# An introduction to free independence

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## 1 C∗-Algebras

By definition, an *algebra* is a linear space $\mathcal{A}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ on which there is defined a product $\mathcal{A}^2 \ni (A,B) \mapsto AB \in \mathcal{A}$ such that

$$\begin{array}{rll} \text{(i)} & (AB)C = A(BC) & (A,B,C \in \mathcal{A}), \\ \text{(ii)} & A(bB + cC) = bAB + cAC & (A,B,C \in \mathcal{A},\ b,c \in \mathbb{K}), \\ \text{(iii)} & (aA + bB)C = aAC + bBC & (A,B,C \in \mathcal{A},\ a,b \in \mathbb{K}) \end{array}$$

Often, it is assumed that $\mathcal{A}$ contains a (necessarily unique) element 1 such that

$$\text{(iv)} \quad 1A = A = A1 \qquad (A \in \mathcal{A}).$$

An algebra is *abelian* if

$$AB = BA \qquad (A,B \in \mathcal{A}).$$

An *adjoint operation* is a map $A \mapsto A^*$ such that

$$\begin{array}{rll} \text{(v)} & (A^*)^* = A & (A \in \mathcal{A}), \\ \text{(vi)} & (aA + bB)^* = \overline{a}A^* + \overline{b}B^* & (A,B \in \mathcal{A},\ a,b \in \mathbb{C}), \\ \text{(vii)} & (AB)^* = B^*A^* & (A,B \in \mathcal{A}). \end{array}$$

In what follows, we reverse the term $*$*-algebra* for an algebra over $\mathbb{C}$ that is equipped with an adjoint operation such that (i)–(vii) hold. A $C*$*-algebra* is a $*$-algebra equipped with a norm $\|\cdot\|$ such that

$$\begin{array}{rl} \text{(viii)} & \mathcal{A} \text{ is complete in the norm } \|\cdot\|, \\ \text{(ix)} & \|AB\| \leq \|A\|\|B\| \qquad (A,B \in \mathcal{A}), \\ \text{(x)} & \|A^*A\| = \|A\|^2. \end{array}$$

A *representation* of a $C*$-algebra is a Hilbert space $\mathcal{H}$ together with a continuous map $\mathcal{A} \times \mathcal{H} \ni (A,\phi) \mapsto A\phi \in \mathcal{H}$ such that

$$\begin{array}{rll} 1. & (AB)\phi = A(B\phi) & (A,B \in \mathcal{A},\ \phi \in \mathcal{H}), \\ 2. & A(b\phi + c\psi) = bA\phi + cA\psi & (A \in \mathcal{A},\ \phi,\psi \in \mathcal{H},\ b,c \in \mathbb{C}), \\ 3. & (aA + bB)\phi = aA\phi + bB\phi & (A,B \in \mathcal{A},\ \phi \in \mathcal{H},\ a,b \in \mathbb{K}), \\ 4. & \langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle & (A \in \mathcal{A},\ \phi,\psi \in \mathcal{H}), \\ 5. & \|A\| = \sup_{\|\phi\| \leq 1} \|A\phi\| & (A \in \mathcal{A}), \\ 6. & 1\phi = \phi & (\phi \in \mathcal{H}). \end{array}$$

Let $\mathcal{L}(\mathcal{H})$ denote the space of bounded linear operators $L : \mathcal{H} \to \mathcal{H}$, equipped with the usual adjoint operation $L \mapsto L^*$ and the operator norm $\|A\| := \sup_{\|x\| \leq 1} \|Ax\|$. If $\mathcal{H}$ is a representation of $\mathcal{A}$, then setting

$$\ell(A)\phi := A\phi \qquad (A \in \mathcal{A},\ \phi \in \mathcal{H})$$

defines a linear map $\ell : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that $\ell(1) = 1$, $\ell(AB) = \ell(A)\ell(B)$, $\ell(A^*) = \ell(A)^*$, and $\|\ell(A)\| = \|A\|$. The image $\ell(\mathcal{A}) := \{\ell(A) : A \in \mathcal{A}\}$ of $\mathcal{A}$ under this map is a closed subset of $\mathcal{L}(\mathcal{H})$ that is isomorphic, as a $C*$-algebra, to $\mathcal{A}$. The *Gelfand-Naimark theorem* says that each *$C*$-algebra* has a representation $\mathcal{H}$. If $\mathcal{A}$ is separable, then $\mathcal{H}$ can be taken separable too.

A map $\tau : \mathcal{A} \to \mathbb{C}$ is a *linear form* if

$$\text{(xi)}\quad \tau(aA + bB) = a\tau(A) + b\tau(B) \quad (A, B \in \mathcal{A},\ a, b \in \mathbb{C}).$$

It is called *real* if

$$\text{(xii)}\quad \tau(A^*) = \overline{\tau(A)} \quad (A \in \mathcal{A}).$$

Equivalently, (xii) says that $\tau(A) \in \mathbb{R}$ for each self-adjoint $A \in \mathcal{A}$. A *positive* linear form is a real linear form such that

$$\text{(xiii)}\quad \tau(A^*A) \geq 0 \quad (A \in \mathcal{A}).$$

Equivalently, (xiii) says that $\tau(A) \geq 0$ for each positive $A \in \mathcal{A}$, i.e., for each self-adjoint $A \in \mathcal{A}$ with $\sigma(A) \subset [0, \infty)$. If moreover

- $\tau(A^*A) = 0 \ \Rightarrow\ A = 0$,

then we say that $\tau$ is *faithful*. A positive linear form that is normalized in the sense that

$$\text{(xiv)}\quad \tau(1) = 1$$

is called a *state*. If moreover

- $\tau(AB) = \tau(BA) \qquad (A, B \in \mathcal{A})$,

then $\tau$ is called a *pseudotrace*. It can be shown that every positive linear form is continuous, and in fact satisfies

$$|\tau(A)| \leq |\tau(1)|\, \|A\|.$$

*Example 1* We can take $\mathcal{A} = M_n(\mathbb{C})$, the space of all complex $n \times n$ matrices, equipped with the usual adjoint and the *normalized trace* $\tau(A) := \frac{1}{n}\mathrm{tr}(A)$. Then $\tau$ is a state, and moreover a faithful pseudotrace.

