} : (1, s) \rightsquigarrow (k', t)\} = k. \tag{3.31}$$

It follows that $R_{s,t}(0) = i$ and $R_{s,t}(1) = k$, and therefore, since $R_{s,t}$ is monotone, there is no $n \in \mathbb{Z}$ with $R_{s,t}(n) = j$.

This 'obviously' violates stationarity. More formally, this follows from the mass transport principle (see [Hag97] or [BLPS99, Section 3]). Fix $s < t$ and define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by

$$f(i, j) := P[i \in \chi_s, j \in \chi_t,\ j = R_{s,t}(i)]. \tag{3.32}$$

Then

$$\begin{aligned} \sum_j f(0, j) &= P[0 \in \chi_s, \exists j \in \chi_t \text{ s.t. } j = R_{s,t}(0)] \\ &= P[0 \in \chi_s] > P[\exists i \in \chi_s \text{ s.t. } 0 \in \chi_t,\ 0 = R_{s,t}(i)] = \sum_i f(i, 0). \end{aligned} \tag{3.33}$$

Since $\sum_j f(0, j) = \sum_j f(-j, 0) = \sum_i f(i, 0)$, we arrive at a contradiction. ∎

3.4 Survival on finitely generated groups

In this section we prove:

Lemma 16 (Survival for low recovery rates) *If Λ is finitely generated, then $\delta_c > 0$.*

Proof Let Δ be a finite generating set for Λ. Since $\{i : a(0,i) > 0\}$ generates Λ, there exists a finite subset $A \subset \{i : a(0,i) > 0\}$ that generates Δ, and thereby all of Λ. Therefore, we can find $i_0, i_1, \ldots \in \Lambda$, all different, such that $\inf_{k \geq 0} a(i_k, i_{k+1}) > 0$. We will use comparison to oriented site percolation to show that $P((i_0, 0) \rightsquigarrow \infty) > 0$ if δ is sufficiently small. Fix $T > 0$. Call a point (n,m) with $n, m \in \mathbb{N}^2$ good if in the grapical representation, in the time interval $[Tm, T(m+1))$, there is an arrow from i_n to i_{n+1} and there are no recoveries in i_n and i_{n+1}. By choosing T large enough and δ small enough, the probability that a point is good can be made arbitrarily high, uniformly in n. If this probability is larger than the critical parameter for independent 2-dimensional oriented site percolation, then with positive probability there is an upward path along good points, and therefore the contact process survives. ∎

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