*Example 2* If $\mathcal{H}$ is a representation of $\mathcal{A}$ and $\psi \in \mathcal{H}$ satisfies $\|\psi\| = 1$, then setting $\tau(A) := \langle \psi, A\psi \rangle$ defines a state on $\mathcal{A}$. Convex combinations of this sort of states are dense in the space of all states on $\mathcal{A}$.

By definition, an element $X \in \mathcal{A}$ is *normal* if $XX^* = X^*X$. An $n \times n$ matrix $X \in M_n(\mathbb{C})$ is normal if and only if it is diagonal with respect to an orthonormal basis of $\mathbb{C}^n$. Equivalently, this says that there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ such that

$$X = \sum_{i=1}^{n} \lambda_i P_{e_i}, \tag{1}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $X$ and $P_{e_i}$ denotes the orthogonal projection operator on $e_i$. We can define a *spectral measure* $\pi_X$ by

$$\pi_X(D) = \sum_{i:\, \lambda_i \in D} P_{e_i} \qquad (D \in \mathcal{B}(\mathbb{C})),$$

where $\mathcal{B}(\mathbb{C})$ denotes the Borel-$\sigma$-algebra on $\mathbb{C}$. Then (1) can formally be written

$$X = \int \lambda\, \pi_X(\mathrm{d}\lambda).$$

More generally,

$$X^k(X^*)^l = \int \lambda^k \overline{\lambda}^l \, \pi_X(\mathrm{d}\lambda) \qquad (k, l \geq 0). \tag{2}$$

By the complex version of the Stone-Weierstrass theorem, this formula determines $\pi_X$ uniquely. It turns out that (2) can be generalized to any normal element of a $C*$-algebra $\mathcal{A}$. More precisely, if $\mathcal{A}$ is a $C*$-algebra and $X \in \mathcal{A}$ satisfies $XX^* = X^*X$, then there exists a unique projection operator-valued measure $\pi_X$ on $\mathbb{C}$ such that (2) holds for each $k, l$. The measure $\pi_X$ is called the *spectral measure* of $X$. One can show that $\pi_X$ is concentrated on the *spectrum* of $X$, defined as

$$\sigma(X) := \big\{\lambda \in \mathbb{C} : X - \lambda \text{ is not invertible}\big\}.$$

One has $\|X\| = \sup\{|\lambda| : \lambda \in \sigma(X)\}$ and $\sigma(X)$ is a compact subset of $\mathbb{C}$. A normal operator $X$ is self-adjoint if and only if its spectrum is real, i.e., $\sigma(X) \subset \mathbb{R}$. More generally than in (2), one has

$$F(X) = \int F(\lambda) \, \pi_X(\mathrm{d}\lambda) \tag{3}$$

for any polynomial $F$ of $X$ and $X^*$. By the Stone-Weierstrass theorem, the polynomials are dense in the space of continuous function $F : \sigma(X) \to \mathbb{C}$, equipped with the supremumnorm. We can use this to take the right-hand side of (3) as the definition of $F(X)$ for any continuous function $F : \mathbb{C} \to \mathbb{C}$.

## 2 Quantum probability

A pair $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a $C*$-algebra and $\tau$ is a state on $\mathcal{A}$ is a *quantum probability space.* Here $\mathcal{A}$ plays more or less the role of a $\sigma$-field, $\tau$ plays more or less the role of a probability measure, and self-adjoint operators correspond to real random variables.

Let $\mathcal{A}$ be a $C*$-algebra. For any set $\mathcal{X} \subset \mathcal{A}$, we let $\alpha(\mathcal{X})$ denote the smallest sub-$C*$-algebra (i.e., linear subspace that contains 1, that is closed under the product and adjoint operation, and is closed in the norm) of $\mathcal{A}$ that contains $\mathcal{X}$. Letting $\mathcal{X}^* := \{X^* : X \in \mathcal{X}\}$, one has

$$\alpha(\mathcal{X}) = \overline{\operatorname{span}\big\{\prod_{i=1}^{n} Y_i : Y_i \in \mathcal{X} \cup \mathcal{X}^*\big\}},$$

where $\overline{\mathcal{B}}$ denotes the closure of a set $\mathcal{B} \subset \mathcal{A}$ in the norm. In particular, we write $\alpha(X) := \alpha(\{X\})$. If $X$ is normal, then one can prove that

$$\alpha(X) = \{F(X) : F : \mathbb{C} \to \mathbb{C} \text{ continuous}\}. \tag{4}$$

If $X \in \mathcal{A}$ is self-adjoint, then we claim that setting

$$\int F(\lambda) \, \mu_X(\mathrm{d}\lambda) := \tau\big(F(X)\big) = \tau\Big(\int F(\lambda) \, \pi_X(\mathrm{d}\lambda)\Big)$$

for any continuous $F : \mathbb{R} \to \mathbb{C}$ defines a probability measure on $\mathbb{R}$ that is concentrated on $\sigma(X)$. Indeed, by (xi) the map $F \mapsto \tau\big(F(X)\big)$ is linear, by (xii) one has $\tau\big(F(X)\big) \in \mathbb{R}$ if $F$ is real, by (xiii) one has $\tau\big(F(X)\big) \geq 0$ if $F \geq 0$, and by (xiv) one has $\tau\big(F(X)\big) = 1$ if $F \equiv 1$. In quantum probability, a self-adjoint operator $X$ is called an *observable*, and $\mu_X$ is interpreted as its law.

If $X_1, \ldots, X_n$ are normal operators that commute with each other, then there exists a measure $\pi_{(X_1,\ldots,X_n)}$ on $\mathbb{C}^n$ such that

$$F(X_1, \ldots, X_n) = \int_{\mathbb{C}^n} F(\lambda_1, \ldots, \lambda_n) \pi_{(X_1,\ldots,X_n)}(\mathrm{d}\lambda). \tag{5}$$

This formula should be interpreted as follows. First, one shows that there exists a unique measure $\pi_{(X_1,\ldots,X_n)}$ on $\mathbb{C}^n$ taking values in the space of projection operators such that (5) holds for each polynomial of $X_1,\ldots,X_n,X_1^*,\ldots,X_n^*$. Next, for any continuous function $F:\mathbb{C}^n\to\mathbb{C}$, one takes (5) as the definition of $F(X_1,\ldots,X_n)$, which is equivalent to what one would if one would define $F(X_1,\ldots,X_n)$ via approximation with polynomials. In general, $\pi_{(X_1,\ldots,X_n)}$ is concentrated on $\sigma(X_1)\times\sigma(X_n)$.

In particular, if $X_1,\ldots,X_n$ are self-adjoint operators that commute with each other, then setting

$$\int F(\lambda_1,\ldots,\lambda_n)\,\mu_{(X_1,\ldots,X_n)}(\mathrm{d}\lambda):=\tau\big(F(X_1,\ldots,X_n)\big)=\tau\Big(\int F(\lambda_1,\ldots,\lambda_n)\,\pi_{(X_1,\ldots,X_n)}(\mathrm{d}\lambda)\Big)$$

for any continuous $F:\mathbb{R}\to\mathbb{C}$ defines a probability measure on $\mathbb{R}^n$ that has the interpretation of the *joint law* of the observables $X_1,\ldots,X_n$. A peculiar feature of quantum probability is that if two observables do not commute, then their joint law is not defined. This is related to the Heisenberg uncertaincy principle, which says that the momentum and position of a particle cannot both simultaneously be determined with arbitrary precision.

Note that if $X_1,\ldots,X_n$ are normal operators that commute with each other, then the sub-$C*$-algebra $\alpha(X_1,\ldots,X_n):=\alpha(\{X_1,\ldots,X_n\})$ that they generate is abelian. More generally, one can show that a quantum probability space $(\mathcal{A},\tau)$ where $\mathcal{A}$ is abelian corresponds to a classical probability space, so quantum probability can be seen as an extension of classical probability.

# 3 Independence

Let $\mathcal{A}_1,\mathcal{A}_2\subset\mathcal{A}$ be sub-$C*$-algebras of some larger $C*$-algebra $\mathcal{A}$. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ *commute* if

$$A_1A_2=A_2A_1\qquad(A_1\in\mathcal{A}_1,\ A_2\in\mathcal{A}_2).$$

We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *logically independent* if

- $\{A_kB_l:1\le k\le n,\ 1\le l\le m\}$ are linearly independent whenever $\{A_1,\ldots,A_n\}\subset\mathcal{A}_1$ and $\{B_1,\ldots,B_m\}\subset\mathcal{A}_2$ are linearly independent.

**Lemma 1 (Product algebra)** *Let $\mathcal{A}_1,\mathcal{A}_2\subset\mathcal{A}$ be $C*$-algebras. Then it is possible to construct a $C*$-algebra $\mathcal{A}$ that contains logically independent sub-$C*$-algebras $\mathcal{A}_1',\mathcal{A}_2'\subset\mathcal{A}$ such that $\mathcal{A}_1'$ is isomorphic to $\mathcal{A}_1$ and $\mathcal{A}_2'$ is isomorphic to $\mathcal{A}_2$.*

**Proof (sketch)** If $\mathcal{H}_1$ and $\mathcal{H}_2$ are representations of $\mathcal{A}_1$ and $\mathcal{A}_2$, then one can make $\mathcal{H}_1\otimes\mathcal{H}_2$ into a representation of both $\mathcal{A}_1$ and $\mathcal{A}_2$ by setting $A_1(\phi_1\otimes\phi_2):=(A_1\phi_1)\otimes\phi_2$ and $A_2(\phi_1\otimes\phi_2):=\phi_1\otimes(A_2\phi_2)$ for $A_i\in\mathcal{A}_i$ $(i=1,2)$. Now $\mathcal{A}:=\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$ has the desired properties. ∎

Let $\tau$ be a state on $\mathcal{A}$. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *independent* under $\tau$ if $\mathcal{A}_1$ and $\mathcal{A}_2$ commute and

$$\tau(A_1A_2)=\tau(A_1)\tau(A_2)\qquad(A_1\in\mathcal{A}_1,\ A_2\in\mathcal{A}_2).$$

**Proposition 2 (Product state)** *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ commute and are logically independent. Let $\tau_i$ be a state on $\mathcal{A}_i$ $(i=1,2)$. Then there exists a unique state $\tau_{12}$ on $\alpha(\mathcal{A}_1\cup\mathcal{A}_2)$ such that its restrictions to $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\tau_1$ and $\tau_2$ and $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent under $\tau_{12}$. Without the assumption of logical independence, the uniqueness statement still holds but existence may fail.*

**Proof (sketch)** We only prove the uniqueness statement. If $\mathcal{A}_1$ and $\mathcal{A}_2$ commute, then

$$\alpha(\mathcal{A}_1\cup\mathcal{A}_2)=\overline{\operatorname{span}\{A_1A_2:A_1\in\mathcal{A}_1,\ A_2\in\mathcal{A}_2\}}.\tag{6}$$

Thus $\tau_{12}$ is uniquely determined by $\tau_{12}(A_1A_2) = \tau_1(A_1)\tau_2(A_2)$ $(A_i \in \mathcal{A}_i,\ i = 1, 2)$. ∎

In the special case that $\mathcal{A}_i$ $(i = 1, 2)$ are of the form $\alpha(X_i)$ with $X_i$ an observable, Proposition 2 reduces to the usual concept of two independent real random variables.

**Lemma 3 (Independent observables)** *Let $\mathcal{A}$ be a $C*$-algebra, let $\tau$ be a state on $\mathcal{A}$, and let $X_1, X_2 \in \mathcal{A}$ be self-adjoint. Then the following statements are equivalent.*

(i) *$\alpha(X_1)$ and $\alpha(X_2)$ are independent.*

(ii) *$X_1$ commutes with $X_2$ and $\mu_{(X_1,X_2)}$ is the product measure of $\mu_{X_1}$ and $\mu_{X_2}$.*

**Proof** If (i) holds, then $X_1$ must commute with $X_2$ and

$$\begin{aligned}\int F_1(\lambda_1)F_2(\lambda_2)\mu_{(X_1,X_2)}(\mathrm{d}\lambda) &= \tau\big(F_1(X_1)F_2(X_2)\big) = \tau\big(F_1(X_1)\big)\tau\big(F_2(X_2)\big) \\ &= \Big(\int F_1(\lambda)\mu_{X_1}(\mathrm{d}\lambda)\Big)\Big(\int F_2(\lambda)\mu_{X_2}(\mathrm{d}\lambda)\Big)\end{aligned}$$

for all continuous $F_1, F_2$, which shows that $\mu_{(X_1,X_2)}$ is the product measure of $\mu_{X_1}$ and $\mu_{X_2}$. Conversely, if (ii) holds, then $\alpha(X_1)$ commutes with $\alpha(X_2)$ and the calculation above and (4) show that $\tau(A_1A_2) = \tau(A_1)\tau(A_2)$ for all $A_1 = \alpha(X_1)$ and $A_2 = \alpha(X_2)$. ∎

If the equivalent conditions of Lemma 3 are satisfied, then we say that that the observables $X_1, X_2$ are independent. Lemma 3 implies that for independent self-adjoint operators,

$$\mu_{X_1+X_2} = \mu_{X_1} * \mu_{X_2},$$

where $*$ denotes convolution of probability measures.

# 4 Free independence

Let $\mathcal{A}$ be a $C*$-algebra and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be sub-$C*$-algebras. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *free* if

- Whenever $1, A_1, \ldots, A_n \in \mathcal{A}_1$ are linearly independent and $1, B_1, \ldots, B_m \in \mathcal{A}_2$ are linearly independent, one has that all alternating products of the form

$$1,\ A_{i_1},\ B_{j_1},\ A_{i_1}B_{j_2},\ B_{j_1}A_{i_2},\ A_{i_1}B_{j_2}A_{i_3},\ B_{j_1}A_{i_2}B_{j_3},\ A_{i_1}B_{j_2}A_{i_3}B_{j_4}, \ldots \tag{7}$$

  with $i_1, i_2, \ldots \in \{1, \ldots, n\}$ and $j_1, j_2 \ldots \in \{1, \ldots, m\}$ are linearly independent.

The use of the word "free" here is similar to its use in the expression "a free group". In a sense, this means that the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are maximally noncommuting. Since both $\mathcal{A}_1$ and $\mathcal{A}_2$ contain the identity, and the identity obviously commutes with itself, we have to give the identity a special role in our definition. To understand why we need linear independence, imagine that there would exist $A_1, A_2 \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$ such that $A_1BA_2 = 2A_1^2B + 1$. Then using such a linear relation, one could try to simplify products of elements of $\mathcal{A}_1$ and $\mathcal{A}_2$. The freeness condition says that no such simplifications are possible. Note that $\alpha(\mathcal{A}_1 \cup \mathcal{A}_2)$ is infinite dimensional as soon as $\mathcal{A}_1$ and $\mathcal{A}_2$ each have dimension $\geq 2$. We skip the somewhat tedious proof of the following fact.

**Lemma 4 (Free product algebra)** *Let $\mathcal{A}_1, \mathcal{A}_2$ be $C*$-algebras. Then it is possible to construct a $C*$-algebra $\mathcal{A}$ that contains free sub-$C*$-algebras $\mathcal{A}'_1, \mathcal{A}'_2 \subset \mathcal{A}$ such that $\mathcal{A}'_1$ is isomorphic to $\mathcal{A}_1$ and $\mathcal{A}'_2$ is isomorphic to $\mathcal{A}_2$.*

Let $\tau$ be a state on $\mathcal{A}$ and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be sub-$C*$-algebras. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *freely independent* if

$$\tau(A_1 B_2) = 0,\ \tau(B_1 A_2) = 0,\ \tau(A_1 B_2 A_3) = 0,\ \tau(B_1 A_2 B_3) = 0,\ \tau(A_1 B_2 A_3 B_4) = 0, \ldots \quad (8)$$

whenever $A_1, A_2, \ldots \in \mathcal{A}_1$ and $B_1, B_2, \ldots \in \mathcal{A}_2$ satisfy $\tau(A_i) = 0$ $i = 1, 2, \ldots$ and $\tau(B_j) = 0$ $j = 1, 2, \ldots$.

**Proposition 5 (Free product state)** *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be free and let $\tau_i$ be states on $\mathcal{A}_i$ $(i = 1, 2)$. Then there exists a unique state $\tau_{12}$ on $\alpha(\mathcal{A}_1 \cup \mathcal{A}_2)$ whose restriction to $\mathcal{A}_i$ is $\tau_i$ $(i = 1, 2)$ such that under $\tau_{12}$, the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are freely independent. If we drop the assumption that $\mathcal{A}_1$ and $\mathcal{A}_2$ are free, then the uniqueness statement still holds (but $\tau_{12}$ may fail to exist in general).*

**Proof (sketch)** We only prove the uniqueness statement. Let $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$. Then $X - \tau(X)1$ has trace zero and hence

$$0 = \tau\big((X - \tau(X))(Y - \tau(Y))\big) = \tau(XY) - \tau(X)\tau(Y),$$

from which we see that

$$\tau(XY) = \tau(X)\tau(Y). \quad (9)$$

Similarly, for $X_1, X_2 \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$,

$$\begin{aligned} 0 &= \tau\big((X_1 - \tau(X_1))(Y - \tau(Y))(X_2 - \tau(X_2))\big) \\ &= \tau(X_1 Y X_2) - \tau(X_1 Y)\tau(X_2) - \tau(X_1 X_2)\tau(Y) - \tau(Y X_2)\tau(X_1) \\ &\quad +3\tau(X_1)\tau(Y)\tau(X_2) - \tau(X_1)\tau(Y)\tau(X_2) \\ &= \tau(X_1 Y X_2) - \tau(X_1 X_2)\tau(Y), \end{aligned}$$

where in the last step we have used (9). It follows that

$$\tau(X_1 Y X_2) = \tau(X_1 X_2)\tau(Y) \qquad (X_1, X_2 \in \mathcal{A}_1,\ Y \in \mathcal{A}_2), \quad (10)$$

which in fact we would also have if $\mathcal{A}_1$ and $\mathcal{A}_2$ were independent (and would commute). Similarly

$$\tau(Y_1 X Y_2) = \tau(Y_1 Y_2)\tau(X) \qquad (X \in \mathcal{A}_1,\ Y_1, Y_2 \in \mathcal{A}_2). \quad (11)$$

However, continuing in the same spirit, we find that for $X_1, X_2 \in \mathcal{A}_1$ and $Y_1, Y_2 \in \mathcal{A}_2$,

$$\begin{aligned} 0 &= \tau\big((X_1 - \tau(X_1))(Y_1 - \tau(Y_1))(X_2 - \tau(X_2))(Y_2 - \tau(Y_2))\big) = \tau(X_1 Y_1 X_2 Y_2) \\ &\quad -\tau(X_1 Y_1 X_2)\tau(Y_2) - \tau(X_1 Y_1 Y_2)\tau(X_2) - \tau(X_1 X_2 Y_2)\tau(Y_1) - \tau(Y_1 X_2 Y_1)\tau(X_1) \\ &\quad +\tau(X_1 Y_1)\tau(X_2)\tau(Y_2) + \tau(X_1 X_2)\tau(Y_1)\tau(Y_2) + \tau(X_1 Y_2)\tau(Y_1)\tau(X_2) \\ &\quad +\tau(Y_1 X_2)\tau(X_1)\tau(Y_2) + \tau(Y_1 Y_2)\tau(X_1)\tau(X_2) + \tau(X_2 Y_2)\tau(X_1)\tau(Y_1) \\ &\quad -4\tau(X_1)\tau(Y_1)\tau(X_2)\tau(Y_2) + \tau(X_1)\tau(Y_1)\tau(X_2)\tau(Y_2). \end{aligned}$$

Using (9), we can simplify this to

$$\begin{aligned} \tau(X_1 Y_1 X_2 Y_2) &= \tau(X_1 Y_1 X_2)\tau(Y_2) + \tau(X_1 Y_1 Y_2)\tau(X_2) + \tau(X_1 X_2 Y_2)\tau(Y_1) + \tau(Y_1 X_2 Y_1)\tau(X_1) \\ &\quad -\tau(X_1 X_2)\tau(Y_1)\tau(Y_2) - \tau(Y_1 Y_2)\tau(X_1)\tau(X_2) - \tau(X_1)\tau(Y_1)\tau(X_2)\tau(Y_2), \end{aligned}$$

and using (10) and (11) we can further simplify this to

$$\tau(X_1 Y_1 X_2 Y_2) = \tau(X_1 X_2)\tau(Y_1)\tau(Y_2) + \tau(Y_1 Y_2)\tau(X_1)\tau(X_2) - \tau(X_1)\tau(X_2)\tau(Y_1)\tau(Y_2). \quad (12)$$

This time, we get something different from the independent case. (In the independent case, we would get $\tau(X_1X_2)\tau(Y_1Y_2)$.) Nevertheless, it is not hard to show by induction that using (8), one can express $\tau$ of any mixed moment of elements of $\mathcal{A}_1$ and $\mathcal{A}_2$ in moments of elements of $\mathcal{A}_1$ and $\mathcal{A}_2$ separately. ∎

We say that two self-adjoint operators $X_1$ and $X_2$ are freely independent if they generate freely independent sub-$C*$-algebras. It follows from Proposition 5 that if $X_1$ and $X_2$ are freely independent, then the law of $\mu_{X_1+X_2}$ (and in fact any reasonable function of $X_1$ and $X_2$) is uniquely determined by the marginal laws $\mu_{X_1}$ and $\mu_{X_2}$, so we can write

$$\mu_{X_1+X_2} = \mu_{X_1} \boxplus \mu_{X_2},$$

where $\boxplus$ is called the *free convolution* of two probability measures.

Free independence of three or more algebras is defined in a similar way as for two algebras. It is not hard to see that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are freely independent if and only if $\mathcal{A}_{i+1}$ is freely independent of $\alpha(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_i)$ for each $i$.

## 5 The Free Central Limit Theorem

We note that if $X$ and $Y$ are freely independent with mean

$$\tau(X) = \int \lambda\, \mu_X(\mathrm{d}\lambda) = 0 \quad \text{and} \quad \tau(Y) = \int \lambda\, \mu_Y(\mathrm{d}\lambda) = 0,$$

then by (9),

$$\int \lambda^2\, \mu_{X+Y}(\mathrm{d}\lambda) = \tau\big((X+Y)^2\big) = \tau(X^2) + \tau(Y^2).$$

We set

$$\mathrm{Var}(X) := \tau\big((X - \tau(X))^2\big) = \mathrm{Var}(\mu_X).$$

Then more generally, the variance of $X+Y$ is the sum of the variances of $X$ and $Y$. We recall that the (standard) semicircle law has mean zero and variance $C_{2/2} = 1$. More generally, we can define semicircle laws with any mean and variance by adding a constant and scaling. The following proposition and theorem show that free independence is indeed very similar to classical independence.

**Proposition 6 (Stability of the semicircle law)** *Assume that $X_1, \ldots, X_k$ are freely independent and that $X_i$ has a semicircle law with mean $\tau(X_i)$ and variance $\mathrm{Var}(X_i)$. Then $\sum_{i=1}^k X_i$ has a semicircle law with mean $\sum_{i=1}^k \tau(X_i)$ and variance $\sum_{i=1}^k \mathrm{Var}(X_i)$.*

**Theorem 7 (Free Central Limit Theorem)** *Let $(X_i)_{i\geq 1}$ be self-adjoint elements of some $C*$-algebra that are freely independent and identically distributed with mean zero and variance 1. Then the law of $\frac{1}{\sqrt{n}}\sum_{i=1}^k X_i$ converges weakly to the semicircle law.*

We will not prove Proposition 6 and Theorem 7 but we will show that the free convolution of the standard semicircle law with itself yields a semicircle law with variance 2.

Consider the Hilbert space $\mathcal{H} := \ell^2(\mathbb{N})$ of square integrable functions $f : \mathbb{N} \to \mathbb{C}$ with the usual inner product. Let $e_0, e_1, \ldots$ denote the usual basis. Let $\tau_0$ be the state on $\mathcal{A} := \mathcal{L}(\mathcal{H})$ defined as $\tau_0(A) := \langle e_0, Ae_0\rangle$. Define $U : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $Ue_n := e_{n+1}$ $(n \geq 0)$. It is not hard to see that $U$ is a unitary operator[1] and $U^*e_n = e_{n-1}$ $(n \geq 1)$ while $U^*e_0 = 0$. Set

[1] In the sense that $\langle U\phi, U\psi\rangle = \langle \phi, \psi\rangle$ and the left inverse $U^{-1}$ equals $U^*$. Note that $U$ does not have a right inverse!

$X = U + U^*$, which is self-adjoint. We claim that $\mu_X$ is the semicircle law.[2] To see this, we calculate its moments. Removing the brackets in

$$\tau_0\big((U+U^*)^n\big) = \langle e_0, (U+U^*)^n e_0\rangle,$$

we obtain $2^n$ terms of the form $\langle e_0, V_n \cdots V_i e_0\rangle$ with $V_i \in \{U, U^*\}$ for all $1 \le i \le n$. Each term gives a contribution zero or one. The nonzero terms are precisely those for which there exists a random walk path $S : \{0,\dots,n\} \to \mathbb{N}$ such that

$$V_k \cdots V_i e_0 = e_{S_k} \qquad (0 \le k \le n)$$

with $S_0 = S_n = 0$. This means that $\tau_0\big((U+U^*)^n\big) = 0$ if $n$ is odd and $\tau_0\big((U+U^*)^n\big) = C_{n/2}$ if $n$ is even, where $C_{k/2}$ is the Catalan number we have seen before. We recognize the moments of the (standard) semicircle law, so we conclude that $\mu_X$ is the semicircle law.

Let $\mathbb{T}$ be the set whose elements are finite words of the form $\mathbf{i} = i_1 \cdots i_n$ made from the alphabet $\{1,2\}$. We call $|\mathbf{i}| := n$ the length of the word and let $\varnothing$ denote the word of length zero. Consider the Hilbert space $\mathcal{F} := \ell^2(\mathbb{T})$ of square integrable functions $f : \mathbb{T} \to \mathbb{C}$, with the usual inner product. Let $\mathbb{B}_1$ be the set of all $\mathbf{j} \in \mathbb{T}$ such that either $\mathbf{j} = \varnothing$ or $|\mathbf{j}| = n$ for some $n \ge 1$ and $j_n = 2$. Then for each $\mathbf{j} \in \mathbb{B}_1$ and $f \in \mathcal{F}$, we can define $f_\mathbf{j} \in \ell(\mathbb{N})$ by

$$f_\mathbf{j}(k) := f\big(\mathbf{j}\ \underbrace{1\cdots1}_{k \text{ times}}\ \big) \qquad (\mathbf{j} \in \mathbb{B}_1,\ k \in \mathbb{N}).$$

A function $f \in \mathcal{F}$ is uniquely characterised by the functions $\{f_\mathbf{j} : \mathbf{j} \in \mathbb{B}_1\}$, so that we can view $\mathcal{F}$ as the direct sum $\mathcal{F} = \bigoplus_{\mathbf{j}\in\mathbb{B}_1} \ell(\mathbb{N})$. We define $\ell_1 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{F})$ by

$$\big(\ell_1(A)f\big)_\mathbf{j} := Af_\mathbf{j} \qquad (\mathbf{j} \in \mathbb{B}_1).$$

Similarly, we let $\mathbb{B}_2$ be the set of all $\mathbf{j} \in \mathbb{T}$ such that either $\mathbf{j} = \varnothing$ or $|\mathbf{j}| = n$ for some $n \ge 1$ and $j_n = 1$, and set

$$\big(\ell_2(A)f\big)_\mathbf{j} := Af_\mathbf{j} \qquad (\mathbf{j} \in \mathbb{B}_2) \quad \text{with} \quad f_\mathbf{j}(k) := f\big(\mathbf{j}\ \underbrace{2\cdots2}_{k \text{ times}}\ \big) \quad (k \ge 0).$$

We let $\mathcal{A}_i := \ell_i(\mathcal{A}) = \{\ell_i(A) : A \in \mathcal{L}(\mathcal{H})\}$ be the sub-$C*$-algebra of $\mathcal{L}(\mathcal{F})$ generated by elements of the form $\ell_i(A)$ $(A \in \mathcal{L}(\mathcal{H}),\ i = 1,2)$. Let $e_\varnothing, e_1, e_2, e_{11}, e_{12}, \dots$ denote the obvious orthonormal basis and let $\tau_\varnothing$ be the state on $\mathcal{A} := \mathcal{L}(\mathcal{F})$ defined as $\tau_\varnothing(A) := \langle e_\varnothing, Ae_\varnothing\rangle$. We claim that

- $\ell_i$ is a $C*$-algebra isomorphism from $\mathcal{A} = \mathcal{L}(\mathcal{H})$ to $\mathcal{A}_i = \ell_i(\mathcal{A})$ $(i = 1,2)$.
- $\tau_\varnothing\big(\ell_i(A)\big) = \tau_0(A)$ $(A \in \mathcal{A})$.
- $\mathcal{A}_1$ and $\mathcal{A}_2$ are freely independent under $\tau_\varnothing$.

To see the third point, we note that $\tau_0(X) = 0$ if and only if $(Xe_0)(0) = 0$. Now if $X_1, X_2, X_3, \dots \in \mathcal{L}(\mathcal{H})$ satisfy

$$\begin{aligned}
&\tau_\varnothing\big(\ell_1(X_1)\big) = \tau_0(X_1) = 0,\\
&\tau_\varnothing\big(\ell_2(X_2)\big) = \tau_0(X_2) = 0,\\
&\tau_\varnothing\big(\ell_1(X_3)\big) = \tau_0(X_3) = 0,\\
&\dots,
\end{aligned}$$

[2]This implies that $X$ has a purely continuous spectrum.

then $\ell_1(X_1)e_\varnothing$ is concentrated on $\{1,11,111,1111,\ldots\}$, hence $\ell_2(X_2)\ell_1(X_1)e_\varnothing$ is concentrated on

$$\big\{12,122,1222,\ldots,112,1122,11222,\ldots,1112,11122,\ldots\big\},$$

and so on, proving free independence. In particular, these functions are zero in $\varnothing$ so that

$$\tau_\varnothing\big(\ell_1(X_3)\ell_2(X_2)\ell_1(X_1)\big)=0,$$

and so on. Now let $U_i=\ell_i(U)$ and $U_i^*=\ell_i(U^*)=\ell_i(U)^*$. Then

$$U_ie_{\mathbf{j}}:=e_{\mathbf{j}i}\quad\text{and}\quad U_i^*e_{\mathbf{j}}=\begin{cases}e_{j_2\cdots i_n} & \text{if } n\geq 1 \text{ and } i_1=j,\\ 0 & \text{otherwise}\end{cases}\qquad(\mathbf{j}=j_1\cdots j_n\in\mathbb{T}).$$

Let $X_i:=U_i+U_i^*$. We have already seen that $X_1$ and $X_2$ are freely independent and that $\mu_{X_i}$ is the standard semicircle law ($i=1,2$). We claim that $\mu_{X_1+X_2}$ is a semicircle law with variance 2. To see this, we calculate moments. Removing the brackets in

$$\tau_\varnothing\big((U_1+U_2+U_1^*+U_2^*)^n\big)=\langle e_0,(U_1+U_2+U_1^*+U_2^*)^ne_0\rangle,$$

we obtain $4^n$ terms of the form $\langle e_\varnothing,V_n\cdots V_ie_\varnothing\rangle$ with $V_i\in\{U_1,U_2,U_1^*,U_2^*\}$ for all $1\leq i\leq n$. Each term gives a contribution zero or one. The nonzero terms are precisely those for which there exists a random walk path $S:\{0,\ldots,n\}\to\mathbb{T}$ such that

$$V_k\cdots V_ie_\varnothing=e_{S_k}\qquad(0\leq k\leq n)$$

with $S_0=S_n=\varnothing$. This means that $\tau_\varnothing\big((X_1+X_2)^n\big)=0$ if $n$ is odd and $\tau_\varnothing\big((X_1+X_2)^n\big)=2^{n/2}C_{n/2}$ if $n$ is even, where $C_{k/2}$ is the Catalan number we have seen before, and the factor $2^{n/2}$ comes from the fact that each time we make a step deeper into the tree, we have two choices where to go. (On the way back there is no such choice, so we get $2^{n/2}$ and not $2^n$.). This means that

$$\tau_\varnothing\big((X_1+X_2)^n\big)=\tau_0\big((\sqrt{2}X)^n\big),$$

where $X$ has a standard semicircle law. We conclude that $\mu_{X_1+X_2}$ is a semicircle law with variance 2.

# 6 Convergence in law

Let $m\geq 1$ be an integer, and for each $n$, let $(\mathcal{A}_n,\tau_n,X_{n,1},\ldots,X_{n,m})$ be a quantum probability space that contains $m$ self-adjoint elements $X_{n,1},\ldots,X_{n,m}$. By definition, we say that the sequence $(\mathcal{A}_n,\tau_n,X_{n,1},\ldots,X_{n,m})$ converges in law to a limit $(\mathcal{A},\tau,X_1,\ldots,X_m)$, which we denote as

$$(\mathcal{A}_n,\tau_n,X_{n,1},\ldots,X_{n,m})\underset{n\to\infty}{\Longrightarrow}(\mathcal{A},\tau,X_1,\ldots,X_m),$$

if

(i) $\|X_{n,k}\|\underset{n\to\infty}{\longrightarrow}\|X_k\|$ for all $1\leq k\leq m$,

(ii) $\tau\big(X_{n,k_1}\cdots X_{n,k_d}\big)\underset{n\to\infty}{\longrightarrow}\tau\big(X_{k_1}\cdots X_{k_d}\big)\quad\forall(k_1,\ldots,k_d)\in\{1,\ldots,m\}^d,\ d\geq 1.$

**Proposition 8 (Weak convergence)** *Assume that the self-adjoint elements $X_{n,1},\ldots,X_{n,m}$ commute for each $n$, and that also $X_1,\ldots,X_m$ commute. Then the conditions* (i) *and* (ii) *are equivalent to*

(i) $\|X_{n,k}\|\underset{n\to\infty}{\longrightarrow}\|X_k\|$ *for all* $1\leq k\leq m$, (ii)' $\mu_{X_{n,1},\ldots,X_{n,m}} \underset{n\to\infty}{\Longrightarrow} \mu_{(X_1,\ldots,X_m)}$,

*where in* (ii)', $\Rightarrow$ *denotes weak convergence of probability measures on* $\mathbb{R}^m$.

**Proof** Thanks to condition (i), there exists a compact set $C \subset \mathbb{R}^m$ so that the measures $\mu_{X_{n,1},\ldots,X_{n,m}}$ and $\mu_{(X_1,\ldots,X_m)}$ are all concentrated on $C$. As a result, condition (ii)' is equivalent to

$$\int \lambda_1^{p_1} \cdots \lambda_m^{p_m} \mu_{X_{n,1},\ldots,X_{n,m}} \underset{n\to\infty}{\longrightarrow} \int \lambda_1^{p_1} \cdots \lambda_m^{p_m} \mu_{X_1,\ldots,X_m} \quad \forall p_1,\ldots,p_m \geq 0.$$

Due to the commutativity assumption, filling in the definition of $\mu_{(X_1,\ldots,X_m)}$, we see that this is equivalent to condition (ii). ∎

## 7 Relation to random matrix theory

Let $M = (\xi_{ij})_{i,j\in\mathbb{N}_+}$ be an infinite hermitian Wigner matrix and let $M' = (\xi'_{ij})_{i,j\in\mathbb{N}_+}$ be an independent copy of $M$. For each $n \geq 1$, let $M_n := (\xi_{ij})_{1\leq i,j\leq n}$ and $M'_n := (\xi_{ij})_{1\leq i,j\leq n}$. Let $\mathcal{A}_n$ denote the $C*$-algebra consisting of all complex $n \times n$ matrices and let $\tau_n(A) := n^{-1}\mathrm{tr}(A)$ denote the normalised trace on $\mathcal{A}_n$. I believe the following theorem probably holds, though I do not know an exact reference.

**Theorem 9 (Independent Wigner matrices)** *Assume that* $(\xi_{ij})_{i<j}$ *are i.i.d. with mean zero and variance one. Let* $X_n := M_n/\sqrt{n}$ *and* $X'_n := M'_n/\sqrt{n}$. *Then almost surely, one has*

$$(\mathcal{A}_n, \tau_n, X_n, X'_n) \underset{n\to\infty}{\Longrightarrow} (\mathcal{A}, \tau, X, X'), \tag{13}$$

*where* $X$ *and* $X'$ *are freely independent and distributed according to the standard semicircle law.*

The almost sure convergence $(\mathcal{A}_n, \tau_n, X_n) \underset{n\to\infty}{\Longrightarrow} (\mathcal{A}, \tau, X)$ has, of course, been shown in the book. Theorem 9 boosts this to convergence of the "joint law" of two independent Wigner matrices. It is not so hard to intuitively understand why the algebras generated by $X_n$ and $X'_n$ should be "asymptotically free", in the sense of being as non-commuting as they can possibly be. It is less clear why they should be asymptotically freely independent. In the book by Tao, you can find a sketch of a proof, that is based on moment calculations and in fact proves the asymptotic free independence together with the convergence to the semicircle law. Is there an easier way to intuitively understand why free independence should arise in the limit?

Theorem 9 helps us somewhat to understand why the semicircle law occurs in random matrix theory. Imagine that we would only know that (13) holds for some $(\mathcal{A}, \tau, X, X')$ such that $X$ and $X'$ are freely independent. Since $(X_n + X'_n)/\sqrt{2}$ are also Wigner matrices, we then see that

$$\mu_X = \mu_{(X+X')/\sqrt{2}} = \mu_X \boxplus \mu_{X'} = \mu_X \boxplus \mu_X,$$

so this tells us that the limit law of the spectrum of random matrices has to be stable under free convolution.

In Tao's book, a lot of time is spent on showing that rescaled random matrices with i.i.d. entries converge to the semicircle law. From an operator perspective, the assumption of i.i.d. entries is not so natural, since it is basis-dependent, except for special ensembles such as GOE or GUE. I believe that with the help of the concept of free independence, it has been shown that some matrix ensembles that do not have i.i.d. entries also have the semicircle law in the limit.